Lipschitz quotient maps

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1 Definition If (X,ρ) and (Y,σ) are metric spaces, a function $f:X\to Y$ is a **Lipschitz quotient** function if there are γ_1, $\gamma_2>0$ such that

$$B(f(x),\gamma_1\delta)\subseteq f[B(x,\delta)]\subseteq B(f(x),\gamma_2\delta)$$

for all $x\in X$ and $\delta\ge 0$.

2 Definition If (X,ρ) is a metric space, $A\subseteq X$, $x\in X$ and $r>0$, I will say that x is an r**-dimensional density point** of A if $\lim_{\delta\downarrow 0}\frac{\mu_{Hr}^*(B(x,\delta)\setminus A)}{\delta^r}=0$, where μ_{Hr} is r-dimensional Hausdorff measure on X.

Note that if μ_{Hr} measures A, then μ_{Hr}-almost every point of A is an r-dimensional density point of A, while μ_{Hr}-almost no point of $X\setminus A$ is an r-dimensional density point of A (Fremlin 03, 471P).

3 Proposition (O.Maleva) Suppose that (X,ρ) and (Y,σ) are metric spaces, and $f:X\to Y$ is a Lipschitz quotient function. If $r>0$, $A\subseteq X$ and $x\in X$ is an r-dimensional density point of A, then $f(x)$ is an r-dimensional density point of $f[A]$.

proof Let γ_1, γ_2 be constants for f as in Definition 1, and write $\mu_{Hr}^{(X)}$, $\mu_{Hr}^{(Y)}$ for r-dimensional Hausdorff measure on X, Y respectively. Then

$$\begin{aligned}(\mu_{Hr}^{(Y)})^*(B(f(x),\delta)\setminus f[A])&\le(\mu_{Hr}^{(Y)})^*(f[B(f(x),\frac{\delta}{\gamma_1})]\setminus f[A])\\&\le(\mu_{Hr}^{(Y)})^*(f[B(f(x),\frac{\delta}{\gamma_1})\setminus A])\le\gamma_2^r(\mu_{Hr}^{(X)})^*(B(x,\frac{\delta}{\gamma_1})\setminus A)\end{aligned}$$

for every $\delta>0$. So

$$\begin{aligned}\lim_{\delta\downarrow 0}\frac{(\mu_{Hr}^{(Y)})^*(B(f(x),\delta)\setminus f[A])}{\delta^r}&\le\gamma_2^r\lim_{\delta\downarrow 0}\frac{(\mu_{Hr}^{(X)})^*(B(x,\frac{\delta}{\gamma_1})\setminus A)}{\delta^r}\\&\le\frac{\gamma_2^r}{\gamma_1^r}\lim_{\delta\downarrow 0}\frac{(\mu_{Hr}^{(X)})^*(B(x,\delta)\setminus A)}{\delta^r}=0,\end{aligned}$$

as required.

4 Corollary (O.Maleva) Suppose that (X,ρ) is a metric space, $r\ge 1$ is an integer and $f:X\to\mathbb{R}^r$ is a Lipschitz quotient function. Write μ_{Hr} for r-dimensional Hausdorff measure on X, and μ_r for Lebesgue measure on $\mathbb{R}^r$.

(a) If $F\subseteq\mathbb{R}^r$ is Lebesgue negligible then $f^{-1}[F]$ is μ_{Hr}-negligible.
(b) If $A\subseteq X$ is not μ_{Hr}-negligible then $f[A]$ is not μ_r-negligible.

proof (a) It is enough to consider the case in which F is a G_δ set (Fremlin 03, 471Db). Because f is continuous, $f^{-1}[F]$ is a G_δ set so is measured by μ_{Hr}. **?** If $f^{-1}[F]$ is not μ_{Hr}-negligible then it has ar r-dimensional density point x. By Proposition 2, $f(x)$ is an r-dimensional density point of $f[f^{-1}[F]]\subseteq F$. But as μ_r is just a multiple of Hausdorff r-dimensional measure on $\mathbb{R}^r$, and there is a $\beta_r>0$ such that $\mu_rB(f(x),\delta)=\beta_r\delta^r$ for every $\delta\ge 0$, it follows that μ_rF must be non-zero. **X**

(b) now follows at once, just because $A\subseteq f^{-1}[f[A]]$.

