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Extension-measurable families of sets

D.H.Fremlin

University of Essex, Colchester, England

1 Extension-measurable families

1A Definitions Let X be a set.

(a) A family $\mathcal{A}$ of subsets of X is **extension-measurable** if for every measure μ on X there is a measure on X, extending μ, which measures every member of $\mathcal{A}$.

(b) A family $\mathcal{A}$ of subsets of X is **easily extension-measurable** if for every complete totally finite measure μ on X there is a set $B \subseteq X$, of full outer measure for μ, such that $A \cap B$ is measured by the subspace measure on B for every $A \in \mathcal{A}$.

1B Proposition (a)(i) If $\mathcal{A}$, $\mathcal{B}$ are extension-measurable families of subsets of X then $\mathcal{A} \cup \mathcal{B}$ is an extension-measurable family of subsets of X.

(ii) If $\mathcal{A}$ is an extension-measurable family of subsets of X then any subset of $\mathcal{A}$ is extension-measurable.

(iii) If $\mathcal{A}$ is an extension-measurable family of subsets of X then the σ-algebra of subsets of X generated by $\mathcal{A}$ is extension-measurable.

(b)(i) If $\mathcal{A}$, $\mathcal{B}$ are easily extension-measurable families of subsets of X then $\mathcal{A} \cup \mathcal{B}$ is an easily extension-measurable family of subsets of X.

(ii) If $\mathcal{A}$ is an easily extension-measurable family of subsets of X then any subset of $\mathcal{A}$ is extension-measurable.

(iii) If $\mathcal{A}$ is an easily extension-measurable family of subsets of X then the σ-algebra of subsets of X generated by $\mathcal{A}$ is extension-measurable.

proof Elementary.

(a) If μ is a measure on X, extend it to measure every member of $\mathcal{A}$ and then to measure every member of $\mathcal{B}$. (ii) and (iii) are even easier.

(b) If μ is a complete totally finite measure on X, let $C \subseteq X$ be a set of full outer measure such that $A \cap C$ is measured by the subspace measure μ_C for every $A \in \mathcal{A}$. Let μ' be the measure on X defined by saying that $\mu' E = \mu_C(E \cap C)$ whenever $E \subseteq X$ and $E \cap C$ is measured by μ_C; then μ', like μ and μ_C, is complete and totally finite, so there is a $D \subseteq X$, of full outer measure for μ', such that $B \cap D$ is measured by μ' for every $B \in \mathcal{B}$. Note that μ' extends μ, because C has full outer measure for μ. Because $\mu'(X \setminus C) = 0$, $D \cap C$ has full outer measure for μ'. If E is measured by μ and $D \cap C \subseteq E$, then $\mu E = \mu' E = \mu' X = \mu X$; so $D \cap C$ has full outer measure for μ. Now the subspace measure $\mu'_{D\cap C}$ must measure $B \cap D \cap C$ for every $B \in \mathcal{B}$; but $\mu'_{D\cap C} = (\mu_C)_{D\cap C} = \mu_{D\cap C}$. At the same time, of course, $\mu_{D\cap C}$ measures $A \cap D \cap C$ for every $A \in \mathcal{A}$.

(ii) and (iii) are just as in (a).

1C Proposition Suppose that $\mathcal{A}$ is a family of subsets of a set X, and that every totally finite measure on X has an extension measuring every member of $\mathcal{A}$. Then $\mathcal{A}$ is extension-measurable.

proof Let μ be any measure on X. Let μ^* be the associated outer measure and $\check{\mu}$ the measure defined by Carathéodory's method from μ^*. (See Fremlin 01, 213Xa.) Write $\check{\Sigma}$ for the domain of $\check{\mu}$ and Σ^f for $\{E : \mu E < \infty\}$. Then $\check{\mu}$ extends μ, and if $A \subseteq X$ and $A \cap E \in \check{\Sigma}$ for every $E \in \Sigma^f$ then $A \in \check{\Sigma}$. Let $\mathcal{I}$ be the σ-ideal of subsets of X generated by Σ^f, and λ the measure with domain $\mathcal{P}X$ defined by setting $\lambda A = 0$ for $A \in \mathcal{I}$, $\lambda A = \infty$ for other $A \subseteq X$.

Now let $\mathcal{E} \subseteq \Sigma^f$ be a maximal family such that $\mu(E \cap F) = 0$ for all distinct E, $F \in \mathcal{E}$. For each $E \in \mathcal{E}$, let ν_E be the measure on X defined by setting $\nu_E A = \mu(A \cap E)$ whenever $A \subseteq X$ and $A \cap E \in \Sigma$. Then ν_E is a totally finite measure on X; let ν'_E be a measure on X extending ν_E and measuring every member of $\mathcal{A}$. Note that $\Sigma \subseteq \operatorname{dom} \nu_E \subseteq \operatorname{dom} \nu'_E$.

Let μ' be the sum $\lambda+\sum_{E\in\mathcal{E}}\nu'_E$, defined on $\bigcap_{E\in\mathcal{E}}\operatorname{dom}\nu'_E$. Then μ' is a measure on X, and $\Sigma\cup\mathcal{A}\subseteq\operatorname{dom}\mu'$. Finally, μ' extends μ. **P** Take $F\in\Sigma$. Set $\mathcal{E}'=\{E:E\in\mathcal{E},\ \mu(E\cap F)>0\}$,

$$u=\sum_{E\in\mathcal{E}'}\mu(E\cap F)=\sum_{E\in\mathcal{E}}\nu_EF=\sum_{E\in\mathcal{E}}\nu'_EF.$$

Then $u\le\mu F$ and $u\le\mu'F$, so if $u=\infty$ we have $\mu'F=\mu F$. Otherwise, $\mathcal{E}'$ is countable; set $G=\bigcup\mathcal{E}'$, so that $u=\mu(F\cap G)$. If $u=\mu F$, then $F\setminus G$ is μ-negligible therefore λ-negligible and $u=\mu F=\mu'F$. Suppose that $u<\mu F$, that is, $\mu(F\setminus G)>0$. **?** If $F\setminus G\in\mathcal{I}$, there is an $H\in\Sigma^f$ such that $\mu(H\cap F\setminus G)>0$. By the maximality of $\mathcal{E}$, there is an $E\in\mathcal{E}$ such that $\mu(E\cap H\cap F\setminus G)>0$; but in this case $E\in\mathcal{E}'$ and $E\not\subseteq G$, which is impossible. **X** So $F\setminus G\notin\mathcal{I}$ and

$$\mu'F\ge\lambda(F\setminus G)=\infty=\mu F.$$

Thus in all cases we have $\mu'F=\mu F$. **Q**

As μ is arbitrary, $\mathcal{A}$ is extension-measurable.

1D Corollary If $\mathcal{A}$ is an easily extension-measurable family of subsets of X then it is extension-measurable.

proof Let μ be a totally finite measure on X, and $\hat\mu$ its completion. Then there is a set $B\subseteq X$, of full outer measure for $\hat\mu$, such that $A\cap B$ is measured by the subspace measure $\hat\mu_B$ for every $A\in\mathcal{A}$. Setting $\mu'E=\hat\mu_B(E\cap B)$ whenever $E\subseteq X$ and $E\cap B$ is measured by $\hat\mu_B$, we get a measure on X extending $\hat\mu$ and μ and measuring every member of $\mathcal{A}$. By 1C, this is enough to show that $\mathcal{A}$ is extension-measurable.

1E Proposition (a) If $A\subseteq X$ then $\{A\}$ is an easily extension-measurable family of subsets of X.

(b) If $\mathcal{A}$ is an extension-measurable family of subsets of a set X, then $\{A\cap Y:A\in\mathcal{A}\}$ is an extension-measurable family of subsets of Y for any set Y.

(c) If $\mathcal{A}$ is an easily extension-measurable family of subsets of a set X, then $\{A\cap Y:A\in\mathcal{A}\}$ is an easily extension-measurable family of subsets of Y for any set Y.

proof (a) If μ is a totally finite measure on X, with domain Σ, let F be a measurable envelope of A, and consider $B=A\cup(X\setminus F)$. Then B has full outer measure (because, for instance, any measurable envelope of B is a measurable envelope of A, so essentially includes F), and $A\cap B=F\cap B$ is measured by the subspace measure on B.

(b)(i) If $Y\supseteq X$ and μ is a measure on Y, let ν be an extension of μ to a measure on Y measuring X (put (a) just above together with 1D). Let ν_X be the subspace measure on X and let ν'_X be a measure on X extending ν_X and measuring every member of $\mathcal{A}$. Set $\mu'E=\nu'_X(E\cap X)+\nu(E\setminus X)$ whenever the two terms of the sum are defined. Then μ' extends μ and measures every member of $\mathcal{A}$.

