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On a construction of Gitik & Shelah

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I rewrite a result from Farah & Veličković p05, itself based on ideas in Gitik & Shelah 01.

1 Lemma (Farah & Veličković p05) Suppose that κ is a cardinal of uncountable cofinality such that $\square_\kappa$ is true. Then there are a stationary set $S \subseteq \kappa^+$ and a $\square_\kappa$-family $\langle C_\alpha \rangle_{\alpha<\kappa^+}$ such that $\operatorname{cf}\alpha = \omega$ for every $\alpha \in S$ and $C_\alpha \cap S$ is empty for every limit ordinal $\alpha < \kappa^+$.

proof (a) Start with any $\square_\kappa$-family $\langle D_\kappa \rangle_{\alpha<\kappa^+}$, and let S_0 be the set of ordinals less than κ^+ with cofinality ω. Then $\operatorname{otp} D_\alpha \le \kappa$ for every α; if $\alpha \in S_0$, then

$$\operatorname{cf} D_\alpha = \operatorname{cf}\alpha = \omega < \operatorname{cf}\kappa,$$

so $\operatorname{otp} D_\alpha < \kappa$. There is therefore a $\gamma < \kappa$ such that $S = \{\alpha : \alpha \in S_0,\ \operatorname{otp} D_\alpha = \gamma\}$ is stationary.

(b) For $\alpha < \kappa^+$ let D'_α be the derived set of D_α, that is, $\{\delta : \delta \in D_\alpha \setminus \{0\},\ \delta = \sup(\delta \cap D_\alpha)\}$. Then D'_α is a relatively closed subset of α, and is cofinal with α except sometimes when $\operatorname{cf}\alpha \le \omega$. If $\beta \in D'_\alpha$ then $D_\beta = \beta \cap D_\alpha$ so $D'_\beta = \beta \cap D'_\alpha$.

If $\beta \in S \cap D'_\alpha$ then

$$\operatorname{otp}(\beta \cap D_\alpha) = \operatorname{otp} D_\beta = \gamma,$$

so $\#(S \cap D'_\alpha) \le 1$.

(c) Define $\langle C_\alpha \rangle_{\alpha<\kappa^+}$ as follows. If D'_α is cofinal with α and $S \cap D'_\alpha = \emptyset$ set $C_\alpha = D'_\alpha$. If α is a limit ordinal and D'_α is cofinal with α and $S \cap D'_\alpha$ is a singleton $\{\zeta_\alpha\}$ say, set $C_\alpha = \{\xi : \zeta_\alpha < \xi \in D'_\alpha\}$. Otherwise, if $\alpha = \beta+1$ set $C_\alpha = \{\beta\}$ and if $\operatorname{cf}\alpha = \omega$ but D'_α is not cofinal with α take C_α to be a cofinal set of successor ordinals less than α with $\operatorname{otp} C_\alpha = \omega$. Then C_α is always a relatively closed subset of α cofinal with α, and does not meet S if α is a limit ordinal.

If $\alpha < \kappa^+$ and $\beta = \sup(\beta \cap C_\alpha)$, then either $\beta = 0$ or $\beta = \alpha$ or $C_\alpha \subseteq D'_\alpha$. If $\beta = 0$ then

$$C_\beta = D'_\beta = \emptyset = \beta \cap C_\alpha.$$

If $\beta = \alpha$ of course $C_\beta = \beta \cap C_\alpha$. Otherwise, $\beta \in D'_\alpha$ so $D'_\beta = \beta \cap D'_\alpha$; as $\beta = \sup(\beta \cap C_\alpha)$, D'_β is cofinal with β. If D'_α is disjoint from S then D'_β is disjoint from S and

$$C_\beta = D'_\beta = \beta \cap D'_\alpha = \beta \cap C_\alpha.$$

If D'_α is not disjoint from S then $\zeta_\alpha < \beta$ so $\zeta_\alpha \in S \cap D'_\beta$ and $\zeta_\beta = \zeta_\alpha$. Now

$$C_\beta = D'_\beta \setminus (\zeta_\beta + 1) = \beta \cap D'_\alpha \setminus (\zeta_\alpha + 1) = \beta \cap C_\alpha.$$

Thus $\langle C_\alpha \rangle_{\alpha<\kappa^+}$ is a $\square_\kappa$-family.

