Dependent multiple choice and Baire's theorem

I present the main theorem of Fossy & Morillon 98. All arguments here are in ZF without choice.

1 Definition DMC, 'dependent multiple choice', is the statement

whenever A is a non-empty set and R is a relation on A such that $A = R^{-1}[A]$ then there is a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of non-empty finite subsets of A such that $A_n \subseteq R^{-1}[A_{n+1}]$ for every $n \in \mathbb{N}$.

I will write DMC$'$ for the statement

whenever T is a tree of height ω such that every element of T has a successor, T has a subtree of height ω and finite levels.

(See Blass 79.)

2 Proposition If DMC is true, then every countably compact regular topological space is a Baire space.

proof Assume DMC. Let X be a countably compact regular space, $\langle G_n\rangle_{n\in\mathbb{N}}$ a sequence of dense open subsets of X, and G a non-empty open subset of X. For each $k \in \mathbb{N}$ set $G'_k = X \cap \bigcap_{n<k} G_n$, so that each G'_k is a dense open set. Let $\mathcal{V}$ be the family of non-empty open subsets of G, and for U, $V \in \mathcal{V}$ say that $U \succ V$ if there is some k such that $\overline{V} \subseteq U \cap G'_k$ and $U \not\subseteq G'_k$. If $U \in \mathcal{V}$ and $U \subseteq G_k$ for every k then $G \cap \bigcap_{k\in\mathbb{N}} G_k \supseteq U$ is non-empty. Otherwise, for every $U \in \mathcal{V}$ there is a k such that $U \not\subseteq G_k$; because X is regular, there is a $V \in \mathcal{V}$ such that $\overline{V} \subseteq U \cap G'_k$ and $U \succ V$. By DMC, there is a sequence $\langle \mathcal{V}_n\rangle_{n\in\mathbb{N}}$ of finite subsets of $\mathcal{V}$ such that for every $n \in \mathbb{N}$, $U \in \mathcal{V}_n$ there is a $V \in \mathcal{V}_{n+1}$ such that $U \succ V$. Define $\langle \mathcal{V}'_n\rangle_{n\in\mathbb{N}}$ inductively by setting $\mathcal{V}'_0 = \mathcal{V}_0$,

$$\mathcal{V}'_{n+1} = \{V : V \in \mathcal{V}_{n+1},\, U \succ V \text{ for some } U \in \mathcal{V}'_n\};$$

then no $\mathcal{V}'_n$ is empty. Now $\overline{U} \subseteq G'_n$ whenever $n \in \mathbb{N}$ and $U \in \mathcal{V}'_n$. **P** Induce on n. For $n = 0$, $G'_0 = X$. For the inductive step to $n+1$, if $V \in \mathcal{V}'_{n+1}$ there is a $U \in \mathcal{V}'_n$ such that $U \succ V$. Now there is a $k \in \mathbb{N}$ such that $U \not\subseteq G'_k$ and $\overline{V} \subseteq G'_k$. We must have $k > n$ so $\overline{V} \subseteq G'_{n+1}$. Thus the induction proceeds. **Q**

So, setting $W_n = \bigcup \mathcal{V}'_n$, we have $\overline{W}_n \subseteq G'_n$ for each n, and $\langle W_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of non-empty sets. Because X is countably compact, $\bigcap_{n\in\mathbb{N}} \overline{W}_n$ is non-empty, and this is a subset of $G \cap \bigcap_{n\in\mathbb{N}} G_n$.

As G and $\langle G_n\rangle_{n\in\mathbb{N}}$ are arbitrary, X is a Baire space.

3 Lemma If DMC$'$ is true, so is DMC.

proof Assume DMC$'$. Let A be a non-empty set and R a relation on A such that $A = R^{-1}[A]$. Let T be the set of finite sequences $\langle a_i\rangle_{i<n}$ in A such that $a_i R a_{i+1}$ for $0 < i < n-1$, ordered by end-extension. Then T is a tree of height ω in which every element has a successor. Let $T' \subseteq T$ be a subtree of height ω and finite levels. For each $n \in \mathbb{N}$, let A_n be the set of those elements of T' of rank n which have extensions of arbitrarily large rank. Then no A_n is empty and $A_n = R^{-1}[A_{n+1}]$ for each n. As A and R are arbitrary, DMC is true.

