Hausdorff measures

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1 Theorem Let (X,ρ) be an analytic metric space, and suppose that $s \ge 0$ is less than the Hausdorff dimension of X. Then there is a continuous surjection $f: X \to [0,1]$ such that $\mu_{Hs} f^{-1}[\{\alpha\}] \ge 1$ for every $\alpha \in [0,1]$, where μ_{Hs} is s-dimensional Hausdorff measure on X.

proof (a) For $\delta > 0$, $t \ge 0$, $A \subseteq X$ set

$$\theta_{t\delta}(A) = \inf\{\sum_{i=0}^{\infty} (\operatorname{diam} A_i)^t : \langle A_i\rangle_{i\in\mathbb{N}} \text{ is a cover of } A \text{ by sets of diameter at most } \delta\}.$$

We know that there is a $t > s$ such that $\mu_{Ht}X > 0$; by Howroyd's theorem (Fremlin 03, 471S), or otherwise, there is a compact set $K \subseteq X$ such that $\mu_{Ht}K > 0$. Let $\langle K_i\rangle_{i\in I}$ be a maximal disjoint family of compact subsets of K such that $1 < \mu_{Hs}K_i < \infty$ for every i. Then I is uncountable. **P?** Otherwise, we have $\mu_{Ht}K_i = 0$ for every i (Fremlin 03, 471L), so $\mu_{Ht}(K \setminus \bigcup_{i\in I} K_i) > 0$ and $\mu_{Hs}(K \setminus \bigcup_{i\in I} K_i) = \infty$; now by Howroyd's theorem there is a compact set $K' \subseteq K \setminus \bigcup_{i\in I} K_i$ with $1 < \mu_{Hs}K' < \infty$ and we ought to have added K' to $\langle K_i\rangle_{i\in I}$. **XQ**

(b) Since $\mu^*_{Hs}A = \sup_{\delta>0} \theta_{s\delta}A$ for every $A \subseteq X$, there is a $\delta > 0$ such that $J = \{i : i \in I,\ \theta_{s\delta}K_i \ge 1\}$ is uncountable. Let $\mathcal{G}$ be the family of relatively open subsets G of K such that $\{i : i \in J,\ K_i \subseteq G\}$ is uncountable. Then for every $G \in \mathcal{G}$ there are G_0, $G_1 \in \mathcal{G}$ such that $\overline{G}_0$, $\overline{G}_1$ are disjoint and included in G. **P** Let $\mathcal{U}$ be a countable base for the topology of K closed under finite unions. Let $\langle j_\xi\rangle_{\xi<\omega_1}$ be a family of distinct elements of J such that $K_{j_\xi} \subseteq G$ for every $\xi < \omega_1$. For each ξ, there are U_ξ, $V_\xi \in \mathcal{U}$ such that $K_{j_\xi} \subseteq U_\xi$, $K_{j_{\xi+1}} \subseteq V_\xi$, $\overline{U}_\xi \cap \overline{V}_\xi = \emptyset$ and $\overline{U}_\xi \cup \overline{V}_\xi \subseteq G$. Because $\mathcal{U}$ is countable, there are G_0, $G_1 \in \mathcal{U}$ such that $\{\xi : G_0 = U_\xi,\ G_1 \in V_\xi\}$ is uncountable; and these will serve. **Q**

