Partitions of $[0,1]$ into negligible sets

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1 The problem Let $\langle E_\xi\rangle_{\xi<\kappa}$ be a partition of $[0,1]$ into Lebesgue negligible sets. Must there be an $A\subseteq\kappa$ such that $\mu^*(\bigcup_{\xi\in A}E_\xi)=\mu^*(\bigcup_{\xi\in\kappa\setminus A}E_\xi)=1$, where μ is Lebesgue measure on $[0,1]$?

S.Zeberski[1] has shown that if $\langle E_\xi\rangle_{\xi<\kappa}$ is a partition of $[0,1]$ into meager sets, there is an $A\subseteq\kappa$ such that $\bigcup_{\xi\in A}E_\xi$ and $\bigcup_{\xi\in\kappa\setminus A}E_\xi$ both meet every non-meager Borel set.

2 Notation For the rest of this note, $\langle E_\xi\rangle_{\xi<\kappa}$ will represent a partition of $[0,1]$ into Lebesgue negligible sets. Let $(\mathfrak{B},\bar\mu)$ be the measure algebra of μ. Given $\langle E_\xi\rangle_{\xi<\kappa}$, then for $A\subseteq\kappa$ set $\theta A=G^\bullet\in\mathfrak{B}$ for any measurable envelope G of $\bigcup_{\xi<\kappa}E_\xi$, and $\nu A=\bar\mu(\theta A)=\mu^*(\bigcup_{\xi<\kappa}E_\xi)$. We have $\theta(\bigcup_{n\in\mathbb{N}}A_n)=\sup_{n\in\mathbb{N}}\theta A_n$ for every sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of subsets of κ, so ν is a submeasure on $\mathcal{P}\kappa$ and $\nu(\bigcup_{n\in\mathbb{N}}A_n)=\sup_{n\in\mathbb{N}}\nu A_n$ whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of subsets of κ.

Set

$$\mathcal{I}=\{A:A\subseteq\kappa,\,\nu A=0\}=\{A:A\subseteq\kappa,\,\theta A=0\}.$$

Then $\mathcal{I}$ is a σ-ideal of subsets of κ; let $\mathfrak{A}$ be the Dedekind σ-complete Boolean algebra $\mathcal{P}\kappa/\mathcal{I}$. Then ν and θ factor through $\mathfrak{A}$; setting $\bar\theta A^\bullet=\theta A$ for every $A\subseteq\kappa$, $\bar\theta:\mathfrak{A}\to\mathfrak{B}$ preserves countable suprema; setting $\bar\nu=\bar\mu\bar\theta$, $\bar\nu$ is a strictly positive submeasure on $\mathfrak{A}$.

3 Definition Let us say that $\langle E_\xi\rangle_{\xi<\kappa}$ **splits** $b\in\mathfrak{B}$ if there are disjoint A, $B\subseteq\kappa$ such that $b\subseteq\theta A=\theta B$.

4 Lemma If $\langle b_n\rangle_{n\in\mathbb{N}}$ is such that $\langle E_\xi\rangle_{\xi<\kappa}$ splits b_n for each n, then $\langle E_\xi\rangle_{\xi<\kappa}$ splits $\sup_{n\in\mathbb{N}}b_n$.

proof For each $n\in\mathbb{N}$ choose disjoint A_n, $B_n\subseteq\kappa$ such that $b_n\subseteq\theta A_n=\theta B_n$. Set $A'_n=A_n\setminus\bigcup_{i<n}(A_i\cup B_i)$, $B'_n=B_n\setminus\bigcup_{i<n}(A_i\cup B_i)$ for each n, $A=\bigcup_{n\in\mathbb{N}}A'_n$ and $B=\bigcup_{n\in\mathbb{N}}B'_n$. Then A and B are disjoint, and

$$\sup_{n\in\mathbb{N}}b_n\subseteq\sup_{n\in\mathbb{N}}\theta(A_n\cup B_n)=\theta(A\cup B).$$

Next, setting $A^*_n=\bigcup_{i\le n}A'_i$ and $B^*_n=\bigcup_{i\le n}B'_i$ for each n, $\theta A^*_n\subseteq\theta B^*_n$ for every n. **P** Induce on n. For $n=0$,

$$\theta A^*_0=\theta A'_0=\theta A_0=\theta B_0=\theta B^*_0.$$

For the inductive step to $n+1$,

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$$\theta A^*_{n+1} \setminus \theta B^*_{n+1} = (\theta A^*_n \cup \theta(A_{n+1} \setminus C_n)) \setminus (\theta B^*_n \cup \theta(B_{n+1} \setminus C_n))$$

