Poly-linked Boolean algebras

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1 Definition If $\boldsymbol{k} = \langle k_i \rangle_{i\in I}$ is a family in $\mathbb{N}\setminus\{0,1\}$, a partially ordered set P is **$\boldsymbol{k}$-poly-linked upwards** if it is expressible as $\bigcup_{i\in I} P_i$ where each P_i is k_i-linked upwards.

Similarly, a Boolean algebra $\mathfrak{A}$ is **$\boldsymbol{k}$-poly-linked** if $\mathfrak{A}\setminus\{0\}$ is expressible as $\bigcup_{i\in I} A_i$ where each A_i is k_i-linked.

2 Proposition If P and Q are partially ordered sets, $P \preccurlyeq_{\mathrm{T}} Q$ (definition: Fremlin 07?, §512), and Q is $\boldsymbol{k}$-poly-linked upwards, then P is $\boldsymbol{k}$-poly-linked upwards.

proof Let $f : P \to Q$ be a Tukey function, and $\langle Q_i \rangle_{i\in I}$ a cover of Q such that every Q_i is k_i-linked upwards in Q, where $\boldsymbol{k} = \langle k_i \rangle_{i\in I}$. Set $P_i = f^{-1}[Q_i]$; then P_i is k_i-linked upwards in P for each i, and $P = \bigcup_{i\in I} P_i$.

3 Proposition Let $\boldsymbol{k} = \langle k_n \rangle_{n\in\mathbb{N}}$ be any sequence in $\mathbb{N}\setminus\{0,1\}$. Then there is a $\boldsymbol{k}$-poly-linked Boolean algebra $\mathfrak{A}$ which is not σ-centered.

proof (a) We may suppose that $\langle k_n \rangle_{n\in\mathbb{N}}$ is non-decreasing. Let ν_ω be the usual measure on $\{0,1\}^\omega$, and $(\mathfrak{B}_\omega, \bar\nu_\omega)$ its measure algebra. Choose $\langle C_n \rangle_{n\in\mathbb{N}}$, $\langle r_n \rangle_{n\in\mathbb{N}}$, $\langle s_n \rangle_{n\in\mathbb{N}}$ inductively, as follows. $C_0 = \{0,1\}^\omega$, $r_0 = s_0 = 0$. Given that $\{C_i : i \le s_n\}$ is a family of non-empty cylinder sets in $\{0,1\}^\omega$ including all those of the form $\{x : x{\restriction}r_n = z\}$, let $r_{n+1} > r_n$ be such that $2^{n+2}k_i \le 2^{r_{n+1}}\nu_\omega C_i$ for every $i \le s_n$, and let $\langle C_i \rangle_{s_n < i \le s_{n+1}}$ enumerate the cylinder sets of the form $\{x : x{\restriction}r_{n+1} = z\}$.

At the end of the induction, set $c_n = C_n^\bullet \in \mathfrak{B}_\omega$ for each n, and let $\mathfrak{A}$ be the set

$$\{a : a \in \mathfrak{B}_\omega,\ \lim_{n\to\infty} \frac{k_n}{\bar\nu_\omega c_n}\min(\bar\nu_\omega(c_n \cap a), \bar\nu_\omega(c_n \setminus a)) = 0\}.$$

Then $1\setminus a$ and $a \cup b$ belong to $\mathfrak{A}$ for all a, $b \in \mathfrak{A}$, so $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$; observe also that every cylinder set belongs to $\mathfrak{A}$, because if $c \in \mathfrak{B}_\omega$ is (the equivalence class of) a cylinder set then for all but finitely many n one of $c_n \cap c$, $c_n \setminus c$ is zero.

(b) $\mathfrak{A}$ is $\boldsymbol{k}$-poly-linked. **P** Set

$$A_n = \{a : a \in \mathfrak{A},\ \bar\nu_\omega(c_n \setminus a) < \frac{1}{k_n}\bar\nu_\omega c_n\}.$$

Then A_n is k_n-linked. If $a \in \mathfrak{A}\setminus\{0\}$ then there are infinitely many n such that $\bar\nu_\omega(c_n \cap a) > \frac{1}{2}\bar\nu c_n$, and $a \in A_n$ for all but finitely many of these n. **Q**

(c) ? Suppose, if possible, that $\mathfrak{A}\setminus\{0\}$ can be expressed as $\bigcup_{n\in\mathbb{N}} D_n$ where every D_n is centered. We may suppose that every D_n is maximal; because every cylinder set belongs to $\mathfrak{A}$, there must be a $t_n \in \{0,1\}^\omega$ such that, for cylinder sets $C \subseteq \{0,1\}^\omega$, $t_n \in C$ iff $C^\bullet \in D_n$.