(c) Set $S^* = \bigcup_{n\in\mathbb{N}} \{0,1\}^n$. By (b), we can choose $\langle G_\sigma\rangle_{\sigma\in S^*}$ inductively in $\mathcal{G}$ such that $G_\emptyset = K$ and $G_{\sigma^\frown 0}$, $G_{\sigma^\frown 1}$ have disjoint closures included in G_σ for every $\sigma \in S^*$. Since every G_σ includes some K_i, $\theta_{s\delta}\overline{G}_\sigma \ge 1$ for every σ. For $z \in \{0,1\}^{\mathbb{N}}$, set $L_z = \bigcap_{n\in\mathbb{N}} G_{z\restriction n}$; then $\mu_{Hs}L_z \ge 1$. **P?** Otherwise, there is a cover $\langle A_i\rangle_{i\in\mathbb{N}}$ of L_z by sets of diameter at most $\frac{\delta}{2}$ such that $\sum_{i=0}^{n} (\operatorname{diam} A_i)^s < 1$. Expanding the A_i slightly, we can find a cover $\langle H_i\rangle_{i\in\mathbb{N}}$ of L_z by open sets of diameter at most δ with $\sum_{i=0}^{\infty} (\operatorname{diam} H_i)^s < 1$. But now there is some $n \in \mathbb{N}$ such that $\overline{G}_{z\restriction n} \subseteq \bigcup_{i\in\mathbb{N}} H_i$ and $\theta_{s\delta}G_{z\restriction n} \le 1$. **XQ**

(d) Setting

$$L = \bigcup_{z\in\{0,1\}^{\mathbb{N}}} L_z = \bigcap_{n\in\mathbb{N}} \bigcup_{\sigma\in\{0,1\}^{\mathbb{N}}} \overline{G}_\sigma,$$

we have a continuous function $h : L \to \{0,1\}^{\mathbb{N}}$ such that $h(x) = z$ for $x \in L_z$. Let $g : \{0,1\}^{\mathbb{N}} \to [0,1]$ be a continuous surjection, and $f : X \to [0,1]$ a continuous function extending g. Then f has the required property, since $f^{-1}[\{\alpha\}] \supseteq L_z$ whenever $g(z) = \alpha$.

2 Corollary Suppose that X is an analytic space and that $0 \le s < \dim_H X$. Let μ_{Hs} be Hausdorff s-dimensional measure on X and let λ be the c.l.d. product of μ_{Hs} and Lebesgue measure μ on $[0,1]$, as defined in Fremlin 01. Then there is a set $W \subseteq X \times [0,1]$, measured by λ, such that $\{\alpha : \mu_{Hs}W^{-1}[\{\alpha\}] = 0\}$ is not measured by μ.

proof Take $f : X \to [0,1]$ as in Theorem 1; identify f with its graph, so that it is a closed subset of $X \times [0,1]$. Let $A \subseteq [0,1]$ be any non-measurable set. Set $W = f \cap (X \times A)$. Then

$$\{\alpha : \mu_{Hs}W^{-1}[\{\alpha\}] = 0\} = [0,1] \setminus A$$

is not measured by μ. Also W is measured by λ. **P** If $E \subseteq X$ and $\mu_{Hs}E < \infty$, then $C = \{\alpha : \mu_{Hs}(E \cap f^{-1}[\{\alpha\}]) > 0\}$ is countable. Now $V = f \cap (E \times ([0,1] \setminus C))$ belongs to the Borel σ-algebra $\mathcal{B}(X \times [0,1]) = \mathcal{B}(X)\widehat{\otimes}\mathcal{B}([0,1])$, so is measured by λ; and as all the sections $V^{-1}[\{\alpha\}]$ are empty if $\alpha \in C$ and negligible if

$\alpha \in [0,1] \setminus C$, $\lambda V = 0$, by Fubini's theorem applied to the subspace measure on $E \times [0,1]$ (using Fremlin 01, 251P to see that this is itself a product measure). So

$$(E \times [0,1]) \cap W = (E \times A) \cap f = ((E \times C) \cap f) \cup ((X \times A) \cap V)$$

is measured by the (complete) measure λ. As E is arbitrary, W is measured by λ (Fremlin 01, 251H). **Q**

So we have found a suitable set.

3 These ideas can be used to give a generalization of a result of M.Elekes, as follows. We need some notation.

Definitions (a) Write $\mathcal{N}$ for the Lebesgue null ideal and $\operatorname{add}\mathcal{N}$ for its additivity. Note that if (X,ρ) is an analytic metric space, $r > 0$ and $\mu_{Hr}X > 0$, then the additivity of μ_{Hr} is $\operatorname{add}\mathcal{N}$ (Fremlin 07?, 527B).