(where $C_n = A^*_n \cup B^*_n$)

$$\subseteq (\theta A^*_n \setminus \theta B^*_n) \cup ((\theta A_{n+1} \setminus \theta C_n) \setminus (\theta B_{n+1} \setminus \theta C_n))$$

(because $\theta(B_{n+1} \setminus C_n) \supseteq \theta B_{n+1} \setminus \theta C_n$)

$$\subseteq (\theta A^*_n \setminus \theta B^*_n) \cup (\theta A_{n+1} \setminus \theta B_{n+1}) = 0$$

by the inductive hypothesis. **Q**

Accordingly

$$\theta A = \sup_{n\in\mathbb{N}} \theta A^*_n \subseteq \sup_{n\in\mathbb{N}} \theta B^*_n = \theta B.$$

The same formula with A's and B's exchanged must also be valid, and

$$\theta A = \theta B = \theta(A \cup B) \supseteq b.$$

5 Corollary There is a maximal $b^* \in \mathfrak{B}$ such that $\langle E_\xi \rangle_{\xi<\kappa}$ splits b^*.

6 First reduction of problem The original question was: does $\langle E_\xi \rangle_{\xi<\kappa}$ split 1? that is, is $b^* = 1$? Suppose we have an example with a negative answer to this question. Let $G \subseteq [0, 1]$ be a measurable set such that $G^\bullet = 1 \setminus b^*$. Then $\langle E_\xi \cap G \rangle_{\xi<\kappa}$ is a partition of G into negligible sets which does not split any non-zero $b \subseteq G^\bullet$. Since G, with its subspace measure, is isomorphic, up to a scalar multiple of the measure, to $([0, 1], \mu)$, we can transfer this into a partition $\langle E'_\xi \rangle_{\xi<\kappa}$ of $[0, 1]$ into negligible sets which does not split any non-zero member of $\mathfrak{B}$, that is, for which $b^* = 0$. For the rest of this note I will suppose that $\langle E_\xi \rangle_{\xi<\kappa}$ is such a partition; that is, that whenever $A \subseteq B \subseteq \kappa$ and $\theta B \neq 0$, then one of θA, $\theta(B \setminus A)$ is not equal to θB; that is, whenever a, $a' \in \mathfrak{A}$ are non-zero and disjoint, $\bar\theta a \neq \bar\theta a'$.

7 Proposition $\mathcal{I}$ is ω_1-saturated, that is, $\mathfrak{A}$ is ccc; consequently $\mathfrak{A}$ is Dedekind complete and $\bar\theta : \mathfrak{A} \to \mathfrak{B}$ preserves all suprema.

proof ? If $\langle a_\alpha \rangle_{\alpha<\omega_1}$ is a disjoint family in $\mathfrak{A} \setminus \{0\}$, set $b_\alpha = \sup_{\delta\geq\alpha} \bar\theta a_\delta$ for $\alpha < \omega_1$. Because $\mathfrak{B}$ is ccc, there is an α such that $b_\alpha = b_\beta$ for $\beta \geq \alpha$. Next, there are $\gamma > \beta > \alpha$ such that $b_\alpha = \sup_{\alpha\leq\delta<\beta} \bar\theta a_\delta$ and $b_\beta = \sup_{\beta\leq\delta<\gamma} \bar\theta a_\delta$. But now $\bar\theta a = \bar\theta a' \neq 0$ where $a = \sup_{\alpha\leq\delta<\beta} a_\delta$, $a' = \bigcup_{\beta\leq\delta<\gamma} a_\delta$ are disjoint. **X**

We know already that $\mathfrak{A}$ is Dedekind σ-complete; being ccc, it must also be Dedekind complete. If $A \subseteq \mathfrak{A}$ is non-empty, there is a non-empty countable $B \subseteq A$ with the same upper bounds; now

$$\bar\theta(\sup A) = \bar\theta(\sup B) = \sup \bar\theta[B] = \sup \bar\theta[A].$$

8 Second reduction of problem Set $\kappa' = \operatorname{add} \mathcal{I}$. By SOLOVAY 71 or FREMLIN 93, [2], there is a function $f : \kappa \to \kappa'$ such that $\mathcal{I}' = \{A : A \subseteq \kappa', f^{-1}[A] \in \mathcal{I}\}$ is an ω_1-saturated