(ii) If $Y\subseteq X$ and μ is a measure on Y, set $\nu E=\mu(Y\cap E)$ whenever $E\subseteq X$ and μ measures $Y\cap E$. Let ν' be a measure on X extending ν and measuring every member of $\mathcal{A}$. Let μ' be the subspace measure ν'_Y on Y; then μ' extends μ and measures every member of $\{A\cap Y:A\in\mathcal{A}\}$.

(iii) In general, $\mathcal{A}$ is extension-measurable in $X\cup Y$, by (i), and we can apply (ii).

(c) We can repeat the arguments of (b), with minor variations.

(i) If $Y\supseteq X$ and μ is a complete totally finite measure on Y, let μ_X be the subspace measure on X. Then there is a $B\subseteq X$, of full outer measure for μ_X, such that the subspace measure $\mu_B=(\mu_X)_B$ measures $A\cap B$ for every $A\in\mathcal{A}$. Let F be a measurable envelope of X in Y and set $C=B\cup(Y\setminus F)$. If $E\subseteq Y$ and $\mu E>0$, then either $\mu(E\setminus F)>0$ so $\mu^*(E\cap C)>0$, or

$$\mu^*(E\cap C)\ge\mu^*(E\cap B)=\mu_X^*(E\cap B)=\mu_X(E\cap X)=\mu^*(E\cap X)=\mu(E\cap F)>0.$$

Thus C has full outer measure for μ, while $A\cap C=A\cap B$ is measured by μ_B and μ_C for every $A\in\mathcal{A}$. As μ is arbitrary, $\mathcal{A}$ is easily extension-measurable when regarded as a family of subsets of Y.

(ii) If $Y\subseteq X$ and μ is a complete totally finite measure on Y, set $\nu E=\mu(Y\cap E)$ whenever $E\subseteq X$ and μ measures $Y\cap E$. Then ν is a complete totally finite measure on X, so there is a set $B\subseteq X$, of full outer measure for ν, such that the subspace measure ν_B measures $A\cap B$ for every $A\in\mathcal{A}$. Since Y is conegligible for ν, $C = B \cap Y$ has full outer measure for ν, and ν_C measures $A \cap C$ for every $A \in \mathcal{A}$. But the subspace measure ν_X is μ, so $\mu_C = \nu_C$ and C has full outer measure for μ.

(iii) Putting these together we have the result, as before.

Remark Note that (b)-(c) here tell us that if $\mathcal{A}$ is a family of sets, and it is (easily) extension-measurable as a family of subsets of some set including $\bigcup \mathcal{A}$, then it is (easily) extension-measurable as a family of subsets of any set including $\bigcup \mathcal{A}$. I will therefore speak henceforth of 'extension-measurable' or 'easily extension-measurable' families without necessarily specifying an enveloping set.

1F Lemma Let $\langle A_n \rangle_{n \in \mathbb{N}}$ be a non-decreasing sequence of sets. Then $\{A_n : n \in \mathbb{N}\}$ is easily extension-measurable.

proof Let $X = \bigcup_{n \in \mathbb{N}} A_n$ and let μ be a complete totally finite measure on X. For each $n \in \mathbb{N}$ let F_n be a measurable envelope of A_n; setting $G_n = \bigcap_{m \geq n} F_m$, G_n is always a measurable envelope of A_n and $\langle G_n \rangle_{n \in \mathbb{N}}$ is non-decreasing. Set $B = \bigcup_{n \in \mathbb{N}} A_n \setminus G_{n-1}$ (taking G_{-1} to be $\emptyset$). Then

$$\mu^* B = \sum_{n=0}^{\infty} \mu^*(B \cap G_n \setminus G_{n-1}) = \sum_{n=0}^{\infty} \mu^*(A_n \setminus G_{n-1})$$
$$= \sum_{n=0}^{\infty} \mu(G_n \setminus G_{n-1}) = \mu X$$

because $X = \bigcup_{n \in \mathbb{N}} G_n$. So B has full outer measure for μ. And $A_n = B \cap G_n$ is measured by the subspace measure on B for every n. As μ is arbitrary, $\{A_n : n \in \mathbb{N}\}$ is easily extension-measurable.

1G Theorem Let $\mathcal{A}$ be a family of sets.

(a) If $\{A \cap C : A \in \mathcal{A}\}$ is extension-measurable for every $C \in \mathcal{A}$, then $\mathcal{A}$ is extension-measurable.

(b) If $\{A \cap C : A \in \mathcal{A}\}$ is easily extension-measurable for every $C \in \mathcal{A}$, then $\mathcal{A}$ is easily extension-measurable.

proof (a) Set $X = \bigcup \mathcal{A}$, and let μ be a totally finite measure on X. Let $\mathcal{I}$ be the σ-ideal of subsets of X generated by $\mathcal{A}$. Passing over the trivial case $\mathcal{A} = \emptyset$, there is a sequence $\langle C_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{A}$ such that $\mu^*(\bigcup_{n \in \mathbb{N}} C_n) = \sup_{B \in \mathcal{I}} \mu^* B$. Set $B = \bigcup_{n \in \mathbb{N}} C_n$; then there is a measure μ_1 extending μ such that $\mu_1 B = \mu^* B$ (see the proof of Prop. 1Ea); let $\hat{\mu}_1$ be its completion. Now, for any $A \in \mathcal{A}$,

$$\mu_1^*(B \cup A) \leq \mu^*(B \cup A) = \mu^* B = \mu_1 B,$$

so $\mu_1^*(A \setminus B) = 0$ (because μ_1, like μ, is totally finite) and $\hat{\mu}_1$ measures $A \setminus B$.

Set $B_n = \bigcup_{i<n} C_i$ for each n. Then $\{A \cap B_n : A \in \mathcal{A}\}$ is extension-measurable. **P** Induce on n. For $n = 0$ this is trivial. For the inductive step to $n+1$, $\mathcal{A}' = \{A \cap B_n : A \in \mathcal{A}\}$ and $\mathcal{A}'' = \{A \cap C_n : A \in \mathcal{A}\}$ are extension-measurable, by the inductive hypothesis and the hypothesis of the theorem; but $\{A \cap B_{n+1} : A \in \mathcal{A}\}$ is included in the σ-algebra generated by $\mathcal{A}' \cup \mathcal{A}''$, so is extension-measurable, by Prop. 1B. **Q**

By 1F, there is a measure μ_2 on X extending $\hat{\mu}_1$ and measuring every B_n. For each n, let ν_n be the subspace measure on $B_{n+1} \setminus B_n$ induced by μ_2; then we have an extension ν_n' of ν_n measuring $A \cap B_{n+1} \setminus B_n$ for every $A \in \mathcal{A}$, because $\{A \cap B_{n+1} : A \in \mathcal{A}\}$ is extension-measurable. Let ν be the measure on X defined by saying that

$$\nu E = \mu_2(E \setminus B) + \textstyle\sum_{n=0}^{\infty} \nu_n'(E \cap B_{n+1} \setminus B_n)$$

for any $E \subseteq X$ for which the terms in the sum are all defined. Then ν extends μ_2 and μ and measures every member of $\mathcal{A}$.

Thus every totally finite measure on X can be extended to measure all members of $\mathcal{A}$. By Prop. 1Eb, $\mathcal{A}$ is extension-measurable.

(b) Argue as before, but starting from a complete measure μ, so that $\hat{\mu} = \mu$. Take $\langle C_n \rangle_{n \in \mathbb{N}}$ and $\langle B_n \rangle_{n \in \mathbb{N}}$ and B as before. Let F be a measurable envelope of B, and μ_1 the subspace measure on $B \cup (X \setminus F)$. By 1F, there is a set $D \subseteq B \cup (X \setminus F)$ of full outer measure for μ_1 and μ such that the subspace measure μ_D measures $B_n \cap D$ for every n. For $n \in \mathbb{N}$, let ν_n be the subspace measure on $D \cap B_{n+1} \setminus B_n \subseteq C_n$. Then there is a $D_n \subseteq D \cap B_{n+1} \setminus B_n$, of full outer measure for ν_n, such that the subspace measure μ_{D_n} measures $A \cap C_n \cap D_n = A \cap D_n$ for every $A \in \mathcal{A}$. Set $D' = \bigcup_{n\in\mathbb{N}} D_n \cup (D \setminus F)$. Then

$$\mu^* D' = \mu^*_D D' = \sum_{n=0}^{\infty} \mu^*_D D_n + \mu_D(D \setminus B) = \sum_{n=0}^{\infty} \mu_D(D \cap B_n \setminus B_{n-1}) + \mu_D(D \setminus B)$$
$$= \mu_D(D \cap B) + \mu_D(D \setminus B) = \mu^* D = \mu X,$$

so D' has full outer measure for μ; while the subspace measure $\mu_{D'}$ measures $A \cap C_n$ for every $A \in \mathcal{A}$, and therefore measures $A \cap B$ for every $A \in \mathcal{A}$. But now observe that if $A \in \mathcal{A}$ then $\mu^*(A \cup B) = \mu^* B$, so $A \setminus F$ is μ-negligible and $A \cap D' \setminus B = A \cap D' \setminus F$ is measured by the complete measure $\mu_{D'}$. So $\mu_{D'}$ measures $A \cap D'$ for every $A \in \mathcal{A}$. As μ is arbitrary, $\mathcal{A}$ is easily extension-measurable.