(b) In this paragraph and the next, let λ be the c.l.d. product (Fremlin 01, §251) of counting measure on $[0,1]$ with Lebesgue measure on $[0,1]$, so that $\lambda W = \sum_{\alpha\in[0,1]} \mu W[\{\alpha\}]$ whenever $W \subseteq [0,1]^2$ is such that all its vertical sections are measured by Lebesgue measure μ. For $W \in \Lambda = \operatorname{dom}\lambda$ set

$$\begin{aligned}\lambda' W &= \infty \text{ if } \#(W^{-1}[\,[0,1]\,]) = \mathfrak{c},\\ &= \lambda W \text{ otherwise.}\end{aligned}$$

Then $([0,1]^2, \Lambda, \lambda')$ is a measure space (not, of course, semi-finite).

4 Theorem Suppose that $\operatorname{add}\mathcal{N} = \mathfrak{c}$. Then (X, μ_{Hr}) is isomorphic to $([0,1]^2, \lambda')$ whenever (X,ρ) is an analytic metric space and $0 < r < \dim_H(X)$.

proof (a) Because X is separable and metrizable, its Borel σ-algebra has cardinal at most $\mathfrak{c}$; let $\langle E_\xi\rangle_{\xi<\mathfrak{c}}$ run over the Borel sets of finite measure for μ_{Hr}. Set $\hat{E}_\xi = \bigcup_{\eta<\xi} E_\eta$ for $\xi \le \mathfrak{c}$. Write Σ for the domain of μ_{Hr}. Let $f : X \to [0,1]$ be a continuous surjection such that $\mu_{Hr} f^{-1}[\{\alpha\}] \ge 1$ for every α, as in Theorem 1. By Howroyd's theorem, there is a Borel set $H_\alpha \subseteq f^{-1}[\{\alpha\}]$ with $1 \le \mu_{Hr} H_\alpha < \infty$. Choose a disjoint family $\langle F_\xi\rangle_{\xi<\mathfrak{c}}$ of sets of μ_{Hr}-measure 1 inductively, as follows. Given $\langle F_\eta\rangle_{\eta<\xi}$ where $\xi < \mathfrak{c}$, $\hat{F}_\xi = \bigcup_{\eta<\xi} F_\eta$ belongs to Σ, because we are supposing that $\mathfrak{c} = \operatorname{add}\mathcal{N} = \operatorname{add}\mu_{Hr}$. Observe that if $\eta < \xi$, then $\{\alpha : \alpha \in [0,1],$ $\mu_{Hr}(H_\alpha \cap F_\eta) > 0\}$ is countable; accordingly there are distinct α, $\beta \in [0,1]$ such that

$$\mu_{Hr}(H_\alpha \cap \hat{F}_\xi) = \textstyle\sum_{\eta<\xi} \mu_{Hr}(H_\alpha \cap F_\eta) = 0 = \mu_{Hr}(H_\beta \cap \hat{F}_\xi).$$

Since there is some ζ such that $E_\zeta = H_\alpha \cup H_\beta$, there is a first $\zeta_\xi < \mathfrak{c}$ such that $\mu_{Hr}(\hat{E}_{\zeta_\xi+1} \setminus \hat{F}_\xi) > 1$. Let F_ξ be such that

$$\hat{E}_{\zeta_\xi} \setminus \hat{F}_\xi \subseteq F_\xi \subseteq \hat{E}_{\zeta_\xi+1} \setminus F_\xi, \quad \mu_{Hr} F_\xi = 1,$$

and continue.