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normal ideal. For $\eta<\kappa'$, set $E'_\eta=\bigcup\{E_\xi:\xi<\kappa,\,f(\xi)=\eta\}$; then $\langle E'_\eta\rangle_{\eta<\kappa'}$ is a partition of $[0,1]$, and $\mu E'_\eta=0$ for every η because $\mathcal{I}'$ contains all singletons. If we set $\theta'A=H^\bullet$ whenever H is a measurable envelope of $\bigcup_{\eta\in A}E'_\eta$, then $\theta'A=\theta(f^{-1}[A])$ for every $A\subseteq\kappa'$; it follows that $\langle E'_\eta\rangle_{\eta<\kappa'}$ does not split any non-zero element of $\mathfrak{B}$.

I will therefore suppose from now on that $\mathcal{I}$ itself is a normal ideal on κ, so that κ is a quasi-measurable cardinal in the sense of FREMLIN 93.

9 Lemma Let $A\subseteq\kappa$ and $d\in\mathfrak{B}$ be such that $d\cap\theta A\neq 0$. Set $\nu'B=\bar\mu(d\cap\theta B)$ for $B\subseteq A$; then ν' is not exhaustive.

proof (a) If C, $C'\subseteq A$, then

$$\begin{aligned}\nu'(C\cup C')+\nu(C\cap C')&=\bar\mu(d\cap\theta(C\cup C'))+\bar\mu(d\cap\theta(C\cap C'))\\&\le\bar\mu((d\cap\theta C)\cup(d\cap\theta C'))+\bar\mu((d\cap\theta C)\cap(d\cap\theta C'))\\&=\bar\mu(d\cap\theta C)+\bar\mu(d\cap\theta C');\end{aligned}$$

that is, ν' is submodular.

(b) ? If ν' is exhaustive, then (because it is sequentially order-continuous on the left) it is a submodular Maharam submeasure. Set $\mathcal{J}=\{C:C\subseteq A,\,\nu'C=0\}$; then the quotient algebra $\mathfrak{C}=\mathcal{P}A/\mathcal{J}$ has a strictly positive submodular Maharam submeasure, and must be a measurable algebra. Let λ be a probability measure with domain $\mathcal{P}A$ and null ideal $\mathcal{J}$. Since λ is zero on singletons and $\#(A)\le\mathfrak{c}$, λ is atomless. By the Gitik-Shelah theorem (GITIK & SHELAH 89, or FREMLIN 93, 3F), there is a stochastically independent family $\langle C_\xi\rangle_{\xi<\omega_1}$ in $\mathcal{P}A$ with $\lambda C_\xi=\frac{1}{2}$ for every ξ. Exchanging C_ξ with $B\setminus C_\xi$ if necessary, we may suppose that $\nu'C_\xi<\nu'B$ for every ξ, by the remark in §6; so $d\cap\theta C_\xi\neq d\cap\theta A$. Because $\mathfrak{B}$ has countable Maharam type, there is an infinite $I\subseteq\omega_1$ such that $\sup_{\xi\in I}d\cap\theta C_\xi\neq d\cap\theta A$. But now $C=\bigcup_{\xi\in I}C_\xi$ is λ-conegligible, so $A\setminus C\in\mathcal{J}$ and $d\cap\theta C=d\cap\theta A\neq d\cap\sup_{\xi\in I}\theta C_\xi$. **X**

Remark We note that there is an essentially more elementary approach to part (b) of the proof, not relying on the Gitik-Shelah theorem, as follows. Set $X=H\cap\bigcup_{\xi\in B}E_\xi$ where $H\subseteq[0,1]$ is such that $H^\bullet=d$; then $\mu^*X=\bar\mu(d\cap\theta B)>0$. For $\zeta<\kappa$, set $F_\zeta=H\cap\bigcup_{\xi\in B\cap\zeta}E_\xi$; by the Second Reduction, $\zeta\in\mathcal{I}$ and $\mu F_\zeta=0$. Let $G_\zeta\supseteq F_\zeta$ be an open set with $\mu G_\zeta\le\frac{1}{2}\mu^*X$. Consider the set $R=\{(\zeta,x):\zeta\in B,\,x\in G_\zeta\}$. If $\langle U_n\rangle_{n\in\mathbb{N}}$ runs over a base for the topology of $[0,1]$, then $R=\bigcup_{n\in\mathbb{N}}\{\zeta:\zeta\in B,\,U_n\subseteq G_\zeta\}\times U_n$ is measured by the product measure $\lambda\times\mu$ on $B\times[0,1]$. But

$$\int\mu R[\{\zeta\}]\lambda(d\zeta)=\int\mu G_\zeta\lambda(d\zeta)\le\frac{1}{2}\mu^*X,$$

$$\int\lambda R^{-1}[\{x\}]\mu(dx)\ge\mu^*X$$

because if $x\in X$ then $R^{-1}[\{x\}]$ includes $B\setminus\zeta$ for some $\zeta<\kappa$. So the repeated integrals cannot be equal and Fubini's theorem is contradicted.

