Independence-precalibers of measure algebras

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Notation We follow Fremlin 06?. In particular, ν_κ is the usual measure on $\{0,1\}^\kappa$, $\mathcal{N}_\kappa$ its null ideal and $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ its measure algebra.

1 Definition (a) Let $\mathfrak{A}$ be a Boolean algebra and κ, λ cardinals. We say that (κ, λ) is an **independence-precaliber pair** of $\mathfrak{A}$ if whenever $\langle a_\xi \rangle_{\xi<\kappa}$ is a family of distinct elements of $\mathfrak{A}$ then there is a $\Gamma \in [\kappa]^\lambda$ such that $\{a_\xi : \xi \in \Gamma\}$ is Boolean-independent. If (κ, κ) is an independence-precaliber pair of $(\mathfrak{A}, \bar{\mu})$ we will say that κ is an **independence-precaliber** of $(\mathfrak{A}, \bar{\mu})$.

(b) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and κ, λ cardinals. We say that (κ, λ) is a **measure-independence-precaliber pair** of $(\mathfrak{A}, \bar{\mu})$ if whenever $\langle a_\xi \rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}$ such that $\inf_{\eta<\xi<\kappa} \bar{\mu}(a_\xi \triangle a_\eta) > 0$ then there is a $\Gamma \in [\kappa]^\lambda$ such that $\{a_\xi : \xi \in \Gamma\}$ is Boolean-independent. If (κ, κ) is a measure-independence-precaliber pair of $(\mathfrak{A}, \bar{\mu})$ we will say that κ is a **measure-independence-precaliber** of $(\mathfrak{A}, \bar{\mu})$.

Remark Of course any independence-precaliber (pair) of a measure algebra is a measure-independence-precaliber (pair). If (κ, λ) is a measure-independence-precaliber pair of a totally finite measure algebra $(\mathfrak{A}, \bar{\mu})$, and $\theta^\omega < \kappa$ for every $\theta < \kappa$, then (κ, λ) is an independence-precaliber pair of $\mathfrak{A}$. **P** If $\langle a_\xi \rangle_{\xi<\kappa}$ is any family of distinct elements of $\mathfrak{A}$, then $\overline{\{a_\xi : \xi < \kappa\}}$ must have density κ for the measure metric, and therefore must have a metrically isolated subset of size κ, which will have a Boolean-independent subset of size λ. **Q** (See Proposition 11 below.)

2 Proposition ω is a measure-independence-precaliber of every probability algebra.

proof Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\langle a_n \rangle_{n\in\mathbb{N}}$ a sequence in $\mathfrak{A}$ such that $\bar{\mu}(a_m \triangle a_n) \geq \delta > 0$ whenever $m \neq n$. Let $u \in L^2_{\bar{\mu}}$ be a cluster point of $\langle \chi a_n \rangle_{n\in\mathbb{N}}$ for $\mathfrak{T}_s(L^2_{\bar{\mu}}, L^2_{\bar{\mu}})$, and set $c = [\![0 < u < 1]\!]$. Then $c \neq 0$. **P?** Otherwise, $u = \chi a$ for some a. Now there must be infinitely many n such that

$$\bar{\mu}(a \triangle a_n) \int (u - \chi a_n) \times (\chi a - \chi(1 \setminus a)) < \frac{\delta}{2},$$

which is impossible. **XQ**

We can therefore choose inductively a strictly increasing sequence $\langle n_k \rangle_{k\in\mathbb{N}}$ such that $0 < \bar{\mu}(a_{n_k} \cap b) < \bar{\mu}b$ whenever $0 \neq b \subseteq c$ and b belongs to the algebra generated by $\{c\} \cup \{a_{n_i} : i < k\}$; in which case $\{a_{n_k} : k \in \mathbb{N}\}$ will be Boolean-independent.

3 Lemma Let κ be an uncountable cardinal and $(\mathfrak{A}, \bar{\mu})$ a probability algebra. Let $\mathfrak{D}$ be a closed subalgebra of $\mathfrak{A}$ such that $\mathfrak{A}$ is relatively atomless over $\mathfrak{D}$, and $a \in \mathfrak{A} \setminus \mathfrak{D}$; set $\delta = \rho(a, \mathfrak{D})$. Then there are a sequence $\langle e_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ and a $c \in \mathfrak{A}$ such that