1H Theorem Let $\mathcal{A}$ be a family of sets which is well-ordered by $\subseteq$. Then $\mathcal{A}$ is easily extension-measurable.

proof Induce on the order type $\operatorname{otp}\mathcal{A}$ of $\mathcal{A}$.

(a) If $\operatorname{otp}\mathcal{A} = 0$ then $\mathcal{A} = \emptyset$ and the result is trivial.

(b) For the inductive step to a successor ordinal, suppose that $\operatorname{otp}\mathcal{A} = \xi + 1$. Then $\mathcal{A}$ has a greatest member X and $\mathcal{A}' = \mathcal{A} \setminus \{X\}$ has order type ξ. Let μ be a complete totally finite measure on X. Then there is a $B \subseteq X$ of full outer measure for μ such that the subspace measure μ_B measures $A \cap B$ for every $A \in \mathcal{A}'$; so that in fact μ_B measures $A \cap B$ for every $A \in \mathcal{A}$. By 1Ec, this is enough to show that $\mathcal{A}$ is easily extension-measurable.

(b) For the inductive step to a non-zero limit ordinal, use 1Gb.

1I Corollary Any disjoint family of sets is easily extension-measurable.

proof If $\langle A_\xi \rangle_{\xi<\kappa}$ is a disjoint family, where κ is a cardinal, set $B_\xi = \bigcup_{\eta<\xi} A_\eta$ for $\xi \le \kappa$; then $\{B_\xi : \xi \le \kappa\}$ is easily extension-measurable, by Theorem 1H, so the σ-algebra Σ it generates is easily extension-measurable, while $A_\xi = B_{\xi+1} \setminus B_\xi$ belongs to Σ for every $\xi < \kappa$.

1J Remarks In Fremlin 03, 439A, there are three examples of probability measures on σ-subalgebras of the Borel σ-algebra $\mathcal{B}$ of $[0,1]$ which cannot be extended to the whole of $\mathcal{B}$. Accordingly $\mathcal{B}$ is not extension-measurable. It follows that $\mathcal{A} = \{[0,q] : q \in \mathbb{Q}\}$ is not extension-measurable, even though it is countable and totally ordered. Similarly, setting $E_n = \{x : x \in \{0,1\}^{\mathbb{N}},\ x(n) = 1\}$, $\{E_n : n \in \mathbb{N}\}$ is not extension-measurable, even though it is countable and well-founded. Note that if we set $x_n(i) = 0$ for $i \le n$, 1 for $i > n$ and $I_n = \{x_i : i \le n\}$, then $\mathcal{A} = \{E_n : n \in \mathbb{N}\} \cup \{I_n : n \in \mathbb{N}\}$ is not extension-measurable, even though it is countable, well-founded and has levels of size 2.

Because $\mathcal{P}(\omega_1 \times \omega_1)$ is the σ-algebra generated by $\{A \times B : A,\ B \subseteq \omega_1\}$ (Fremlin 03, 419F), there is a countable set $\mathcal{A} \subseteq \mathcal{P}(\omega_1)$ such that the subdiagonal $\{(\xi,\eta) : \xi \le \eta < \omega_1\}$ is in the σ-algebra generated by $\{A \times B : A,\ B \in \mathcal{A}\}$; accordingly $\mathcal{A}$ is not extension-measurable (see the argument in Fremlin 03, 419G).

On the other hand, if X is any set, the countable-cocountable algebra of X (Fremlin 01, 211R) is extension-measurable, because it is included in the σ-algebra generated by a well-ordered family. Thus any family of countable sets is extension-measurable.

1K Consider the following statement:

(†) Whenever $\mathcal{A}$ is a countable extension-measurable family of subsets of a set X, the σ-algebra of subsets of X generated by $\mathcal{A}$ has only countably many atoms.

I have been unable to determine whether (†) is a theorem of ZFC. It is however relatively consistent with ZFC, as the next result shows.

1L Proposition Suppose that the uniformity $\operatorname{non}\mathcal{N}$ of the Lebesgue null ideal is greater than ω_1. Then (†) is true.

proof ? Otherwise, let $\mathcal{A}$ be a countable extension-measurable family of subsets of a set X such that the σ-algebra it generates has uncountably many atoms. In this case there must be a set $Y \subseteq X$, of cardinal

ω_1, such that the sets $\mathcal{A}_y = \{A : y \in A \in \mathcal{A}\}$ are all different for different $y \in Y$. Let μ be the countable-cocountable measure on Y (Fremlin 01, 211R). As $\{A \cap Y : A \in \mathcal{A}\}$ is extension-measurable (1Eb), there is a measure ν on Y extending μ and measuring $A \cap Y$ for every $A \in \mathcal{A}$. Let $\langle A_n \rangle_{n\in\mathbb{N}}$ run over $\mathcal{A}$, and let $f = \frac{2}{3}\sum_{n=0}^{\infty} 3^{-n}\chi(A_n \cap Y)$ be the corresponding Marczewski functional, so that $f : Y \to [0,1]$ is injective and $\operatorname{dom}\nu$-measurable. Let λ be the image measure νf^{-1}, so that λ is a topological probability measure on $[0,1]$. Because λ is countably separated (Fremlin 02, 343D) and zero on singletons, it is atomless (Fremlin 02, 343Xi); but as $\#(f[Y]) \le \omega_1 < \operatorname{non}\mathcal{N}$, $\lambda^* f[Y] = 0$ (Fremlin 03, 439F), which is impossible. **X**

2 Automatically extension-measurable classes We have seen that any well-ordered family of sets is extension-measurable; that is, that knowing only the order structure of the family, and nothing more about the way its sets are composed, we can be sure that measures extend to it. There is more to be said.

2A Definition Let $\mathcal{P}_{\text{em}}$ be the class of partially ordered sets P such that whenever a family $\mathcal{A}$ of sets is order-isomorphic to P then $\mathcal{A}$ is easily extension-measurable.

2B Proposition Let P be a partially ordered set.

(a) Let Q be a subset of P with the induced ordering. If $P \in \mathcal{P}_{\text{em}}$ then $Q \in \mathcal{P}_{\text{em}}$.

(b) Let Q be a partially ordered set and $f : P \to Q$ an order-preserving surjection. If $P \in \mathcal{P}_{\text{em}}$ then $Q \in \mathcal{P}_{\text{em}}$.

(c) Let P' be the reversal of P. If $P \in \mathcal{P}_{\text{em}}$ then $P' \in \mathcal{P}_{\text{em}}$.

(d) If P can be covered by finitely many subsets, each belonging to $\mathcal{P}_{\text{em}}$, then $P \in \mathcal{P}_{\text{em}}$. In particular, any finite partially ordered set belongs to $\mathcal{P}_{\text{em}}$.

(e) Every well-ordered set belongs to $\mathcal{P}_{\text{em}}$.

proof (a) Let $\langle A_q \rangle_{q\in Q}$ be a family of sets such that $A_q \subseteq A_{q'}$ iff $q \le q'$. We may suppose that $X = \bigcup_{q\in Q} A_q$ is disjoint from P. For $p \in P$ set $B_p = \{p' : p' \in P,\ p' \le p\} \cup \bigcup_{q\in Q, q\le p} A_q$. Then $B_p \subseteq B_{p'}$ iff $p \le p'$, and $B_q \cap X = A_q$ if $q \in Q$. So $\{B_p : p \in P\}$, $\{B_p \cap X : p \in P\}$ and $\{A_q : q \in Q\}$ are easily extension-measurable.

(b) Let $\langle A_q \rangle_{q\in Q}$ be a family of sets such that $A_q \subseteq A_{q'}$ iff $q \le q'$. We may suppose that $X = \bigcup_{q\in Q} A_q$ is disjoint from P. For $p \in P$ set $B_p = \{p' : p' \in P,\ p' \le p\} \cup A_{f(p)}$. Then $B_p \subseteq B_{p'}$ iff $p \le p'$, and $B_p \cap X = A_{f(p)}$ for every p. So $\{B_p : p \in P\}$ and $\{B_p \cap X : p \in P\} = \{A_q : q \in Q\}$ are easily extension-measurable, by 1Ec.

(c) If $\mathcal{A}$ is order-isomorphic to P', then $\mathcal{A}' = \{X \setminus A : A \in \mathcal{A}\}$, where $X = \bigcup\mathcal{A}$, is order-isomorphic to P, therefore easily extension-measurable; now $\mathcal{A}$ and $\mathcal{A}'$ generate the same σ-algebra of subsets of X, so $\mathcal{A}$ is easily extension-measurable.