In a significant respect, however, this argument is more limited than the other, because it cannot be applied to more complex measure spaces than $[0,1]$; the previous method can be applied to any measure with 'reasonable' Maharam type in place of μ.

10 Proposition For any $A \subseteq \kappa$, $\epsilon > 0$ there is a $B \subseteq A$ such that $\min(\nu B, \nu(A \setminus B)) \geq \nu A - \epsilon$.

proof Construct inductively a family $\langle (B_\alpha, B'_\alpha)\rangle_{\alpha<\zeta}$ of pairs of subsets of A, as follows. Given $\langle (B_\beta, B'_\beta)\rangle_{\beta<\alpha}$, set $C_\alpha = \bigcup_{\beta<\alpha} B_\beta \cup B'_\beta$. If $\nu C_\alpha = \nu A$, set $\zeta = \alpha$ and stop. Otherwise, set $d_\alpha = \theta A \setminus \theta C_\alpha$ and $C'_\alpha = A \setminus C_\alpha$. Note that $d_\alpha \subseteq \theta C'_\alpha$. Set $\nu'_\alpha C = \bar\mu(d_\alpha \cap \theta C)$ for $C \subseteq C'_\alpha$. By Lemma 9, ν'_α is not exhaustive, so there are a $\delta > 0$ and a disjoint family $\langle D_i\rangle_{i\in\mathbb{N}}$ in $\mathcal{P}C'_\alpha$ with $\nu'_\alpha D_i \geq \delta$ for all i. Now there must be disjoint I, $J \subseteq \mathbb{N}$ such that

$$\min(\bar\mu(d_\alpha \cap \sup_{i\in I} \theta D_i), \bar\mu(d_\alpha \cap \sup_{i\in J} \theta D_i)) \geq (1-\epsilon)\bar\mu(d_\alpha \cap \sup_{i\in I\cup J} \theta D_i).$$

Set $B_\alpha = \bigcup_{i\in I} D_i$, $B'_\alpha = \bigcup_{i\in J} D_i$ and continue. Note that $\bar\mu(d_\alpha \cap \theta(B_\alpha \cup B'_\alpha)) \geq \delta > 0$, so $\nu(C_\alpha \cup B_\alpha \cup B'_\alpha) > \nu C_\alpha$.

Because $\nu C_{\alpha+1} > \nu C_\alpha$ for every α, the induction must stop at some countable ordinal, with $\nu C_\zeta = 1$. For each $\alpha < \zeta$, set $e_\alpha = \theta C_{\alpha+1} \setminus \theta C_\alpha$; bec ause $\theta C_\alpha = \sup_{\beta<\alpha} \theta C_\beta$ for limit ordinals $\alpha \leq \zeta$, $\langle e_\alpha\rangle_{\alpha<\zeta}$ is a partition of θA in $\mathfrak{B}$ and $\sum_{\alpha<\zeta} \bar\mu e_\alpha = \nu A$.

Set $B = \bigcup_{\alpha<\zeta} B_\alpha$, $B' = \bigcup_{\alpha<\zeta} B'_\alpha$. Because B_α and B'_α are disjoint subsets of C'_α for each α, B and B' are disjoint. Next, for each $\alpha < \zeta$,

$$\bar\mu(e_\alpha \cap \theta B) \geq \bar\mu(e_\alpha \cap \theta B_\alpha) = \bar\mu(d_\alpha \cap \theta B_\alpha)$$

(because $e_\alpha = d_\alpha \cap \theta(B_\alpha \cup B'_\alpha)$)

$$\geq (1-\epsilon)\bar\mu(d_\alpha \cap \theta(B_\alpha \cup B'_\alpha)) = (1-\epsilon)\bar\mu e_\alpha.$$

So $\nu B = \sum_{\alpha<\zeta} \bar\mu(e_\alpha \cap \theta B) \geq \nu A - \epsilon$. Similarly, $\nu B' \geq 1 - \epsilon$, so $\nu(A \setminus B) \geq \nu A - \epsilon$.