2 Lemma Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and κ a cardinal such that $\operatorname{cf}\kappa > \mathfrak{c}$. Let $\langle e_\xi \rangle_{\xi<\kappa}$ be a family in $\mathfrak{A}$ and write $\mathfrak{B}_\xi$ for the closed subalgebra of $\mathfrak{A}$ generated by $\{e_\eta : \eta < \xi\}$. Suppose that $a \setminus e_\xi$ is non-zero for every non-zero $a \in \mathfrak{B}_\xi$. Then there is an $A \in [\kappa]^\kappa$ such that $\inf_{\xi\in I} e_\xi = 0$ for every infinite $I \subseteq A$.

proof For each $\xi < \kappa$ let $u_\xi \in L^\infty(\mathfrak{B}_\xi)$ be such that $\int_a u_\xi = \bar\mu(a \cap e_\xi)$ for every $a \in \mathfrak{B}_\xi$. (See Fremlin 02, 365E for this form of the Radon-Nikodým theorem.) Let $(\mathfrak{A}_L, \bar\mu_L)$ be the measure algebra of Lebesgue measure on $[0,\infty[$ and $u^*_\xi \in L^\infty(\mathfrak{A}_L)$ the decreasing rearrangement of u_ξ (Fremlin 02, 373C). Because $\#(L^\infty(\mathfrak{A}_L)) = \mathfrak{c} < \operatorname{cf}\kappa$, there is a v such that $A = \{\xi : \xi < \kappa,\ u^*_\xi = v\}$ has cardinal κ. Set $\zeta = \min A$. Setting

$$b_1 = \inf_{s<1} [\![u_\zeta \ge s]\!] = [\![u_\zeta \ge 1]\!] \in \mathfrak{B}_\zeta,$$

we have $\bar\mu b_1 = \int_{b_1} u_\zeta = \bar\mu(e_\zeta \cap b_1)$, so $b_1 = 0$. If $t > 0$, there must therefore be an $s < 1$ such that $\bar\mu b < t$, where $b = [\![u_\zeta \ge s]\!]$. Now

$$\int_0^t v \le st + \int (u_\zeta - s\chi 1)^+$$

(Fremlin 02, 373Df)

$$= st + \int_b u_\zeta - s\chi 1 \le st + (1-s)\bar\mu b < st + (1-s)t = t.$$

? Suppose, if possible, that $I \subseteq A$ is infinite and $c = \inf_{\xi\in I} e_\xi$ is non-zero. We may suppose that $\operatorname{otp} I = \omega$. Set $t = \bar\mu c > 0$. Then $\int_0^t v < t$ so there is a $t' > t$ such that $\int_0^{t'} v < t$. Now there is a finite $J \subseteq I$ such that $\bar\mu d \le t'$, where $d = \inf_{\eta\in J} e_\eta$. Take any $\xi \in I$ such that $J \subseteq \xi$. Then

$$t = \bar\mu c \le \bar\mu(e_\xi \cap d) = \int_d u_\xi \le \int_0^{\bar\mu d} v$$

(Fremlin 02, 373E)

$$\le \int_0^{t'} v < t,$$

which is absurd. **X**

Thus we have an appropriate set A.

3 Product measures The proof of the next theorem uses measures on product spaces. All the spaces involved will be of the form $\{0,1\}^I$, typically identified with $\prod_{j\in J}\{0,1\}^{I_j}$ where $\langle I_j\rangle_{j\in J}$ is a partition of I.

(a) For definiteness, and to cohere with the prejudices of my treatise *Measure Theory*, I will take it that all the measures involved are Radon probability measures as described in Fremlin 03, §416, and that all the products are τ-additive product measures as described in Fremlin 03, §417. The facts we shall need are

the product of any family of Radon probability measures on compact Hausdorff spaces is again a Radon probability measure;

the product is 'associative' in the sense that if $\langle(X_i,\mu_i)\rangle_{i\in I}$ is a family of Radon probability spaces, and $\langle I_j\rangle_{j\in J}$ is a partition of I, then the product Radon measure $\prod_{i\in I}\mu_i$ on $X = \prod_{i\in I} X_i$ can be identified with the repeated product $\prod_{j\in J}\prod_{i\in I_j}\mu_i$ on $\prod_{j\in J}\prod_{i\in I_j} X_i$ (Fremlin 03, 417J);

if we have a *finite* family $\langle X_i\rangle_{i\in I}$ of Hausdorff spaces, and on each X_i we have Radon measures μ_i, ν_i such that μ_i and ν_i have the same null ideals, then the Radon measures $\prod_{i\in I}\mu_i$ and $\prod_{i\in I}\nu_i$ on $\prod_{i\in I} X_i$ have the same null ideals (use Fubini's theorem, Fremlin 03, 417H, and induction on $\#(I)$).