(b) The construction ensures that $\langle \zeta_\xi\rangle_{\xi<\mathfrak{c}}$ is non-decreasing. For every $\xi < \mathfrak{c}$ we have a $\zeta'_\xi \le \zeta_\xi$ such that $\mu_{Hr}(F_\xi \cap E_{\zeta'_\xi}) > 0$; also $\{\xi : \zeta'_\xi = \zeta\}$ is countable for every ξ; so $\sup_{\xi<\mathfrak{c}} \zeta_\xi = \sup_{\xi<\mathfrak{c}} \zeta'_\xi = \mathfrak{c}$. This means that every E_ζ is included in $\hat{F}_\xi$ for some $\xi < \mathfrak{c}$. In particular, $X = \bigcup_{\xi<\mathfrak{c}} F_\xi$.

(c) If $E \in \Sigma$, then of course $E \cap F_\xi \in \Sigma$ for every $\xi < \mathfrak{c}$. Conversely, if $E \subseteq X$ and $E \cap F_\xi \in \Sigma$ for every ξ, then $E \in \Sigma$. **P** If $A \subseteq X$ and $\mu^*_{Hr} A < \infty$, there is some $\zeta < \mathfrak{c}$ such that $A \subseteq E_\zeta$ and $\mu^*_{Hr} A = \mu_{Hr} E_\zeta$. Now if $\xi < \mathfrak{c}$ is such that $E_\zeta \subseteq \hat{F}_\xi$,

$$\begin{aligned}\mu^*_{Hr}(A \cap E) + \mu^*_{Hr}(A \setminus E) &\le \mu^*_{Hr}(E_\zeta \cap E) + \mu^*_{Hr}(E_\zeta \setminus E)\\ &\le \mu^*_{Hr}(E_\zeta \cap \hat{F}_\xi \cap E) + \mu^*_{Hr}(E_\zeta \cap \hat{F}_\xi \setminus E)\\ &= \sum_{\eta<\xi} \mu_{Hr}(E_\zeta \cap F_\eta \cap E) + \sum_{\eta<\xi} \mu_{Hr} E_\zeta \cap F_\eta \setminus E)\end{aligned}$$

(once again because $\operatorname{add}\mu_{Hr} = \mathfrak{c}$)

$$= \sum_{\eta<\xi} \mu_{Hr}(E_\zeta \cap F_\eta) = \mu_{Hr}(E_\zeta \cap \hat{F}_\xi)$$
$$= \mu_{Hr} E_\zeta = \mu^*_{Hr} A.$$

As A is arbitrary, $E \in \Sigma$. **Q**

(d) If $E \in \Sigma$ and $E \subseteq \hat{F}_\xi$ for some $\xi < \mathfrak{c}$, then

$$\mu_{Hr} E = \sum_{\eta<\xi} \mu_{Hr}(E \cap F_\eta) = \sum_{\eta<\mathfrak{c}} \mu_{Hr}(E \cap F_\eta).$$

If $E \in \Sigma$ and $E \not\subseteq \hat{F}_\xi$ for any $\xi < \mathfrak{c}$, then $E \not\subseteq E_\zeta$ for any $\zeta < \mathfrak{c}$, so $\mu_{Hr} E = \infty$. Note that as $\mathfrak{c} = \operatorname{add}\mathcal{N}$ is regular (Fremlin 07?, 513Ca), there is a $\xi < \mathfrak{c}$ such that $E \subseteq \hat{F}_\xi$ iff $\#(\{\eta : E \cap F_\eta \neq \emptyset\}) < \mathfrak{c}$.

(e) Observe now that, for each $\xi < \mathfrak{c}$, the subspace measure on F_ξ is isomorphic to Lebesgue measure on $[0,1]$, being an atomless Radon probability measure on a separable metrizable space (put 471Dg, 471F, 416Wa and 433B of Fremlin 03 together with 344I of Fremlin 02). Let $\langle \alpha_\xi \rangle_{\xi<\mathfrak{c}}$ be an enumeration of $[0,1]$; for each $\xi < \mathfrak{c}$, let $\phi_\xi : F_\xi \to [0,1]$ be a measure space isomorphism; and define $\phi : X \to [0,1]^2$ by setting $\phi(x) = (\alpha_\xi, \phi_\xi(x))$ if $x \in F_\xi$. Then (c)-(d) tell us that ϕ is an isomorphism between (X, μ_{Hr}) and $([0,1]^2, \lambda')$, because $\mathfrak{c} = \operatorname{add}\mathcal{N}$ is regular.