5 Lemma Let $r\ge 1$ be an integer, and $f:\mathbb{R}^r\to\mathbb{R}^r$ a Lipschitz quotient function with constants γ_1, γ_2. Let μ be Lebesgue measure on $\mathbb{R}^r$. If $E\subseteq\mathbb{R}^r$ is measurable and $\alpha>\dfrac{1}{\gamma_1^r}$, then there is an $F\subseteq E$ such that $f[F]=f[E]$ and $\mu F\le\alpha\mu f[E]$.

proof (a) Note that $f[E]$ is measurable because E is expressible as the union of a K_σ set and a negligible set, so $f[E]$ also is. Also the result is trivial if $\mu E=0$ or $\mu f[E]=\infty$, so by Corollary 4b we can suppose that $0<\mu f[E]<\infty$.

(b) Suppose that $G\subseteq E$ is measurable and that $\eta>0$. Then there is an $F\subseteq G$ such that

$$\mu F\le\tfrac{1+\eta}{\gamma_1^r}\mu f[G],$$

$$\mu(f[G]\setminus f[F])\le\tfrac{\gamma_2^r\eta(1+\eta)}{\gamma_1^r}\mu f[G].$$

P Let G_0 be the set of density points of G; then $G\setminus G_0$ and $f[G]\setminus f[G_0]\subseteq f[G\setminus G_0]$ are negligible. Let H be the set of density points of $f[G_0]$, and $G_1=G_0\cap f^{-1}[H]$; then $f[G_0]\setminus H$ and $G_0\setminus G_1\subseteq f^{-1}[f[G_0]\setminus H]$ are negligible, by Corollary 4a. Consequently every point of G_1 is a density point of G_1, while every point of H is a density point of H. Also, of course, $f[G]\setminus H$ is negligible, while $f[G_1]=H$.

Let Q be the set of pairs (x,δ) where $x\in G_1$, $\delta>0$, $\mu B(x,\delta)\le(1+\eta)\mu(G\cap B(x,\delta))$ and $\mu B(f(x),\gamma_1\delta)\le(1+\eta)\mu(f[G]\cap B(f(x),\gamma_1\delta))$. Set $\mathcal{I}=\{B(f(x),\gamma_1\delta):(x,\delta)\in Q\}$. Then every point of $H=f[G_1]$ is the centre of arbitrarily small members of $\mathcal{I}$. There is therefore a countable family $\langle(x_i,\delta_i)\rangle_{i\in J}$ in Q such that $\langle B(f(x_i),\gamma_1\delta_i)\rangle_{i\in J}$ is disjoint and $H\setminus\bigcup_{i\in J}B(f(x_i),\gamma_1\delta_i)$ is negligible (Fremlin 01, 261B). Set $F=G\cap\bigcup_{i\in J}B(x_i,\delta_i)$. Then

$$\begin{aligned}\mu F&\le\sum_{i\in J}\mu B(x_i,\delta_i)=\frac{1}{\gamma_1^r}\sum_{i\in J}\mu B(f(x_i),\gamma_1\delta_i)\\&\le\tfrac{1+\eta}{\gamma_1^r}\sum_{i\in J}\mu(H\cap B(f(x_i),\gamma_1\delta_i))=\tfrac{1+\eta}{\gamma_1^r}\mu H=\tfrac{1+\eta}{\gamma_1^r}\mu f[G].\end{aligned}$$

At the same time,

$$f[G]\setminus f[F]\subseteq(f[G]\setminus\textstyle\bigcup_{i\in J}B(f(x_i),\gamma_1\delta_i))\cup\bigcup_{i\in J}(f[B(x_i,\delta_i)]\setminus f[G\cap B(x_i,\delta_i)])$$

has measure at most

$$\sum_{i\in J}\mu f[B(x_i,\delta_i)\setminus G]\le\gamma_2^r\sum_{i\in J}\mu(B(x_i,\delta_i)\setminus G)$$
$$\le\gamma_2^r\eta\sum_{i\in J}\mu B(x_i,\delta_i)\le\tfrac{\gamma_2^r\eta(1+\eta)}{\gamma_1^r}\mu f[G],$$

as required. **Q**

(c) Now let $\eta>0$ and $\langle\epsilon_k\rangle_{k\in\mathbb{N}}$ be such that $\epsilon_k>0$ for every k and $\frac{1+\eta}{\gamma_1^r}(\mu f[E]+\sum_{k=0}^{\infty}\epsilon_k)\le\alpha\mu f[E]$. Choose $\langle E_k\rangle_{k\in\mathbb{N}}$ and $\langle F_k\rangle_{k\in\mathbb{N}}$ inductively, as follows. $E_0=E$. Given that E_k is measurable, use (b) to find a measurable $F_k\subseteq E_k$ such that

$$\mu F_k\le\tfrac{1+\eta}{\gamma_1^r}\mu f[E_k],\quad \mu(f[E_k]\setminus f[F_k])\le\epsilon_k$$