(d) Use Prop. 1B(b-i) and Prop. 1Ea.

(e) This is a re-statement of Theorem 1H.

2C Lemma Let $\mathcal{P}'_{\text{em}}$ be the class of partially ordered sets P such that whenever a family $\mathcal{A}$ of sets is order-isomorphic to P then $\mathcal{A}$ is extension-measurable. If $P \in \mathcal{P}'_{\text{em}}$ and $Q \subseteq P$ then $Q \in \mathcal{P}'_{\text{em}}$.

proof Repeat the argument of 2Ba, but using 1Eb instead of 1Ec.

2D Incomparability indices Recall that if P is a partially ordered set, a **weak antichain** in P is a set $C \subseteq P$ such that if $p, q \in C$ and $p \le q$ then $p = q$ (Fremlin 84). Now suppose that P has no infinite weak antichains. In this case, if $\mathcal{C}$ is the set of all weak antichains in P, we have a rank function $s_P : \mathcal{C} \to \mathrm{On}$ such that, for any $C \in \mathcal{C}$ and ordinal ξ, $s_P(C) \le \xi$ iff $s_P(C') < \xi$ for every weak antichain $C' \supset C$. The **incomparability index** $\tilde{s}(P)$ is now $s_P(\emptyset)$.

Note that if $p \in P$, $Q = P \setminus (]-\infty, p] \cup [p, \infty[)$ and $C \subseteq Q$, then $C \cup \{p\}$ is a weak antichain in P iff C is a weak antichain in Q. Consequently $s_Q(C) = s_P(C \cup \{p\})$ for every weak antichain $C \subseteq Q$ and $\tilde{s}(Q) = s_P(\{p\}) < \tilde{s}(P)$.

2E Theorem Let P be a partially ordered set. Then the following are equiveridical:

(i) $P \in \mathcal{P}_{\text{em}}$;

(ii) any family of sets which is order-isomorphic to P is extension-measurable;
(iii) P has no infinite weak antichains and $\mathbb{Q}$ does not embed into P.

proof (i)⇒(ii) is immediate from 1D.

(ii)⇒(iii) The examples in 1I show that we can have (α) a non-extension-measurable family of sets order-isomorphic to $\mathbb{Q}$ (β) a non-extension-measurable family of sets which is a countable weak antichain. So 2C gives the result.

(iii)⇒(i) Suppose that P has no weak antichains and that $\mathbb{Q}$ does not embed into P.

(α) The proof proceeds by induction on the incomparability index $\tilde{s}(P)$. So let us suppose that whenever Q is a partially ordered set, not including a copy of $\mathbb{Q}$, and $\tilde{s}(Q) < \tilde{s}(P)$, then $Q \in \mathcal{P}_{\mathrm{em}}$. In particular, if $p \in P$ then $P'_p = P \setminus (]-\infty, p] \cup [p, \infty[) \in \mathcal{P}_{\mathrm{em}}$, by the remark in §2D.

Now the point of the proof is the following fact: if $Q \subseteq P$, then $R = \{q : q \in Q, Q \cap]-\infty, q[\in \mathcal{P}_{\mathrm{em}}\} \in \mathcal{P}_{\mathrm{em}}$. **P** Note that $q \in R$ whenever $q \in Q$ and $q \leq r \in R$, by Prop. 2Ba as usual. Let $\mathcal{A}$ be a family of sets which is order-isomorphic to R. Take any $C \in \mathcal{A}$. Then $\{A : A \in \mathcal{A},\ A \subset C\}$ is order-isomorphic to an initial segment $R \cap]-\infty, r[$ of R and is therefore easily extension-measurable. Also $\{A : A \in \mathcal{A},\ A \not\subseteq C,\ C \not\subseteq A\}$ is order-isomorphic to a subset of some P'_p, so is easily extension-measurable. Consequently

$$\{A \cap C : A \in \mathcal{A}\} = \{C\} \cup \{A : A \in \mathcal{A},\ A \subset C\} \cup \{A \cap C : A \in \mathcal{A},\ A \not\subseteq C,\ C \not\subseteq A\}$$

is easily extension-measurable. As C is arbitrary, $\mathcal{A}$ is easily extension-measurable, by Theorem 1Gb. As $\mathcal{A}$ is arbitrary, $R \in \mathcal{P}_{\mathrm{em}}$. **Q**

(β) It follows that if $Q \subseteq P$ and $Q \notin \mathcal{P}_{\mathrm{em}}$ there is a $B_0 \in Q$ such that neither $Q \cap]-\infty, B_0]$ nor $Q \cap [B_0, \infty[$ belongs to $\mathcal{P}_{\mathrm{em}}$. **P** Set $R = \{q : q \in Q,\ Q \cap]-\infty, q] \in \mathcal{P}_{\mathrm{em}}\}$, $R' = \{q : q \in Q,\ Q \cap [q, \infty[\in \mathcal{P}_{\mathrm{em}}\}$. Then $R \in \mathcal{P}_{\mathrm{em}}$, by (b); similarly, applying (b) to the reversal of Q, $R' \in \mathcal{P}_{\mathrm{em}}$. So $Q \neq R \cup R'$; take $B_0 \in Q \setminus (R \cup R')$. **Q** Note that in this case neither $Q \cap]-\infty, B_0[$ nor $Q \cap]B_0, \infty[$ can belong to $\mathcal{P}_{\mathrm{em}}$.

(γ) **?** If $P \notin \mathcal{P}_{\mathrm{em}}$, we can choose inductively a sequence $\langle I_n \rangle_{n \in \mathbb{N}}$ of finite subsets of $P \cup \{-\infty, \infty\}$, as follows. $I_0 = \{-\infty, \infty\}$. Given that $I_n \subseteq P \cup \{-\infty, \infty\}$ is finite and totally ordered and that none of the open intervals $]u, v[$, for $u < v$ in I_n, belong to $\mathcal{P}_{\mathrm{em}}$, enumerate I_n in increasing order as $u_0, u_1, \ldots, u_m$ and for $j < m$ choose $q_j \in]u_j, u_{j+1}[$ such that neither $]u_j, q_j[$ nor $]q_j, u_{j+1}[$ belongs to $\mathcal{P}_{\mathrm{em}}$; (c) tells us that this is possible. Set $I_{n+1} = I_n \cup \{q_j : j \leq m\}$ and continue.

At the end of the induction, set $D = P \cap \bigcup_{n \in \mathbb{N}} I_n$. Then D is countable, totally ordered, not empty, has neither greatest nor least element, and between any two members of D there is another; so $D \cong \mathbb{Q}$, which is supposed to be impossible. **X**

So $P \in \mathcal{P}_{\mathrm{em}}$, and the induction continues, as required.

2F Corollary If P is a partially ordered set, then $P \in \mathcal{P}_{\mathrm{em}}$ iff every countable subset of P belongs to $\mathcal{P}_{\mathrm{em}}$.

Remark There is no corresponding result for extension-measurable families, at least not in ZFC (see 4H).

2G I note that for families of the type examined here we automatically have a version of (†) as described in 1K.

Lemma Let X be a set and $\mathcal{A}$ a countable family of subsets of X which, regarded as a partially ordered set, belongs to $\mathcal{P}_{\mathrm{em}}$. Then the σ-algebra of subsets of X generated by $\mathcal{A}$ has countably many atoms.

proof I adapt ideas from the proof of 2E.

(a) Let $\mathbb{A}$ be the family of those countable subfamilies $\mathcal{A}$ of $\mathcal{P}X$ such that the σ-algebra of subsets of X generated by $\mathcal{A}$ has countably many atoms. Observe that if $\mathcal{A} \in \mathbb{A}$ then $\{A \cap B : A \in \mathcal{A}\} \in \mathbb{A}$ for every $B \subseteq X$. Set $\mathbb{P} = \{\mathcal{A} : \mathcal{A} \subseteq \mathcal{P}X \text{ is countable}, \mathcal{A} \in \mathcal{P}_{\mathrm{em}}\}$. I seek to show by induction on ζ that if $\mathcal{A} \in \mathbb{P}$ and $\tilde{s}(\mathcal{A})$, as defined in 2D, is at most ζ, then $\mathcal{A} \in \mathbb{A}$. The induction starts with the trivial case $\zeta = 0$. For the inductive step to $\zeta > 0$, assume that $\mathcal{A} \in \mathbb{A}$ whenever $\mathcal{A} \in \mathbb{P}$ and $\tilde{s}(\mathcal{A}) < \zeta$. Note that this will mean that if $\mathcal{A} \in \mathbb{P}$ and $\tilde{s}(\mathcal{A}) \leq \zeta$, then $\{C : C \in \mathcal{A},\ C \not\subseteq A,\ A \not\subseteq C\}$ will be in $\mathbb{A}$ for every $A \in \mathcal{A}$.