4 The measurable envelope property Let μ_{H1} be one-dimensional Hausdorff measure on $\mathbb{R}^2$ with the Euclidean metric. I do not know whether μ_{H1} ever has the measurable envelope property (Fremlin 02, 213Xl). However under certain special axioms it does not.

(a) Suppose that $\operatorname{non}\mathcal{N} = \operatorname{cf}\mathcal{N}$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$. Then μ_{H1} does not have the measurable envelope property. **P** Set $\kappa = \operatorname{non}\mathcal{N}$. Then there is a universally negligible set $Y \subseteq \mathbb{R}$ of cardinal κ (Fremlin 03, 439Fc); enumerate Y as $\langle y_\xi \rangle_{\xi<\kappa}$. Also, writing Σ for the family of Lebesgue measurable subsets of $\mathbb{R}$, $\operatorname{ci}(\Sigma \setminus \mathcal{N}) = \operatorname{cf}\mathcal{N} = \kappa$ (Fremlin 07?, 524Pb); let $\langle F_\xi \rangle_{\xi<\kappa}$ enumerate a coinitial family in $\Sigma \setminus \mathcal{N}$. Finally, take a set $D \subseteq \mathbb{R}$ such that both D and its complement meet every uncountable compact set in a set of cardinal $\mathfrak{c}$, and set $A = D \times Y$.

? If $E \supseteq A$ is a measurable envelope of A for μ_{H1}, then for each $\xi < \kappa$ the set $E^{-1}[\{y_\xi\}]$ is Lebesgue measurable and includes D, so is conegligible and $F_\xi \cap E^{-1}[\{y_\xi\}] \setminus D$ has cardinal $\mathfrak{c}$. We can therefore choose a family $\langle x_\xi \rangle_{\xi<\kappa}$ of distinct points such that $x_\xi \in F_\xi \cap E^{-1}[\{y_\xi\}] \setminus D$ for every ξ. Consider the set $H = \{(x_\xi, y_\xi) : \xi < \kappa\}$. Then $H \subseteq E \setminus A$. Because the projection of H on the vertical axis is universally negligible and every horizontal section of H is universally negligible, H is universally negligible and is measured by μ_{H1}. But the projection of H on the horizontal axis is not Lebesgue negligible so $\mu_{H1} H > 0$. **XQ**

(b) Suppose that $\operatorname{non}\mathcal{N} < \operatorname{cov}\mathcal{N}$. Then μ_{H1} does not have the measurable envelope property. **P** Again, set $\kappa = \operatorname{non}\mathcal{N}$. This time, enumerate a D of full outer measure for Lebesgue measure on $\mathbb{R}$ as $\langle x_\xi \rangle_{\xi<\kappa}$ (Fremlin 07?, 523Dc) and set $A = \{(x_\xi, x_\eta) : \eta \leq \xi < \kappa\}$. **?** If $E \supseteq A$ is a measurable envelope of A for μ_{H1}, then for each $\xi < \kappa$ the set $E^{-1}[\{x_\xi\}]$ is Lebesgue measurable and includes all but a negligible subset of D, so must be conegligible. Because $\kappa < \operatorname{cov}\mathcal{N}$ there is an $x \in \bigcap_{\xi<\kappa} E^{-1}[\{x_\xi\}]$, so that D is included in the vertical section $E[\{x\}]$. Accordingly $E[\{x\}]$ is not Lebesgue negligible. But every vertical section of A is Lebesgue negligible, so $F = E[\{x\}] \setminus A[\{x\}]$ is Lebesgue measurable and not Lebesgue negligible and $H = \{x\} \times F \subseteq E \setminus A$ is measured by μ_{H1} but not negligible. **XQ**

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