(b) When we come to the actual constructions, however, it will be apparent that all the work could equally well be done with the non-topological notion of product measure set out in Fremlin 01, §254. And even if we use Radon measures, as I suggest, then it is easy to check that they are all strictly positive, and therefore completion regular, by Grekas' theorem (Fremlin 07?, 535Xd). So in fact you can do the whole thing with Baire measures, if you use uncompleted product measures. You will observe that in (d) of the proof below I dodge some technical considerations by, in effect, moving to Baire measures at that stage.

4 Theorem Suppose that $\kappa \ge \mathfrak{c}$ is a cardinal of uncountable cofinality such that

(α) $\square_\kappa$ is true,

(β) $\diamondsuit(\kappa^+, S)$ is true for every stationary set $S \subseteq \{\alpha : \alpha < \kappa^+, \operatorname{cf}\alpha = \omega\}$.

Then there is a Dedekind complete ccc Boolean algebra $\mathfrak{A}$, of Maharam type κ^+, which is not itself a measurable algebra but is such that whenever $\mathfrak{B} \subseteq \mathfrak{A}$ is an order-closed subalgebra of Maharam type at most κ then $\mathfrak{B}$ is measurable.

proof (a) By Lemma 1, there are a stationary set $S_1 \subseteq \kappa^+$ and a $\Box_\kappa$-family $\langle C_\alpha\rangle_{\alpha<\kappa^+}$ such that $\operatorname{cf}\alpha = \omega$ for every $\alpha \in S_1$ and $C_\alpha \cap S_1$ is empty if α is a limit ordinal. Let $\langle A_\alpha\rangle_{\alpha\in S_1}$ be a $\diamondsuit(\kappa^+, S_1)$-family. Let S be the set of those $\alpha \in S_1$ such that A_α is cofinal with α.

(b) Define a family $\langle\mu_{\beta\alpha}\rangle_{\beta\le\alpha<\kappa^+}$ inductively, as follows. μ_{00} is the (unique) probability measure on $\{\emptyset\} = \{0,1\}^0$. Given that $\mu_{\beta\gamma}$ is a (Radon) probability measure on $\{0,1\}^{\gamma\setminus\beta}$ for $\beta \le \gamma < \alpha$, proceed as follows.

case 1 Suppose that $\alpha = \gamma + 1$ is a successor ordinal. In this case, for $\beta < \alpha$, let $\mu_{\beta\alpha}$ be the product of $\mu_{\beta\gamma}$ with the usual measure, that is, the fair-coin probability measure, on the two-element set $\{0,1\}^{\{\gamma\}}$.

case 2 Suppose that α is a non-zero limit ordinal not belonging to S. In this case, let $\langle\delta(\xi)\rangle_{\xi<\zeta}$ be the increasing enumeration of C_α. If $\beta < \alpha$, set $\xi_0 = \min\{\xi : \beta \le \delta(\xi)\}$, and let $\mu_{\beta\alpha}$ be the product

$$\mu_{\beta,\delta(\xi_0)} \times \prod_{\xi_0\le\xi<\zeta} \mu_{\delta(\xi),\delta(\xi+1)},$$

interpreted as a measure on $\{0,1\}^{\alpha\setminus\beta}$, observing that (because C_α is cofinal and relatively closed in α) $\alpha\setminus\beta$ is the disjoint union of $\delta(\xi_0)\setminus\beta$ and $\bigcup_{\xi_0\le\xi<\zeta} \delta_{\xi+1}\setminus\delta_\xi$.

case 3 Suppose that $\alpha \in S$. Let $\langle\beta_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence in A_α with supremum α. Write ν_n for $\mu_{\beta_n+1,\beta_{n+1}+1}$ for each n. Then ν_n is the product of $\mu_{\beta_n+1,\beta_{n+1}}$ with the usual measure on $\{0,1\}^{\{\beta_{n+1}\}}$; in particular, $\nu_n E_n > 0$ where $E_n = \{x : x \in \{0,1\}^{(\beta_{n+1}+1)\setminus(\beta_n+1)},\ x(\beta_{n+1}) = 1\}$. There is therefore a (Radon) probability measure ν'_n on $\{0,1\}^{(\beta_{n+1}+1)\setminus(\beta_n+1)}$ such that ν'_n and ν_n have the same null ideals and $\nu'_n E_n \ge 1 - 2^{-n-1}$.