(b) If $\mathcal{B} \in \mathbb{P}$ is countable and $\tilde{s}(\mathcal{B}) \le \zeta$, then $\mathcal{C} = \{B : B \in \mathcal{B}, \mathcal{B} \cap \mathcal{P}C \in \mathbb{A}\} \in \mathbb{A}$. **P** Let $\langle C_n \rangle_{n\in\mathbb{N}}$ run over $\mathcal{C}$, and set $C'_n = \bigcup_{i<n} C_i$ for $i \le n$; let T_n be the σ-algebra generated by $\{C'_n \cap B : B \in \mathcal{B}\}$ for each n. Observe that T_{n+1} is included in the σ-algebra generated by

$$\mathrm{T}_n \cup (\mathcal{B} \cap \mathcal{P}C_n) \cup \{C_n \cap B : B \in \mathcal{B},\, C_n \not\subseteq B,\, B \not\subseteq C_n\}$$

for each n; so, inducing on n, we see that T_n has countably many atoms, for each n. Because $\langle C'_n \rangle_{n\in\mathbb{N}}$ is non-decreasing, the σ-algebra generated by $\bigcup_{n\in\mathbb{N}} \mathrm{T}_{n+1} \cap \mathcal{P}(C'_{n+1} \setminus C'_n)$ has countably many atoms. But this σ-algebra includes $\mathcal{C}$. **Q**

(c) It follows that if $\mathcal{B} \in \mathbb{P} \setminus \mathbb{A}$ then there is a $B_0 \in \mathcal{B}$ such that neither $\{B : B_0 \supseteq B \in \mathcal{B}$ nor $\{B : B_0 \subseteq B \in \mathcal{B}$ belongs to $\mathbb{A}$. **P** Set $\mathcal{C} = \{B : B \in \mathcal{B}, \mathcal{B} \cap \mathcal{P}B \in \mathbb{A}\}$, $\mathcal{C}' = \{B : B \in \mathcal{B}, \{B' : B \subseteq B' \in \mathcal{B}\} \in \mathbb{A}\}$. Then $\mathcal{C} \in \mathbb{A}$, by (b); similarly, applying (b) to $\{X \setminus B : B \in \mathcal{B}\}$, $\mathcal{C}' \in \mathbb{A}$. So $\mathcal{B} \ne \mathcal{C} \cup \mathcal{C}'$; take $B_0 \in \mathcal{B} \setminus (\mathcal{C} \cup \mathcal{C}')$. **Q** Note that in this case neither $\{B : B_0 \supseteq B \in \mathcal{B}\}$ nor $\{B : B_0 \subseteq B \in \mathcal{B}\}$ can belong to $\mathbb{A}$.

(d) ? If $\mathcal{A} \in \mathbb{P}$, $\tilde{s}(\mathcal{A}) = \zeta$, but $\mathcal{A} \notin \mathbb{A}$, we can choose inductively a sequence $\langle \mathcal{I}_n \rangle_{n\in\mathbb{N}}$ of finite subsets of $\mathcal{A} \cup \{\emptyset, X\}$, as follows. $\mathcal{I}_0 = \{\emptyset, X\}$. Given that $\mathcal{I}_n \subseteq \mathcal{A} \cup \{\emptyset, X\}$ is finite and totally ordered and that none of the intervals $\{A : A \in \mathcal{A}, I \subset A \subset J\}$, for $I \subset J$ in $\mathcal{I}_n$, belong to $\mathbb{A}$, enumerate I_n in increasing order as $I_0, I_1, \dots, I_m$ and for $j < m$ choose $B_j \in \mathcal{A}$ such that $I_j \subset B_j \subset I_{j+1}$ and neither $\{A : A \in \mathcal{A}, I_j \subset A \subset B_j\}$ nor $\{A : A \in \mathcal{A}, B_j \subset A \subset I_{j+1}\}$ belongs to $\mathbb{A}$; (c) tells us that this is possible. Set $\mathcal{I}_{n+1} = \mathcal{I}_n \cup \{B_j : j \le m\}$ and continue.

At the end of the induction, set $\mathcal{D} = \mathcal{A} \cap \bigcup_{n\in\mathbb{N}} \mathcal{I}_n$. Then $\mathcal{D}$ is countable, totally ordered, not empty, has neither greatest nor least element, and between any two members of $\mathcal{D}$ there is another; so $\mathcal{D} \cong \mathbb{Q}$, which is supposed to be impossible. **X**

So $\mathcal{A} \in \mathbb{A}$, and the induction continues, as required.

(e) Accordingly $\mathbb{P} \subseteq \mathbb{A}$, as required.

3 Null-ideal-preserving extension-measurable families In the definition of 'extension-measurablefamily' above, arbitrary countably additive extensions of measures have been accepted. There are contexts in which we wish to maintain control of some feature of our original measure μ; e.g., its measure algebra or its null ideal $\mathcal{N}(\mu)$. I look first at null ideals.

3A Preserving null ideals (a) Of course any extension of a measure leaves previously negligible sets negligible; the question is, whether it creates new negligible sets. The construction in the proof of 1D normally does. Let us therefore say that a family $\mathcal{A}$ of subsets of a set X is **null-ideal-preserving extension-measurable** if for every measure μ on X there is an extension of μ to a measure ν measuring every member of $\mathcal{A}$ and with the same null ideal as μ.

(c) I offer immediately a couple of examples to add to those mentioned in §1J.

(i) Let X be any set and $\langle D_\xi \rangle_{\xi<\omega_1}$ a disjoint family of non-empty subsets of X. Then $\{D_\xi : \xi < \omega_1\}$ is easily extension-measurable, by Theorem 1G. But it is not null-ideal-preserving extension-measurable. **P** Let μ be the trivial probability measure with domain $\{\emptyset, X\}$. Then no D_ξ is μ-negligible. But if ν is any extension of μ measuring every D_ξ, then there must be some ξ such that $\nu D_\xi = 0$. **Q** Setting $A_\xi = \bigcup_{\eta<\xi} D_\eta$ for $\xi < \omega_1$, we have a well-ordered family which is not null-ideal-preserving extension-measurable.

(ii) Let $f : [0,1] \to [0,1]$ be a function such that (α) $\mu^* f^{-1}[\{t\}] = 1$ for every $t \in [0,1]$, where μ is Lebesgue measure on $[0,1]$ (β) $\mu_2^*\{(x,t) : f(x) = t\} = 1$, where μ_2 is Lebesgue measure on $[0,1]^2$. Let $\mathcal{U}$ be a countable base for the topology of $[0,1]$, and set $\mathcal{A} = \{f^{-1}[U] : U \in \mathcal{U}\}$. Then there is an extension of μ to a measure ν measuring every member of $\mathcal{A}$. **P** Set $g(x) = (x, f(x))$ for $x \in [0,1]$, so that $\mu_2^* g[\,[0,1]\,] = 1$, and set $\nu F = \mu_2 H$ whenever $H \subseteq [0,1]^2$ is measured by μ_2 and $F = g^{-1}[H]$. **Q** But of course any extension of μ measuring every set of the form $f^{-1}[\{t\}]$ must change the null ideal.

3B Using the same paths as before, we have the following easy results.

Proposition (a) If $\mathcal{A}$, $\mathcal{B}$ are null-ideal-preserving extension-measurable families of subsets of X then $\mathcal{A} \cup \mathcal{B}$ is a null-ideal-preserving extension-measurable family of subsets of X.

(b) If $\mathcal{A}$ is null-ideal-preserving extension-measurable then any subset of $\mathcal{A}$ is null-ideal-preserving extension-measurable.

(c) If $\mathcal{A}$ is null-ideal-preserving extension-measurable and $\bigcup\mathcal{A} \subseteq X$ then the σ-algebra of subsets of X generated by $\mathcal{A}$ is null-ideal-preserving extension-measurable.

(d) If $A \subseteq X$ then $\{A\}$ is a null-ideal-preserving extension-measurable family of subsets of X.

(e) If $\mathcal{A}$ is an null-ideal-preserving extension-measurable family of subsets of a set X, then $\{A \cap Y : A \in \mathcal{A}\}$ is a null-ideal-preserving extension-measurable family of subsets of Y for any set Y.

(f) Suppose that $\mathcal{A}$ is a family of subsets of a set X, and that every probability measure on X has an extension, with the same null ideal, measuring every member of $\mathcal{A}$. Then $\mathcal{A}$ is null-ideal-preserving extension-measurable.

proof (a)-(c) Use the same ideas as in Proposition 1B.