and take $E_{k+1}=E_k\setminus f^{-1}[f[F_k]]$. Note that this will ensure that $f[E_{k+1}]=f[E_k]\setminus f[F_k]$ has measure at most ϵ_k. So $\bigcap_{k\in\mathbb{N}}f[E_k]$ is negligible, and if we set $F=\bigcup_{k\in\mathbb{N}}F_k\cup(E\cap f^{-1}[\bigcap_{k\in\mathbb{N}}f[E_k]])$ then F will be a subset of E, $f[F]=f[E]$ and

$$\mu F\le\textstyle\sum_{k=0}^{\infty}\mu F_k\le\frac{1+\eta}{\gamma_1^r}\sum_{k=0}^{\infty}\mu f[E_k]\le\frac{1+\eta}{\gamma_1^r}(\mu f[E]+\sum_{k=0}^{\infty}\epsilon_k)\le\alpha\mu f[E],$$

as required.

6 Proposition Let $r\ge1$ be an integer, and $f:\mathbb{R}^r\to\mathbb{R}^r$ a Lipschitz quotient function with constants γ_1, γ_2. Let μ be Lebesgue measure on $\mathbb{R}^r$ and set

$$m=\sup\{\#(f^{-1}[\{y\}]):y\in\mathbb{R}^r\},$$

$$M=\sup\{\frac{\mu E}{\mu(f[E])}:E\subseteq\mathbb{R}^r\text{ is measurable, }\mu f[E]>0\}.$$

(Allow ∞ for either m or M if appropriate.) Then

$$m\le\gamma_2^rM,\quad \gamma_1^rM\le m.$$

proof (a) Take any integer $n\le m$. Then there is a $y\in\mathbb{R}^r$ such that $\#(f^{-1}[\{y\}])\ge n$; take distinct $x_1,\dots,x_n\in f^{-1}[\{y\}]$. For $\delta>0$ set $E_\delta=\bigcup_{1\le i\le n}B(x_i,\delta)$. Then $f[E_\delta]\subseteq B(y,\gamma_2\delta)$, so $\mu f[E_\delta]\le\beta_r\gamma_2^r\delta^r$, where $\beta_r=\mu B(\mathbf{0},1)$. On the other hand, for all small δ, $\mu E_\delta=\sum_{i=1}^n\mu B(x_i,\delta)=n\beta_r\delta^r$. So

$$M\ge\sup_{\delta>0}\frac{\mu E_\delta}{\mu f[E_\delta]}\ge\frac{n\beta_r\delta^r}{\beta_r\gamma_2^r\delta^r}=\frac{n}{\gamma_2^r}.$$

As n is arbitrary, $M\ge\frac{m}{\gamma_2^r}$.

(b) ? Suppose, if possible, that $m<\gamma_1^rM$. Let $\alpha>\frac{1}{\gamma_1^r}$ be such that $M>\alpha m$. Take a measurable $E\subseteq\mathbb{R}^r$ such that $\frac{\mu E}{\mu f[E]}>\alpha m$. Choose $E_0,\dots,E_m$ and $F_0,\dots,F_m$ as follows.

$E_0 = E$. Given that $i < m$, $E_i \subseteq E$ is measurable and $\mu E_i > \alpha(m-i)\mu f[E]$, Lemma 5 tells us that there is an $F_i \subseteq E_i$ such that $f[F_i] = f[E_i]$ and $\mu F_i \le \alpha\mu f[E_i] \le \alpha\mu f[E]$. Set $E_{i+1} = E_i \setminus F_i$; then $\mu E_{i+1} > \alpha(m-i-1)\mu f[E]$, so the induction continues.

At the end, we have non-empty disjoint sets $F_0, \dots, F_m$ such that $f[F_i] = f[E_i] \supseteq f[F_{i+1}]$ for each $i < m$; so that if we take $y \in f[F_m]$ then $\#(f^{-1}[\{y\}]) \ge m+1$, which is impossible. **X**

So $\gamma_1^r M \le m$.

References

Csörnyei M. [01] 'Can one squash the space into the plane without squashing?', Geom. Funct. Anal. 11 (2001) 933-952.

Fremlin D.H. [01] *Measure Theory, Vol. 2: Broad Foundations.* Torres Fremlin, 2001.

Fremlin D.H. [03] *Measure Theory, Vol. 4: Topological Measure Theory.* Torres Fremlin, 2003.

Johnson W.B., Lindenstrauss J., Preiss D. & Schechtman G. [00] 'Uniform quotient mappings of the plane', Mich. Math. J. 47, (2000) 15-31.

Maleva O. [02] 'Lipschitz quotient mappings with good ratio of constants', Mathematika 49, (2002) 159-165.