Given $\beta < \alpha$, let m be minimal subject to $\beta \le \beta_m$, and take $\mu_{\beta\alpha}$ to be the product measure

$$\mu_{\beta,\beta_m+1} \times \prod_{n\ge m} \nu'_n.$$

(In all three cases, of course, $\mu_{\alpha\alpha}$ is to be the unique probability measure on $\{\emptyset\} = \{0,1\}^{\alpha\setminus\alpha}$.)

(c) (The key.) Whenever $\beta \le \gamma < \alpha$, $\mu_{\beta\alpha}$ has the same null sets as the product $\mu_{\beta\gamma} \times \mu_{\gamma\alpha}$. **P** Induce on α.

case 0 The case $\alpha = 0$ is vacuous.

case 1 Suppose that α is a successor ordinal; say $\alpha = \alpha' + 1$. Then $\mu_{\beta\alpha} = \mu_{\beta\alpha'} \times \mu_{\alpha'\alpha}$ has the same null sets as

$$(\mu_{\beta\gamma} \times \mu_{\gamma\alpha'}) \times \mu_{\alpha'\alpha} = \mu_{\beta\gamma} \times (\mu_{\gamma\alpha'}) \times \mu_{\alpha'\alpha}) = \mu_{\beta\gamma} \times \mu_{\gamma\alpha}.$$

case 2 Suppose that α is a non-zero limit ordinal not in S. Let $\langle\delta(\xi)\rangle_{\xi<\zeta}$ be the increasing enumeration of C_α.

case 2a Suppose that $C_\alpha \cap \gamma \setminus \beta$ is finite. Then there are $\xi_0 < \zeta$ and $n \in \mathbb{N}$ such that $\delta(\xi_0)$ is the least member of $C_\alpha \setminus \beta$ and $\delta(\xi_0 + n)$ is the least member of $C_\alpha \setminus \gamma$. Now the inductive hypothesis tells us that

$$\mu_{\beta\alpha} = \mu_{\beta,\delta(\xi_0)} \times \prod_{\xi_0\le\xi<\zeta} \mu_{\delta(\xi),\delta(\xi+1)},$$

$$\mu_{\beta,\delta(\xi_0)} \times \prod_{i<n} \mu_{\delta(\xi_0+i),\delta(\xi_0+i+1)} \times \mu_{\delta(\xi_0+n),\gamma} \times \mu_{\gamma,\delta(\xi_0+n+1)} \times \prod_{\xi_0+n<\xi<\zeta} \mu_{\delta(\xi),\delta(\xi+1)}$$

and

$$\mu_{\beta\gamma} \times \mu_{\gamma,\delta(\xi_0+n+1)} \times \prod_{\xi_0+n<\xi<\zeta} \mu_{\delta(\xi),\delta(\xi+1)} = \mu_{\beta\gamma} \times \mu_{\gamma\alpha}$$

have the same null sets.

case 2b Suppose that $C_\alpha \cap \gamma \setminus \beta$ is infinite. Then there is a greatest non-zero limit ordinal $\eta < \zeta$ such that $\delta(\eta) \le \gamma$, and $\beta < \delta(\eta)$. As $\delta(\eta) = \sup(\delta(\eta) \cap C_\alpha)$, $C_{\delta(\eta)} = \delta(\eta) \cap C_\alpha$; also $\delta(\eta) \notin S$. So $\langle\delta(\xi)\rangle_{\xi<\eta}$ is the increasing enumeration of $C_{\delta(\eta)}$, and

$$\begin{aligned}\mu_{\beta\alpha} &= \mu_{\beta,\delta(\xi_0)} \times \prod_{\xi_0\le\xi<\zeta} \mu_{\delta(\xi),\delta(\xi+1)} \\ &= \mu_{\beta,\delta(\xi_0)} \times \prod_{\xi_0\le\xi<\eta} \mu_{\delta(\xi),\delta(\xi+1)} \times \prod_{\eta\le\xi<\zeta} \mu_{\delta(\xi),\delta(\xi+1)} \\ &= \mu_{\beta,\delta(\eta)} \times \mu_{\delta(\eta),\alpha}\end{aligned}$$

for an appropriate $\xi_0 < \eta$.