(d) Let μ be a measure on X, and Σ its domain. For E, $F \in \Sigma$, set

$$\nu_1((E \cap A) \cup (F \setminus A)) = \mu^*(E \cap A) + \sup\{\mu G : G \in \operatorname{dom}\mu,\, G \subseteq F \setminus A\}$$

$$\nu_2((E \cap A) \cup (F \setminus A)) = \mu^*(F \setminus A) + \sup\{\mu G : G \in \operatorname{dom}\mu,\, G \subseteq E \cap A\}.$$

It is easy to check that both ν_1 and ν_2 are measures extending μ and measuring A. Set $\nu = \frac{1}{2}(\nu_1 + \nu_2)$; then ν is a measure extending μ and measuring A. If now $D \in \mathcal{N}(\nu)$, there are E, $F \in \Sigma$ such that $D \subseteq (E \cap A) \cup (F \setminus A)$ and $\nu(E \cap A) = \nu(F \setminus A) = 0$. But this means that $\mu^*(E \cap A) = \mu^*(F \setminus A) = 0$ so $E \cap A$ and $F \setminus A$ and D are all μ-negligible. As μ is arbitrary, $\{A\}$ is null-ideal-preserving extension-measurable.

(e) Use the arguments of the proof of 1Eb, but at each point checking that extensions preserve null ideals.

(f) Of course the hypothesis ensures that if μ is any totally finite measure on X then there is an extension of μ with the same null ideal as μ and measuring every member of $\mathcal{A}$. I can therefore follow the method of the proof of 1C. In that argument, note that $\mathcal{N}(\check{\mu}) = \mathcal{N}(\mu) = \{F : \mu^* F = 0\}$. When choosing ν'_E for $E \in \mathcal{E}$, of course we ensure that $\mathcal{N}(\nu'_E) = \mathcal{N}(\nu_E)$; otherwise. proceed as before to get the extension μ' of μ. Now $\mathcal{N}(\mu') = \mathcal{N}(\mu)$. **P** If $D \in \mathcal{N}(\mu')$, there is an $H \in \operatorname{dom}\mu'$ such that $D \subseteq H$ and $\mu' H = 0$. In particular, $\lambda H = 0$, so H must be included in $\bigcup_{n\in\mathbb{N}} H_n$ for some sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ in Σ^f. For each n, $\mathcal{E}_n = \{E : E \in \mathcal{E},\, \mu(E \cap H_n) > 0\}$ is countable, and $H_n \setminus \bigcup\mathcal{E}_n$ is μ-negligible, by the maximality of $\mathcal{E}$. Next, for $E \in \mathcal{E}$, $\nu'_E H \le \mu' H = 0$, so $\nu_E H = 0$ and $H \cap E$ is μ-negligible; but this means that $H \cap H_n$ must be μ-negligible for each n and $D \subseteq H \in \mathcal{N}(\mu)$. As D is arbitrary, $\mathcal{N}(\mu') \subseteq \mathcal{N}(\mu)$, and the two null ideals are equal. **Q**

Remark As before, (e) here tells us that if $\mathcal{A}$ is a family of sets, and it is null-ideal-preserving extension-measurable as a family of subsets of some set including $\bigcup\mathcal{A}$, then it is null-ideal-preserving extension-measurable as a family of subsets of any set including $\bigcup\mathcal{A}$. We can therefore speak of 'null-ideal-preserving extension-measurable families' without necessarily specifying an enveloping set.

3C As remarked in §3A, we do not have a result as strong as Theorem 1H. But we have the following limited version.

Theorem Let $\mathcal{A}$ be a countable family of sets which is well-ordered by $\subseteq$. Then $\mathcal{A}$ is null-ideal-preserving extension-measurable.

proof Of course we may suppose that $\emptyset \in \mathcal{A}$. Induce on the order type $\operatorname{otp}\mathcal{A}$ of $\mathcal{A}$.

(a) If $\mathcal{A} = \{\emptyset\}$, the result is trivial. For the inductive step to $\operatorname{otp}\mathcal{A} = \zeta + 1$ where $\zeta \ge 1$, let A^* be the greatest member of $\mathcal{A}$. Then $\mathcal{A} \setminus \{A^*\}$ and $\{A^*\}$ are null-ideal-preserving extension-measurable by the inductive hypothesis and 3Bd, so $\mathcal{A}$ is null-ideal-preserving extension-measurable, by 3Ba.

(b) For the case $\operatorname{otp}\mathcal{A} = \omega$, let $\langle A_n\rangle_{n\in\mathbb{N}}$ be the increasing enumeration of $\mathcal{A}$, and μ a probability measure on $X = \bigcup_{n\in\mathbb{N}} A_n$. Choose $\langle n_k\rangle_{k\in\mathbb{N}}$ and $\langle \mu_k\rangle_{k\in\mathbb{N}}$ inductively, as follows. $\mu_0 = \mu$, $n_0 = 0$. Given that μ_k is an extension of μ with the same null sets as μ, measuring A_i for $i \le n_k$, let $n_{k+1} > n_k$ be such that $\mu_k^* A_{n_{k+1}} \ge 1 - 2^{-k}$. Use the recipe of the proof of 3Bd to construct an extension μ'_k of μ_k, with the same null sets as μ_k, such that $\mu'_k A_{n_{k+1}}$ is defined and equal to $\frac{1}{2}(\mu_k^* A_{n_{k+1}} + \sup\{\mu_k G : G \subseteq A_{n_{k+1}}$ is measured by $\mu_k\}$). Of course $\{A_i : i < n_{k+1}\}$ is null-ideal-preserving extension-measurable, so we can extend μ'_k to a measure μ_{k+1}, still with the same null sets as μ'_k, μ_k and μ, measuring A_i for every $i \le n_{k+1}$. Cutting back the domain of μ_{k+1} if we have been careless, we can ensure that $\mu_{k+1}\restriction\mathcal{P}A_{n_k} = \mu_k\restriction\mathcal{P}A_{n_k}$. Continue.

Observe that for each k we shall have

$$\mu_{k+1}A_{n_{k+1}} \ge \frac{1}{2}(\mu_k^*A_{n_{k+1}} + \mu_k A_{n_k}) \ge \frac{1}{2} - \frac{1}{2^{k+1}} + \frac{1}{2}\mu_k A_{n_k};$$

so if we set $\alpha = \sup_{k\in\mathbb{N}} \mu_k A_{n_k} = \lim_{k\to\infty} \mu_k A_{n_k}$, $\alpha \ge \frac{1}{2}(1+\alpha)$ and $\alpha = 1$. This means that if we set

$$\nu E = \sum_{k=0}^{\infty} \mu_{k+1}(E \cap A_{n_{k+1}} \setminus A_{n_k})$$

whenever $E \subseteq X$ and the sum is defined, $\nu X = 1$ and ν extends every μ_k. Moreover (because $X = \bigcup_{k\in\mathbb{N}} A_{n_k}$), any ν-null set will be a countable union of μ-null sets and $\mathcal{N}(\nu) = \mathcal{N}(\mu)$.

By 3Bf, this is enough to show that $\mathcal{A}$ is null-ideal-preserving extension-measurable.

(c) For the inductive step to $\mathcal{A} = \zeta$, where ζ is a non-zero countable limit ordinal, let $\langle A_\xi\rangle_{\xi<\zeta}$ be the increasing enumeration of $\mathcal{A}$. Let $\langle\zeta_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence with supremum ζ, starting from $\zeta_0 = 0$, and μ a measure on $X = \bigcup\mathcal{A}$. By (b), we have an extension of μ to a measure μ' with the same null ideal and measuring A_{ζ_n} for each n. By the inductive hypothesis, we have for each n an extension μ'_n of μ' with the same null ideal and measuring A_ξ for $\xi \le \zeta_{n+1}$. Now set

$$\nu E = \sum_{n=0}^{\infty} \mu'_n(E \cap A_{\zeta_{n+1}} \setminus A_{\zeta_n})$$

for those $E \subseteq X$ for which the sum is defined; then ν is a measure extending μ, measuring every A_ξ, and with the same null ideal as μ.

3D We now have an easy characterization of null-ideal-preserving extension-measurable families.

Theorem Let $\mathcal{A}$ be a family of subsets of a set X. Then $\mathcal{A}$ is null-ideal-preserving extension-measurable iff the σ-algebra T of subsets of X which it generates has only countably many atoms.

proof (a) If the set $\mathcal{D}$ of atoms of T is countable, enumerate it as $\langle D_i\rangle_{i<\zeta}$ for some $\zeta \le \omega$, and set $B_n = \bigcup_{i\le n} D_i$ for $n < \zeta$. Then $\langle B_n\rangle_{n<\zeta}$ is null-ideal-preserving extension-measurable, by Theorem 3C. But the σ-algebra generated by $\{B_n : n < \zeta\}$ includes $\mathcal{D}$ and therefore includes $\mathcal{A}$. So $\mathcal{A}$ is null-ideal-preserving extension-measurable.

(b) Suppose that $\mathcal{A}$ is null-ideal-preserving extension-measurable. Then there is a probability measure ν on X, extending the trivial probability measure, with domain T and null ideal $\{\emptyset\}$. Now ν must be purely atomic; moreover, for any atom D of ν, $D \cap A$ is either empty or D for every $A \in \mathcal{A}$. So the atoms of ν are also the atoms of T. As ν has only countably many atoms, T has countably many atoms, as declared.