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$\bar{\mu}e_n = \frac{1}{2}$ for every n

the e_n are stochastically independent of each other and $\mathfrak{D}$

a, c belong to the closed subalgebra $\mathfrak{B}$ generated by $\mathfrak{D} \cup \{e_n : n \in \mathbb{N}\}$

taking $\psi : \mathfrak{B} \to \mathfrak{B}$ to be the measure-preserving automorphism such that $\psi d = d$ for every $d \in \mathfrak{D}$, $\psi e_0 = 1 \setminus \psi e_0$, $\psi e_n = e_n$ for $n > 0$, then $\psi c = c$

$c \cap e_0 \subseteq 1 \setminus a$, $c \setminus e_0 \subseteq a$

$\bar{\mu}c \geq \delta$.

proof Define $u \in L^\infty(\mathfrak{D})$ by saying that $\int_d u = \bar{\mu}(a \cap d)$ for every $d \in \mathfrak{D}$. Set $d_0 = [\![u > \frac{1}{2}]\!]$. Let $e_0' \subseteq a$ be such that $\bar{\mu}(e_0' \cap d) = \frac{1}{2}\bar{\mu}(d \cap d_0)$ for every $d \in \mathfrak{D}$ (331B); let $e_0'' \subseteq 1 \setminus a$ be such that $\bar{\mu}(e_0' \cap d) = \frac{1}{2}\bar{\mu}(d \setminus d_0)$ for every $d \in \mathfrak{D}$; set $e_0 = (d_0 \setminus e_0') \cup e_0''$, so that $\bar{\mu}(e_0 \cap d) = \frac{1}{2}\bar{\mu}d$ for every $d \in \mathfrak{D}$. Now choose e_n, for $n \geq 1$, such that $\bar{\mu}(e_n \cap d) = \frac{1}{2}\bar{\mu}d$ for every d in the algebra generated by $\mathfrak{D} \cup \{e_i : i < n\}$, and a belongs to the closed subalgebra $\mathfrak{B}$ generated by $\mathfrak{D} \cup \{e_n : n \in \mathbb{N}\}$. [To see that this is possible, start from any sequence $\langle b_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}$ of elements of measure $\frac{1}{2}$ stochastically independent of each other and of $\mathfrak{D}$, and let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{D} \cup \{a, e_0\} \cup \{b_n : n \in \mathbb{N}\}$; use Fremlin 02, 333C to see that $(\mathfrak{B}, \bar{\mu}\restriction\mathfrak{B})$ can be identified with the probability algebra free product of $(\mathfrak{D}_1, \bar{\mu}\restriction\mathfrak{D}_1)$ and $(\mathfrak{B}_\omega, \bar{\nu}_\omega)$, where $\mathfrak{D}_1$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{D} \cup \{e_0\}$.] Because $(\mathfrak{B}, \bar{\mu}\restriction\mathfrak{B})$ is equally isomorphic to the probability algebra free product of $(\mathfrak{D}, \bar{\mu}\restriction\mathfrak{D})$ and $(\mathfrak{B}_\omega, \bar{\nu}_\omega)$, we have a unique measure-preserving automorphism $\psi : \mathfrak{B} \to \mathfrak{B}$ such that $\psi d = d$ for every $d \in \mathfrak{D}$, $\psi e_0 = 1 \setminus \psi e_0$ and $\psi e_n = e_n$ for $n > 0$.

Set $c' = d_0 \setminus a$, $c'' = a \setminus d_0$. Then at least one of c', c'' has measure at least $\frac{1}{2}\delta$; call this c^*; set $c = c^* \cup \psi c^*$, so that $c \in \mathfrak{B}$ and $\psi c = c$.

If $c^* = c'$ then $c^* \subseteq d_0 \cap e_0$, $\psi c^* \subseteq d_0 \setminus e_0 \subseteq a$, $c \cap e_0 = c'$ is disjoint from a, $c \setminus e_0 = \psi c' \subseteq a$ and $\bar{\mu}c = 2\bar{\mu}c' \geq \delta$. If $c^* = c''$ then $c^* \cap e_0 = 0$,

$$c \cap e_0 = \psi c^* \subseteq e_0 \setminus d_0 = e_0'' \subseteq 1 \setminus a,$$

$c \setminus e_0 = c'' \subseteq a$ and $\bar{\mu}c = 2\bar{\mu}c'' \geq \delta$.. So we have what we need.