Next, there is an $n \in \mathbb{N}$ such that $\delta(\eta+n) \le \gamma < \delta(\eta+n+1)$, so that $\mu_{\delta(\eta),\alpha} = \prod_{\eta\le\xi<\zeta} \mu_{\delta(\xi),\delta(\xi+1)}$ has the same null sets as

$$\prod_{i<n} \mu_{\delta(\eta+i),\delta(\eta+i+1)} \times \mu_{\delta(\eta+n),\gamma} \times \mu_{\gamma,\delta(\eta+n+1)} \times \prod_{\eta+n<\xi<\zeta} \mu_{\delta(\xi),\delta(\xi+1)}$$

and $\mu_{\delta(\eta),\gamma} \times \mu_{\gamma\alpha}$; so that $\mu_{\beta\alpha}$ has the same null sets as $\mu_{\beta,\delta(\eta)} \times \mu_{\delta(\eta),\gamma} \times \mu_{\gamma\alpha}$ and $\mu_{\beta\gamma} \times \mu_{\gamma\alpha}$.

case 3 Suppose that $\alpha \in S$. Let $\langle\beta_n\rangle_{n\in\mathbb{N}}$ and $\langle\nu'_n\rangle_{n\in\mathbb{N}}$ be the sequences used in the construction of $\mu_{\beta\alpha}$ and $\mu_{\gamma\alpha}$ as described in case 3 of (b) above. Let $l \le m$ be such that $l = \min\{n : \beta \le \beta_n\}$ and $m = \min\{n : \gamma \le \beta_n\}$.

case 3a If $l = m$ then $\mu_{\beta\gamma} \times \mu_{\gamma\alpha} = \mu_{\beta\gamma} \times \mu_{\gamma,\beta_m+1} \times \prod_{n\ge m} \nu'_n$ has the same null sets as $\mu_{\beta,\beta_m+1} \times \prod_{n\ge m} \nu'_n = \mu_{\beta\alpha}$.

case 3b If $l < m$ then

$$\mu_{\beta\gamma} \times \mu_{\gamma\alpha},$$

$$\mu_{\beta,\beta_n+1} \times \prod_{l\le n<m-1} \mu_{\beta_n+1,\beta_{n+1}+1} \times \mu_{\beta_{m-1}+1,\gamma} \times \mu_{\gamma,\beta_m+1} \times \prod_{n\ge m} \nu'_n,$$

$$\mu_{\beta,\beta_n+1} \times \prod_{l\le n<m} \mu_{\beta_n+1,\beta_{n+1}+1} \times \prod_{n\ge m} \nu'_n$$

and

$$\mu_{\beta,\beta_n+1} \times \prod_{l\le n<m} \nu'_n \times \prod_{n\ge m} \nu'_n = \mu_{\beta\alpha}$$

all have the same null sets. **Q**

(d) In particular, if $\beta \le \alpha < \kappa^+$, a set $E \subseteq \{0,1\}^\beta$ is $\mu_{0\beta}$-negligible iff $E \times \{0,1\}^{\alpha\setminus\beta}$ is $\mu_{0\alpha}$-negligible.

Let $\mathcal{I}$ be the ideal of Baire subsets of $\{0,1\}^{\kappa^+}$ generated by sets of the form $E \times \{0,1\}^{\kappa^+\setminus\alpha}$ where $\alpha < \kappa^+$, $E \subseteq \{0,1\}^\alpha$ is a Baire set and $\mu_{0\alpha}E = 0$. Then $\mathcal{I}$ is a σ-ideal of the Baire σ-algebra $\mathcal{B}\mathfrak{a}$ of $\{0,1\}^{\kappa^+}$. Let $\mathfrak{A}$ be the quotient $\mathcal{B}\mathfrak{a}/\mathcal{I}$, so that $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra. Setting $E_\xi = \{x : x \in \{0,1\}^{\kappa^+}, x(\xi) = 1\}$, $\mathcal{B}\mathfrak{a}$ is the σ-subalgebra of itself generated by $\{E_\xi : \xi < \kappa^+\}$, so that $\mathfrak{A}$ is τ-generated by $\{E^\bullet_\xi : \xi < \kappa\}$ and $\tau(\mathfrak{A}) \le \kappa^+$.