3E Corollary If (†) of 1K is true, then every countable extension-measurable family is null-ideal-preserving extension-measurable.

4 Measure-algebra-preserving extensions At the other extreme from null-ideal-preserving extensions we have measure-algebra-preserving extensions, which work entirely by adding new negligible sets.

4A Definition I will say that a family $\mathcal{A}$ of subsets of a set X is **measure-algebra-preserving extension-measurable** if for every *localizable* measure μ on X (Fremlin 01, 211G, and Fremlin 02, 322Be) there is an extension of μ to a measure ν measuring every member of $\mathcal{A}$ and such that the embedding of $\operatorname{dom}\mu$ into $\operatorname{dom}\nu$ identifies the measure algebras; that is, for every $F \in \operatorname{dom}\nu$ there is an $E \in \operatorname{dom}\mu$ such that $\nu(F \triangle E) = 0$.

4B I should explain at once why I have adjusted the definition to consider only localizable measures.

Example (cf. Fremlin 01, 216D) Set $X = \omega_2 \times \omega_2$. Let μ_0 be the countable-cocountable algebra of ω_2. Let Σ be the set of those subsets E of X such that

$$\mu E = \sum_{\xi<\omega_2} \mu_0 E[\{\xi\}] + \sum_{\xi<\omega_2} \mu_0 E^{-1}[\{\xi\}]$$

is defined. Note that $\#(E)=\omega_2$ whenever $\mu E>0$; we therefore have an extension μ' of μ to the σ-algebra $\{E\triangle I : I\in[X]^{\le\omega_1}\}$ which gives measure 0 to every set of cardinal less than or equal to ω_1.

Let A be the set $\{(\xi,\eta):\xi\le\eta<\omega_2\}$. If ν is any measure on X extending μ' and measuring A, consider $A^\bullet$ in the measure algebra of ν. This includes $a_\xi=(\{\xi\}\times\omega_2)^\bullet$ and is disjoint from $b_\xi=(\omega_2\times\{\xi\})^\bullet$ for every $\xi<\omega_2$. **?** If $F\in\operatorname{dom}\mu'$ and $\nu(A\triangle F)=0$, express F as $E\triangle I$ where $E\in\Sigma$ and $\#(I)\le\omega_1$, and let $\alpha<\omega_2$ be such that $I\subseteq\alpha\times\alpha$. Let $J\subseteq\omega_2\setminus I$ have cardinal ω_1. For $\xi\in J$, $E[\{\xi\}]=F[\{\xi\}]$ must have cardinal ω_2, so is in fact cocountable in ω_2. There must therefore be a $\zeta\in\omega_2\setminus\alpha$ such that $\zeta\in E[\{\xi\}]$ for every $\xi\in J$, and $F^{-1}[\{\zeta\}]=E^{-1}[\{\zeta\}]$ is cocountable. But this means that $F^\bullet\supseteq b_\zeta$. **X**

So the measure algebras of ν and μ' cannot be the same.

4C As in 3B, we can recycle some of the previous ideas.

Proposition (a) If $\mathcal{A}$ and $\mathcal{B}$ are measure-algebra-preserving extension-measurable families of subsets of X then $\mathcal{A}\cup\mathcal{B}$ is a measure-algebra-preserving extension-measurable family of subsets of X.

(b) If $\mathcal{A}$ is measure-algebra-preserving extension-measurable then any subset of $\mathcal{A}$ is null-ideal-preserving extension-measurable.

(c) If $\mathcal{A}$ is measure-algebra-preserving extension-measurable and $\bigcup\mathcal{A}\subseteq X$ then the σ-algebra of subsets of X generated by $\mathcal{A}$ is measure-algebra-preserving extension-measurable.

(d) If $A\subseteq X$ then $\{A\}$ is a measure-algebra-preserving extension-measurable family of subsets of X.

(e) If $\mathcal{A}$ is a measure-algebra-preserving extension-measurable family of subsets of a set X, then $\{A\cap Y : A\in\mathcal{A}\}$ is a measure-algebra-preserving extension-measurable family of subsets of Y for any set Y.

(f) Suppose that $\mathcal{A}$ is a family of subsets of a set X, and that every probability measure on X has an extension, with the same measure algebra, measuring every member of $\mathcal{A}$. Then $\mathcal{A}$ is measure-algebra-preserving extension-measurable.

proof (a)-(c) Use the ideas of 1B.

(d) Start by taking the c.l.d. extension $\tilde\mu$ of μ (Fremlin 01, 213D). This does not change the measure algebra (Fremlin 01, 213Hb). Now use the construction of 1Ea, based on $\tilde\mu$. To check that the extension ν, as defined there, has the same measure algebra as μ and $\tilde\mu$, we need note only that $A^\bullet$, interpreted in the measure algebra $\mathfrak{B}$ of ν, corresponds to some member of the measure algebra $\mathfrak{A}$ of $\tilde\mu$. But $\tilde\mu$ is complete, locally determined and localizable, so A has a measurable envelope H for $\tilde\mu$ (Fremlin 01, 213K); and now $\nu(H\setminus A)=0$, so $A^\bullet=H^\bullet$ in $\mathfrak{B}\cong\mathfrak{A}$.

(e) As in 3Be, work through the argument for 1Eb, checking that measure algebras can be preserved at each stage.

(f) I select ideas from the proof of 1C. Let μ be a localizable measure on X, Σ its domain, and $\Sigma^f\subseteq\Sigma$ the family of sets of finite measure. Let $\mathcal{E}\subseteq\Sigma^f$ be a maximal family such that $\mu(E\cap F)=0$ for all distinct E, $F\in\mathcal{E}$. For each $E\in\mathcal{E}$, let ν_E be the measure on X defined by setting $\nu_E G=\mu(G\cap E)$ whenever $G\subseteq X$ and $A\cap G\in\Sigma$. Then ν_E is a totally finite measure on X; let ν'_E be a measure on X extending ν_E, measuring every member of $\mathcal{A}$, and with the same measure algebra as ν_E. Now set $\Sigma'=\bigcap_{E\in\mathcal{E}}\operatorname{dom}\nu'_E$ and $\mu'G=\sum_{E\in\mathcal{E}}\nu_E G$ for $G\in\Sigma'$. If $G\in\Sigma$, then

$$\mu G=\sup_{F\in\Sigma^f}\mu(G\cap F)=\textstyle\sum_{E\in\mathcal{E}}\mu(G\cap E)=\sum_{E\in\mathcal{E}}\nu'_E G=\mu' G$$

because μ is semi-finite; so μ' extends μ. If $G\in\Sigma'$, then for each $E\in\mathcal{E}$ there is a $G_E\in\operatorname{dom}\nu_E$ such that $\nu'_E(G\triangle G_E)=0$. Because μ is localizable, there is an $H\in\Sigma$ such that $H^\bullet=\sup_{E\in\mathcal{E}}(E\cap G_E)^\bullet$ in the measure algebra of μ. But (because $\langle E^\bullet\rangle_{E\in\mathcal{E}}$ is disjoint) this means that $\mu((E\cap H)\triangle(E\cap G_E))=0$ for every $E\in\mathcal{E}$, and

$$\begin{aligned}\mu'(H\triangle G)&=\sum_{E\in\mathcal{E}}\nu'_E(H\triangle G)=\sum_{E\in\mathcal{E}}\nu'_E(H\triangle G_E)\\&=\sum_{E\in\mathcal{E}}\mu(E\cap(H\triangle G_E))=0.\end{aligned}$$

Thus μ' and μ have the same measure algebra, and μ' is an extension of the kind required.

4D Corollary Any easily extension-measurable family is measure-algebra-preserving extension-measurable.

proof Use the construction of 1D, observing that μ, μ_B and μ' there have naturally isomorphic measure algebras, and 4Cf.