4 Theorem If every compact Hausdorff space is a Baire space, then DMC$'$ is true.

proof Let T be a tree of height ω in which every element has a successor. Let ∞ be any object not belonging to T. Give T its discrete topology and $A = T \cup \{\infty\}$ the one-point-compactification topology. Let X be $A^{\mathbb{N}}$ with the product topology, so that X is Hausdorff. Let K be

$$\{x : x \in X, \text{ whenever } m \leq n \text{ either } x(m) = \infty \text{ or } x(n) = \infty$$
$$\text{or } x(m) \in T,\, x(n) \in T \text{ and } x(m) < x(n)\}.$$

Then K is a closed set in X. For $n \in \mathbb{N}$, $x \in X$ set $\pi_n(x) = x(n)$.

case 1 Suppose that K is compact. For each $n \in \mathbb{N}$, set $G_n = \{x : x \in K, \exists\, i \geq n, x(i) \neq \infty\}$. Then G_n is an open set in K; also G_n is dense. **P** Let G be a non-empty open set in K. Take $z \in G$; then there is an $m > n$ and a family $\langle B_i \rangle_{i<m}$ of sets such that (i) $z{\restriction}m \in \prod_{i<m} B_i$ (ii) for each $i < m$ either $B_i = \{z(i)\} \subseteq T$ or $z(i) = \infty$ and B_i is a cofinite subset of A (iii) $\{x : x \in K,\, x{\restriction}m \in \prod_{i<m} B_i\} \subseteq G$. Set $J = \{j : j < m, z(j) \neq \infty$ and $k = \max J$, counting $\max \emptyset$ as -1. If $0 \leq i \leq k$, set $s_i = z(i)$. If $k = m-1$, jump to the next paragraph. If $0 \leq k < m-1$, note that $\{t : z(k) < t\}$ must be infinite, so meets the cofinite set B_{k+1} and we can choose $s_{k+1} \in B_{k+1} \cap T$ such that $z(k) < s_{k+1}$; if $k = -1$, take any $s_0 \in B_0$. If $k+1 = m-1$, jump to the next paragraph. Otherwise, choose $\langle s_i \rangle_{k+1<i<m}$ inductively so that $s_i \in B_i \cap T$ and $s_{i-1} < s_i$ for each i.

Set $y(i) = s_i$ for $i < m$ and $y(i) = \infty$ for $i \geq m$. Then $y \in G$ and $y(m-1) \neq \infty$, so $y \in G_n$. As G is arbitrary, G_n is dense. **Q**

There must therefore be an $x \in \bigcap_{n\in\mathbb{N}} G_n$. (Of course K is non-empty because the sequence with constant value ∞ belongs to K.) Set $D = \{i : x(i) \neq \infty\}$; then D is infinite. Now $\langle x(i) \rangle_{i\in D}$ is a strictly increasing family in T so T has an infinite branch; *a fortiori*, T has a subtree with finite levels and height ω.

case 2 Suppose that K is not compact. Then there is a family of closed subsets of K with the finite intersection property and empty intersection. Accordingly there is a downwards-directed family $\mathcal{E}$ of non-empty closed subsets of X, with empty intersection and containing all sets of the form $X \setminus \{x : x(m) = s,\, x(n) = t\}$ where $m < n$, s, $t \in T$ and $s \not< t$. We may suppose that every member of $\mathcal{E}$ belongs to the algebra of subsets of X generated by sets of the form $\{x : x(n) = t\}$ where $n \in \mathbb{N}$ and $t \in T$, since every basic open set in X belongs to this algebra.

Construct $\langle \mathcal{E}_n \rangle_{n\in\mathbb{N}}$ as follows. $\mathcal{E}_0 = \mathcal{E}$. If $\infty \in \pi_n[E]$ for every $E \in \mathcal{E}_n$, set $\mathcal{E}_{n+1} = \mathcal{E}_n \cup \{E \cap \pi_n^{-1}[\{\infty\}] : E \in \mathcal{E}_n\}$; otherwise, set $\mathcal{E}_{n+1} = \mathcal{E}_n$. Inducing on n, we see that every element of $\mathcal{E}_n$ belongs to the algebra of subsets of X generated by

$$\{\pi_i^{-1}[\{t\}] : i \in \mathbb{N},\, t \in T\} \cup \{\pi_i^{-1}[\{\infty\}] : i < n\}.$$

Consequently, for $E \in \mathcal{E}_n$, $\pi_n[E]$ must be a closed subset of $T \cup \{\infty\}$, and if it does not contain ∞ must be a finite subset of T. Set $\mathcal{E}^* = \bigcup_{n\in\mathbb{N}} \mathcal{E}_n$, so that $\mathcal{E}^*$ is again a downwards-directed family of non-empty closed sets with empty intersection.