(e) $\mathfrak{A}$ is ccc. **P?** Otherwise, let $\langle F_\xi\rangle_{\xi<\omega_1}$ be a disjoint family in $\mathcal{B}\mathfrak{a} \setminus \mathcal{I}$. Each F_ξ is determined by coordinates in a countable set, and $\operatorname{cf}\kappa^+ > \omega_1$, so there is an $\alpha < \kappa^+$ such that every F_ξ is determined by coordinates less than α. Expressing F_ξ as $F'_\xi \times \{0,1\}^{\kappa^+\setminus\alpha}$ where $F'_\xi \subseteq \{0,1\}^\alpha$ is a Baire set for each ξ, we see that $\langle F'_\xi\rangle_{\xi<\omega_1}$ is disjoint, so that there is some ξ such that $\mu_{0\alpha}F'_\xi = 0$ and $F_\xi \in \mathcal{I}$, which is impossible. **XQ**

(f) If $\mathfrak{B} \subseteq \mathfrak{A}$ is an order-closed subalgebra and $\tau(\mathfrak{B}) \le \kappa$, then $\mathfrak{B}$ is measurable. **P** Let $\langle c_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{B}$ which τ-generates $\mathfrak{B}$, and for $\xi < \kappa$ let $F_\xi \in \mathcal{B}\mathfrak{a}$ be such that $F^\bullet_\xi = c_\xi$. Let $\alpha < \kappa^+$ be such that every F_ξ is generated by coordinates less than α, and set

$$\mathfrak{C}_\alpha = \{(F \times \{0,1\}^{\kappa^+\setminus\alpha})^\bullet : F \subseteq \{0,1\}^\alpha \text{ is a Baire set}\}.$$

Because $\mathfrak{A}$ is ccc, $\mathfrak{C}_\alpha$ is order-closed, so includes $\mathfrak{B}$. Now we have a probability measure $\bar\nu$ on $\mathfrak{C}_\alpha$ defined by saying that

$$\bar\nu(F \times \{0,1\}^{\kappa^+\setminus\alpha})^\bullet = \mu_{0\alpha}F$$

for every Baire set $F \subseteq \{0,1\}^\alpha$, so that $\mathfrak{C}_\alpha$ and $\mathfrak{B}$ are measurable algebras. **Q**

(g) $\mathfrak{A}$ is not a measurable algebra. **P?** Suppose, if possible, that $(\mathfrak{A},\bar\mu)$ is a probability algebra. Consider the subalgebras $\mathfrak{C}_\alpha$ defined as in (f) and the elements $e_\xi = E_\xi^\bullet$ where E_ξ is defined as in (d). The construction of $\mu_{\alpha+1}$ shows that if $F\subseteq\{0,1\}^\alpha$ is a Baire set which is not $\mu_{0\alpha}$-negligible then neither $F\cap E_\xi$ nor $F\setminus E_\xi$ is $\mu_{0,\alpha+1}$-negligible; that is, if $c\in\mathfrak{C}_\alpha\setminus\{0\}$ then neither $c\cap e_\xi$ nor $c\setminus e_\xi$ is zero.

By Lemma 2, there is an $A\subseteq\kappa^+$ such that $\#(A)=\kappa^+$ and $\inf_{\xi\in I}e_\xi=0$ for every infinite $I\subseteq A$. Now $S'=\{\alpha:\alpha\in S_1,\ A\cap\alpha=A_\alpha\}$ is stationary, so there is an $\alpha\in S'$ such that $\alpha\cap A$ is cofinal with α, and $\alpha\in S$. Let $\langle\beta_n\rangle_{n\in\mathbb{N}}$ and $\langle\nu'_n\rangle_{n\in\mathbb{N}}$ be the sequences used in the construction of $\mu_{\beta\alpha}$ in case 3 of (b) above. Then

$$\mu_{0\alpha}\{x:x\in\{0,1\}^\alpha,\ x(\beta_n)=1\text{ for every }n\}\ge\prod_{n\in\mathbb{N}}(1-2^{-n})>0,$$

so $\bigcap_{n\in\mathbb{N}}E_{\beta_n}\notin\mathcal{I}$ and $\inf_{n\in\mathbb{N}}e_{\beta_n}\ne0$ in $\mathfrak{A}$; but $\{\beta_n:n\in\mathbb{N}\}$ is an infinite subset of A. **XQ**

Thus $\mathfrak{A}$ is not a measurable algebra.