For each $n\in\mathbb{N}$ there is an i_n such that $s_n<i_n\le s_{n+1}$ and $t_n\in C_{i_n}$. Set $a=\sup_{n\in\mathbb{N}}c_{i_n}$. For each $j>0$, if $s_m<j\le s_{m+1}$, then $c_j\cap a$ is either c_j or $c_j\cap\sup_{n>m}c_{i_n}$, which has measure at most

$$\sum_{n=m+1}^{\infty}2^{-r_{n+1}}\le\sum_{n=m+1}\frac{1}{2^{n+2}k_j}\bar\nu_\omega c_j=\frac{1}{2^{m+2}k_j}\bar\nu_\omega c_j.$$

So $a\in\mathfrak{A}$. Also $\bar\nu_\omega a<1$, so there ought to be some $n\in\mathbb{N}$ such that $1\setminus a\in D_n$; in which case D_n cannot be linked, because $c_{i_n}\in D_n$. **X**

4 Proposition If $\boldsymbol{k}=\langle k_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{N}\setminus\{0,1\}$ such that $\lim_{n\to\infty}k_n=\infty$, then $\mathfrak{B}_\omega$ is not $\boldsymbol{k}$-poly-linked.

proof (a) If $\langle D_r\rangle_{r\in R}$ is a finite family of linked sets in $\mathfrak{B}_\omega$, there is an element of measure $\frac{1}{4}$ not belonging to any D_r. **P?** Otherwise, for $i<j$ in $\mathbb{N}$, set $c_{ij}=\{x:x(i)=0,\ x(j)=1\}^\bullet$, and set $\phi(\{i,j\})=r$ for some $r\in R$ such that $c_{ij}\in D_r$. By Ramsey's theorem, there is an infinite $M\subseteq\mathbb{N}$ such that $\phi\restriction[M]^2$ is constant; so we have an r such that $c_{ij}\in D_r$ whenever $i<j$ in M. But now observe that if $i<j<k$ in M then $c_{ij}\cap c_{jk}=0$, so D_r cannot be linked. **X**

(b) Suppose that $m\ge2$ and that $\langle D_r\rangle_{r\in R}$ is a finite family of subsets of $\mathfrak{B}\setminus\{0\}$ such that every element of measure $1-2^{-m+1}$ belongs to some D_r. Then there is an $r\in R$ such that D_r is not $\frac{(2m)!}{(m!)^2}$-linked. **P** For $I\in[\mathbb{N}]^m$ set $d_I=\{x:x\restriction I\text{ is constant}\}^\bullet$, and let $\phi(I)\in R$ be such that $1\setminus d_I\in D_{\phi(I)}$. By Ramsey's theorem again, there are $M\in[\mathbb{N}]^\omega$ and $r\in R$ such that $\phi(I)=r$ for every $I\in[M]^m$. Take $K\in[M]^{2m}$; then $\sup_{I\in[K]^m}d_I=1$, so $\{1\setminus d_I:I\in[K]^m\}$ witnesses that D_r is not $\frac{(2m)!}{(m!)^2}$-linked. **Q**

(c) Now suppose that $k_n\ge2$ for every n and that $\lim_{n\to\infty}k_n=\infty$. **?** If $\mathfrak{B}_\omega=\bigcup_{n\in\mathbb{N}}D_n$ where D_n is k_n-linked for every n, expand each D_n to a maximal k_n-linked set in $\mathfrak{B}_\omega$. Choose a strictly increasing sequence $\langle r_m\rangle_{m\in\mathbb{N}}$ such that $k_n\ge\frac{(2m)!}{(m!)^2}$ for $n\ge r_m$. By (a), we have an $a_0\in\mathfrak{B}_\omega$ such that $\bar\nu_\omega a_0\ge\frac{1}{4}$ and $a_0\notin D_n$ for $n<r_5$; by (b), we can find for each $m\ge1$ an a_m such that $\bar\nu_\omega a_m=1-2^{-m-3}$ and $a_m\notin D_n$ for $r_{m+4}\le n<r_{m+5}$. Now $a=\inf_{m\in\mathbb{N}}a_m$ is non-zero and belongs to no D_n. **X**

References

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