Set $D = \{n : \pi_n^{-1}[\{\infty\}] \notin \mathcal{E}^*\}$. For each $n \in D$ we must have had an $E \in \mathcal{E}_n$ such that $\infty \notin \pi_n[E]$ and $\pi_n[E]$ is a finite subset of T; accordingly $F_n = \bigcap\{\pi_n[E] : E \in \mathcal{E}^*\}$ is a finite subset of T and (because $\mathcal{E}^*$ is downwards-directed) there is an $E \in \mathcal{E}^*$ such that $\pi_n[E] = F_n$. Now D is infinite. **P?** Otherwise, $\bigcup_{n\in D} F_n$ is finite. So we can enlarge $\mathcal{E}^*$ to a family $\mathcal{E}_1^*$, still a downwards-directed family of closed sets in X, such that for every $n \in D$ there is an $s_n \in T$ such that $\pi_n^{-1}[\{s_n\}] \in \mathcal{E}_1^*$. Set $z(n) = s_n$ for $n \in D$, $z(n) = \infty$ for $n \in \mathbb{N} \setminus D$; then for every neighbourhood U of z there is an $E \in \mathcal{E}_1^*$ meeting U; because $\mathcal{E}_1^*$ is downwards-directed and consists of closed sets, $z \in \bigcap \mathcal{E}_1^* \subseteq \bigcap \mathcal{E}$, which is supposed to be impossible. **XQ**

If m, $n \in D$, $m < n$ and $s \in F_m$, there is a $t \in F_n$ such that $s < t$. **P** Let $E \in \mathcal{E}^*$ be such that $\pi_n[E] = F_n$. **?** If $s \not< t$ for every $t \in F_n$, then $E_t = X \setminus \{x : x(m) = s, x(n) = t\} \in \mathcal{E}$ for every $t \in F_n$, so there is an $E' \in \mathcal{E}^*$ such that $E' \subseteq E \cap \bigcap_{t\in F_n} E_t$. Now there must be an $x \in E'$ such that $x(m) = s$; but in this case $x(n) \in F_n$ and $x(n) \neq t$ for every $t \in F_n$, which is absurd. **XQ**

Enumerate D in increasing order as $\langle m_i\rangle_{i\in\mathbb{N}}$ and define $\langle F_i'\rangle_{i\in\mathbb{N}}$ by setting $F_0' = F_{m_0}$,

$$F_{i+1}' = \{t : t \in F_{m_{i+1}},\ s < t \text{ for some } s \in F_i'\}.$$

Then for every $s \in F_i'$ there is a $t \in F_{i+1}'$ such that $s < t$, and for every $t \in F_{i+1}'$ there is an $s \in F_i'$ such that $s < t$; also every F_i' is finite and not empty, and $\mathrm{rank}(s) \geq i$ for every $i \in \mathbb{N}$ and $s \in F_i$. Consider

$$T' = \{s : s \in T,\ s \leq t \text{ for some } t \in \textstyle\bigcup_{i\in\mathbb{N}} F_i'\}.$$

Then T' is a subtree of T of height ω. Also, if $s \in T'$ and $\mathrm{rank}(s) = i$, there is some $t \in F_i'$ such that $s \leq t$. **P** There are $j \in \mathbb{N}$, $t \in F_j$ such that $s \leq t$; now there is a $t' \in F_i$ such that $t' \leq t$; as $\mathrm{rank}(t') \geq \mathrm{rank}(s)$, $s \leq t'$. **Q** Because every F_i' is finite, T' has finite levels.

As T is arbitrary, DMC$'$ is true.

5 Theorem In ZF, DMC is equiveridical with 'every compact Hausdorff space is a Baire space'.

proof Put §§2-4 together.

References

Blass A. [79] 'Injectivity, projectivity and the axiom of choice', Trans. Amer. Math. Soc. 225 (1979) 31-59.

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