5 Scholium The algebra $\mathfrak{A}$ of Theorem 4 has the following further properties:

(a) $\mathfrak{A}$ is weakly (σ,∞)-distributive;

(b) (λ,λ,k) is a precaliber triple of $\mathfrak{A}$ (definition: Fremlin 07?, 511E) whenever $\lambda\le\kappa$ is a cardinal of uncountable cofinality and $k\in\mathbb{N}$;

(c) in particular, $\mathfrak{A}$ satisfies Knaster's condition;

(d) (Farah & Veličković p05) $\mathfrak{A}$ is not a Maharam algebra, that is, there is no strictly positive Maharam submeasure on $\mathfrak{A}$;

(e) $\mathfrak{A}$ is not σ-finite-cc, that is, if $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of sets with union $\mathfrak{A}\setminus\{0\}$ then there is some n such that A_n includes an infinite disjoint set.

proof (a) Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence of maximal antichains in $\mathfrak{A}$. Because $\mathfrak{A}$ is ccc, every A_n is countable, and the order-closed subalgebra $\mathfrak{B}$ generated by $\bigcup_{n\in\mathbb{N}}A_n$ has countable Maharam type. By (f), $\mathfrak{B}$ is measurable, therefore weakly (σ,∞)-distributive, and there is a maximal antichain C in $\mathfrak{B}$ such that $\{a:a\in A_n,\ a\cap c\ne0\}$ is finite for every $n\in\mathbb{N}$ and $c\in C$. Of course C is also a maximal antichain in $\mathfrak{A}$. As $\langle A_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ,∞)-distributive. **Q**

(b) Let $\langle a_\xi\rangle_{\xi<\lambda}$ be a family in $\mathfrak{A}\setminus\{0\}$. Let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $\{a_\xi:\xi<\lambda\}$. Then $\mathfrak{B}$ is a measurable algebra. If $\bar\mu$ is such that $(\mathfrak{B},\bar\mu)$ is a probability algebra, then (λ,λ,k) is a measure-precaliber triple of $(\mathfrak{B},\bar\mu)$ (Fremlin 07?, 524U); because $\operatorname{cf}\lambda>\omega$, (λ,λ,k) is a precaliber triple of $\mathfrak{B}$ (Fremlin 07?, 524Eb). There is therefore a $\Gamma\in[\lambda]^\lambda$ such that $\inf_{\xi\in I}a_\xi\ne0$ for every $I\in[\Gamma]^k$. As $\langle a_\xi\rangle_{\xi<\lambda}$ is arbitrary, (λ,λ,k) is a precliber triple of $\mathfrak{A}$.

(c) Setting $k=2$ and $\lambda=\omega_1$ in (b), we have Knaster's condition.

(d) ? If ν is a strictly positive Maharam submeasure on $\mathfrak{A}$, then it cannot be uniformly exhaustive, by the Kalton-Roberts theorem (Fremlin 02, 392F). So there is a family $\langle a_{ni}\rangle_{i\le n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\langle a_{ni}\rangle_{i\le n}$ is disjoint for each n and $\inf_{i\le n\in\mathbb{N}}\nu a_{ni}>0$. Let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $\{a_{ni}:i\le n\in\mathbb{N}\}$; then $\mathfrak{B}$ is measurable; let $\bar\mu$ be such that $(\mathfrak{B},\bar\mu)$ is a probability algebra. Now $\nu\restriction\mathfrak{B}$ is a strictly positive Maharam submeasure on $\mathfrak{B}$, so is absolutely continuous with respect to $\bar\mu$ (Fremlin 02, 393E). Since $\inf_{i\le n\in\mathbb{N}}\bar\mu a_{ni}\le\inf_{n\in\mathbb{N}}\frac{1}{n+1}=0$, $\inf_{i\le n\in\mathbb{N}}\nu a_{ni}=0$. **X**

(e) By Todorčević's theorem (Fremlin n04, 3C), a Dedekind complete weakly (σ,∞)-distributive σ-finite-cc Boolean algebra is a Maharam algebra; as $\mathfrak{A}$ is weakly (σ,∞)-distributive, Dedekind complete and not a Maharam algebra, it is not σ-finite-cc.

6 Remarks If V=L, the hypotheses of Theorem 4 are satisfied by any cardinal $\kappa\ge\mathfrak{c}$ of uncountable cofinality (). But they are also compatible with $\mathfrak{m}=\mathfrak{c}>\omega_1$, for instance; see Farah & Veličković p05.

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