11 Proposition $\mathfrak{A}$ is not weakly σ-distributive.

proof (a) There are an $a \in \mathfrak{A}$ and an $\eta > 0$ such that $\bar\nu a = 1$ and $\bar\nu(a \setminus b) \leq 1 - \eta$ whenever $b \subseteq a$ and $\bar\nu b = 1$. **P?** Otherwise, we can choose $\langle c_n\rangle_{n\in\mathbb{N}}$ inductively so that $c_0 = 1$, $\nu c_n = 1$ and $\nu(c_n \setminus c_{n+1}) \geq 1 - 2^{-n}$ for each n. Now $a' = \sup_{n\in\mathbb{N}} c_{2n}$, $a'' = \sup_{n\in\mathbb{N}} c_{2n+1}$ are disjoint and $\bar\nu a' = \bar\nu a'' = 1$, that is, $\bar\theta a' = \bar\theta a'' = 1$ in $\mathfrak{B}$, and we have a splitting. **XQ**

(b) Let $\mathfrak{A}_a$ be the principal ideal of $\mathfrak{A}$ generated by a. Using Proposition 10 repeatedly, we can find for each $n \in \mathbb{N}$ a disjoint sequence $\langle a_{nk}\rangle_{k\in\mathbb{N}}$ in $\mathfrak{A}_a$ such that $\bar\nu a_{nk} \geq 1 - 2^{-n}$ for all k and n. Expanding a_{n0} if necessary, we may suppose that each $\langle a_{nk}\rangle_{k\in\mathbb{N}}$ is a maximal antichain in $\mathfrak{A}_a$. **?** If $\mathfrak{A}$ is weakly σ-distributive, so is $\mathfrak{A}_a$, and there is a maximal antichain $\langle a_k\rangle_{k\in\mathbb{N}}$ in $\mathfrak{A}_a$ such that $\{i : a_k \cap a_{ni} \neq 0\}$ is finite for all n, $k \in \mathbb{N}$. In this case, setting $e_m = \sup_{k\leq m} a_k$ for each m, there must be some m such that $\bar\nu e_m > 1 - \eta$. So $\bar\nu(a \setminus e_m) < 1$; let n be such that $\bar\nu(a \setminus e_m) < 1 - 2^{-n}$. But $a_{nk} \subseteq a \setminus e_m$ for all large k, so this is impossible. **X**

So $\mathfrak{A}$ is not weakly σ-distributive, as claimed.

12 The argument of §9 gives the following property of the algebra $\mathfrak{A}$.

Proposition Let $\langle a_\alpha\rangle_{\alpha<\omega_1}$ be any family in $\mathfrak{A}$. Then either there is an infinite $I \subseteq \omega_1$ such that $\sup_{\alpha\in I} a_\alpha \neq 1$, or there is an infinite $I \subseteq \omega_1$ such that $\inf_{\alpha\in I} a_\alpha \neq 0$.

proof Suppose first that $A \subseteq \omega_1$ is an uncountable set such that $\bar{\theta} a_\alpha \neq 1$ for every $\alpha \in A$. Then there is an infinite $I \subseteq A$ such that $\sup_{\alpha \in I} \bar{\theta} a_\alpha \neq 1$ in $\mathfrak{B}$ and therefore $\sup_{\alpha \in I} a_\alpha \neq 1$ in $\mathfrak{A}$.

If this is not the case, then there must be a cocountable set $A \subseteq \omega_1$ such that $\bar{\theta}(1 \setminus a_\alpha) \neq 1$ for every $\alpha \in A$. Now there is an infinite $I \subseteq A$ such that $\sup_{\alpha \in I} 1 \setminus a_\alpha \neq 1$, that is, $\inf_{\alpha \in I} a_\alpha \neq 0$.

Remark In Balcar Jech & Pazák p03 it is pointed out that (if there can be measurable cardinals at all) there can be an ω_1-saturated σ-ideal $\mathcal{I}$ of a power set $\mathcal{P}\kappa$ such that $\mathcal{P}\kappa/\mathcal{I}$ has the stronger property that it is sequentially compact for the order-sequential topology, that is, every sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ has a subsequence $\langle b_n \rangle_{n \in \mathbb{N}}$ such that $\inf_{n \in \mathbb{N}} \sup_{i \geq n} b_i = \sup_{n \in \mathbb{N}} \inf_{i \geq n} b_i$.

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Zeberski S.[3]

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