4E Theorem Let $\mathcal{A}$ be an extension-measurable family of subsets of a set X, and suppose that for every countable $\mathcal{A}' \subseteq \mathcal{A}$ the σ-algebra of subsets of X generated by $\mathcal{A}'$ has countably many atoms. Then $\mathcal{A}$ is measure-algebra-preserving extension-measurable.

proof Let μ be a probability measure on X, with domain Σ and measure algebra $\mathfrak{A}$, and ν an extension of μ to the σ-algebra T generated by $\Sigma \cup \mathcal{A}$, with measure algebra $\mathfrak{B}$; we have a natural embedding of $\mathfrak{A}$ in $\mathfrak{B}$. Now let $\mathfrak{C}$ be the closed subalgebra of $\mathfrak{B}$ generated by $\{A^\bullet : A \in \mathcal{A}\}$. Then $\mathfrak{C}$ is purely atomic. **P?** Otherwise, $\mathfrak{C}$ has an atomless closed subalgebra which is countably τ-generated, and therefore included in the closed subalgebra $\mathfrak{D}$ generated by $\{A^\bullet : A \in \mathcal{A}'\}$ for some countable $\mathcal{A}' \subseteq \mathcal{A}$. Now $\mathfrak{D}$, being a measure algebra with an atomless closed subalgebra, is also atomless. But the σ-algebra of subsets of X generated by $\mathcal{A}'$ has countably many atoms, at least one of which has non-zero ν-measure and must correspond to an atom of $\mathfrak{D}$. **XQ**

Let $\langle c_n \rangle_{n \in \mathbb{N}}$ run over the atoms of $\mathfrak{C}$, and for each $n \in \mathbb{N}$ let $a_n \in \mathfrak{A}$ be the upper envelope $\inf\{a : c_n \subseteq a \in \mathfrak{A}\}$. Set $a'_n = a_n \setminus \sup_{i<n} a_i$ for each n, so that $\langle a'_n \rangle_{n \in \mathbb{N}}$ is a partition of unity in $\mathfrak{A}$. Note that $\mathfrak{B}$ is the closed subalgebra of itself generated by $\mathfrak{A} \cup \{c_n : n \in \mathbb{N}\}$, so the principal ideal $\mathfrak{B}_{c_n}$ generated by c_n is always just $\{c_n \cap a : a \in \mathfrak{A}\}$. If a, $a' \in \mathfrak{A}$ are such that $c_n \cap a = c_n \cap a'$, then $1 \setminus (a \triangle a')$ includes c_n and a_n, and $a_n \cap a = a_n \cap a'$. So we can define $\pi_n : \mathfrak{B}_{c_n} \to \mathfrak{A}$ by setting $\pi_n(c_n \cap a) = a'_n \cap a$ for every $a \in \mathfrak{A}$. Each π_n is an order-continuous Boolean homomorphism onto the principal ideal $\mathfrak{A}_{a'_n}$. Since $\sup_{n \in \mathbb{N}} a'_n = \sup_{n \in \mathbb{N}} a_n = 1$, we can assemble the π_n into an order-continuous Boolean homomorphism $\pi : \mathfrak{B} \to \mathfrak{A}$ (Fremlin 02, 381C). We can therefore define $\nu' : \mathrm{T} \to [0,1]$ by setting $\nu' F = \bar{\mu}(\pi F^\bullet)$; because $F \to F^\bullet$, π and $\bar{\mu}$ are sequentially order-continuous, ν' is countably additive. If $a \in \mathfrak{A}$, then

$$\pi a = \sup_{n \in \mathbb{N}} \pi_n(c_n \cap a) = \sup_{n \in \mathbb{N}} \pi_n(a'_n \cap a) = a,$$

so $\nu' E = \mu E$ for every $E \in \Sigma$. If $F \in \mathrm{T}$ there is an $E \in \Sigma$ such that $\pi F^\bullet = E^\bullet$, so that

$$\pi(F^\bullet \triangle E^\bullet) = \pi F^\bullet \triangle \pi E^\bullet = \pi F^\bullet \triangle E^\bullet = 0$$

and

$$\nu'(F \triangle E) = \bar{\mu}(\pi(F^\bullet \triangle E^\bullet)) = 0.$$

Thus ν' is an extension of μ to T which does not change the measure algebra. By 4Cf, this is enough to show that $\mathcal{A}$ is measure-algebra-preserving extension-measurable.

4F Corollary Suppose that (†) of 1K is true. Then every extension-measurable family is measure-algebra-preserving extension-measurable.

4G For all I know, it is a theorem of ZFC that every extension-measurable family is easily extension-measurable. The following examples indicate some of the obstacles to sorting this out.

(a) Example Suppose that $\operatorname{non}\mathcal{N} = \mathfrak{c}$. Then there is a function $f : [0,1] \to [0,1]$ such that (α) the graph X of f has full outer measure for Lebesgue measure μ_2 on $[0,1]^2$ (β) for every Borel measurable $g : [0,1] \to [0,1]$ the set $\{t : f(t) = g(t)\}$ is negligible for Lebesgue measure μ_1 on $[0,1]$. Let μ be the image measure $\mu_1 \tilde{f}^{-1}$ where $\tilde{f}(t) = (t, f(t))$ for $t \in [0,1]$, Σ its domain, ν the subspace measure $(\mu_2)_X$ derived from μ_2; set $A_q = \{(t,u) : (t,u) \in X,\ u \le q\}$ for $q \in \mathbb{Q}$, and $\mathcal{A} = \{A_q : q \in \mathbb{Q}\}$. Then ν is an extension of μ which measures every member of $\mathcal{A}$.

? Suppose that there is an extension ν_1 of μ which measures every member of $\mathcal{A}$ and has the same measure algebra as μ. Because μ is a copy of Lebesgue measure, it has a lifting $\phi : \Sigma \to \Sigma$ (Fremlin 02, §341). For each $q \in \mathbb{Q}$, let $F_q \in \Sigma$ be such that $\nu_1(A_q \triangle F_q) = 0$, and set $H_q = \phi F_q$. Define $h : [0,1] \to [0,1]$ by setting $\{h(t) = \sup\{q : (t, f(t)) \in H_q\}$ for $t \in [0,1]$. Then h is Lebesgue measurable, so there is a Borel function g such that $h = g$ μ_1-a.e.; by the choice of f, $\{t : g(t) = f(t)\}$ and $\{t : h(t) = f(t)\}$ are μ_1-negligible. So $V = \{(t, f(t)) : t \in [0,1],\ h(t) = f(t)\}$ is ν_1-negligible. But $V \supseteq X \setminus \bigcup_{q \in \mathbb{Q}} (A_q \triangle H_q)$ is also ν_1-conegligible. **X**

Thus here we have an example of a measure μ and a countable family $\mathcal{A}$ such that μ can be extended to measure every member of $\mathcal{A}$, but not in a way which leaves the measure algebra unchanged.

(b) Example Suppose that there is an atomlessly-measurable cardinal. Then there is an extension of Lebesgue measure on $\mathbb{R}$ which measures every subset of $\mathbb{R}$, but by the Gitik-Shelah theorem (Fremlin 07?, 543E), no such extension can have the same measure algebra.

4H Example There is a family $\mathcal{A}$ of sets such that every countable subfamily of $\mathcal{A}$ is easily extension-measurable, but $\mathcal{A}$ itself is not extension-measurable.

proof Let X be an Aronszajn tree, that is, an uncountable tree in which every totally ordered subset and every level is countable. Let $\mathcal{A}$ be

$$\{\{x\} : x \in X\} \cup \{[x, \infty[: x \in X\}.$$

Let $r : X \to \omega_1$ be the rank function of X, that is, $r(x) = \operatorname{otp}(]-\infty, x[)$ for $x \in X$. If $\mathcal{A}' \subseteq \mathcal{A}$ is countable, set

$$\zeta = \max(\sup\{r(x) + 1 : \{x\} \in \mathcal{A}'\}, \sup\{r(x) : [x, \infty[\in \mathcal{A}'\}).$$

Then

$$\mathcal{C}_\zeta = \{\{x\} : r(x) < \zeta\} \cup \{[x, \infty[: r(x) = \zeta\}$$

is a countable partition of X, so is easily extension-measurable (1I), and every member of $\mathcal{A}'$ is the union of countably many members of $\mathcal{C}_\zeta$, so $\mathcal{A}'$ is easily extension-measurable (1Bb). Note that the set of atoms of the σ-algebra generated by $\mathcal{A}'$ is countable because $\mathcal{C}_\zeta$ is countable.

However, $\mathcal{A}$ is not measure-algebra-preserving extension-measurable. **P?** Otherwise, let μ be the countable-cocountable measure on X and ν an extension of μ, measuring every member of $\mathcal{A}$, and such that for every $F \in \operatorname{dom} \nu$ there is an $E \in \operatorname{dom} \mu$ such that $\nu(E \triangle F) = 0$. Then ν, like μ, takes only the values 0 and 1. Each $\mathcal{C}_\zeta$, as defined above, is a countable partition of X into members of $\mathcal{A}$, so we have a (unique) $C_\zeta \in \mathcal{C}_\zeta$ such that $\nu C_\zeta = 1$. If $x \in X$ then $\nu\{x\} = \mu\{x\} = 0$, so there must be an x_ζ such that $r(x_\zeta) = \zeta$ and $\nu\,[x_\zeta, \infty[= 1$. But this means that $[x_\xi, \infty[\cap [x_\zeta, \infty[\neq \emptyset$, that is, that either $x_\xi \le x_\zeta$ or $x_\zeta \le x_\xi$, for all ξ, $\zeta < \omega_1$, and we have an uncountable branch in X. **XQ**

By Theorem 4E, $\mathcal{A}$ is not extension-measurable.

5 Problems

5A Is every measure-algebra-preserving extension-measurable family easily extension-measurable?

5B Is it consistent to suppose that there is an extension-measurable family which is not measure-algebra-preserving extension-measurable? (See 4F.)

5C Is (†) of 1K provable in ZFC?

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