4 Corollary If κ is an infinite cardinal, $(\mathfrak{A}, \bar{\mu})$ a Maharam homogeneous probability algebra, and $\langle a_\xi \rangle_{\xi < \kappa}$ a family in $\mathfrak{A}$ such that $\inf_{\eta < \xi < \kappa} \bar{\mu}(a_\eta \triangle a_\xi) = \delta > 0$, then there are a family $\langle e_{\xi n} \rangle_{\xi < \kappa, n \in \mathbb{N}}$, a function $\alpha : \kappa \to \kappa$ and a family $\langle c_\xi \rangle_{\xi < \kappa}$ in $\mathfrak{A}$ such that

$\bar{\mu}e_{\xi n} = \frac{1}{2}$ for all ξ, n,

$\langle e_{\xi n} \rangle_{\xi < \kappa, n \in \mathbb{N}}$ is stochastically independent,

$a_{\alpha(\xi)}$, c_ξ belong to the closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by $\{e_{\eta n} : \eta < \kappa, n \in \mathbb{N}\}$ for every ξ,

if $\phi_\xi : \mathfrak{B} \to \mathfrak{B}$ is the measure-preserving automorphism defined by setting $\phi_\xi(e_{\xi 0}) = 1 \setminus e_{\xi 0}$ and $\phi_\xi(e_{\eta n}) = e_{\eta n}$ for all $(\eta, n) \neq (\xi, 0)$, then $\phi_\xi c_\xi = c_\xi$,

$\bar{\mu}c_\xi \geq \frac{1}{2}\delta$ for every ξ,

$c_\xi \cap e_{\xi 0} \subseteq 1 \setminus a_{\alpha(\xi)}$, $c \setminus e_{\xi 0} \subseteq a_{\alpha(\xi)}$ for every ξ.

proof If $\mathfrak{D} \subseteq \mathfrak{A}$ is a closed subalgebra with Maharam type less than κ, then there must be a $\xi < \kappa$ such that $\rho(a_\xi, \mathfrak{D}) \geq \frac{1}{2}\delta$. We can therefore choose $\mathfrak{D}_\xi$, $e_{\xi n}$, c_ξ, ψ_ξ and $\alpha(\xi)$ inductively, as follows. $\mathfrak{D}_0 = \{0, 1\}$. Given that $\mathfrak{D}_\xi$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{e_{\eta n} : \eta < \xi, n \in \mathbb{N}\}$, let $\alpha(\xi) < \kappa$ be such that $\rho(a_{\alpha(\xi)}, \mathfrak{D}_\xi) \geq \frac{1}{2}\delta$; using Lemma 3, choose $e_{\xi n}$, of measure $\frac{1}{2}$, stochastically independent of each other and of $\mathfrak{D}_\xi$, and c_ξ, such that

$a_{\alpha(\xi)}$ and c_ξ belong to the closed subalgebra $\mathfrak{D}_{\xi+1}$ generated by $\mathfrak{D}_\xi \cup \{e_{\xi n} : n \in \mathbb{N}\}$,

$\bar{\mu} c_\xi \ge \frac{1}{2}\delta$,

$c_\xi \cap e_{\xi 0} \subseteq 1 \setminus a_{\alpha(\xi)}$, $c_\xi \setminus e_{\xi 0} \subseteq a_{\alpha(\xi)}$,

taking $\psi_\xi : \mathfrak{D}_{\xi+1} \to \mathfrak{D}_{\xi+1}$ to be the measure-preserving automorphism such that $\psi d = d$ for every $d \in \mathfrak{D}_\xi$, $\psi_\xi e_{\xi 0} = 1 \setminus \psi_\xi e_{\xi 0}$, $\psi_\xi e_{\xi n} = e_{\xi n}$ for $n > 0$, then $\psi_\xi c = c_\xi$.

Let $\mathfrak{B} = \mathfrak{D}_\kappa$ be the closed subalgebra of $\mathfrak{A}$ generated by the stochastically independent family $\langle e_{\xi n}\rangle_{\xi<\kappa, n\in\mathbb{N}}$. This works.

5 Proposition If λ, κ are infinite cardinals, $\kappa \ge \omega_2$ and (κ, λ) is a measure-precaliber pair of a totally finite measure algebra $(\mathfrak{A}, \bar{\mu})$ then it is a measure-independence-precaliber pair of $(\mathfrak{A}, \bar{\mu})$.

proof (a) To begin with, suppose that $(\mathfrak{A}, \bar{\mu})$ is a Maharam homogeneous probability algebra, that (κ, λ) is a measure-precaliber pair of $(\mathfrak{A}, \bar{\mu})$ and that $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}$ such that $\bar{\mu}(a_\eta \triangle a_\xi) \ge \delta > 0$ whenever $\eta < \xi$. Take $\langle e_{\xi n}\rangle_{\xi<\kappa, n\in\mathbb{N}}$, $\mathfrak{B}$, $\alpha : \kappa \to \kappa$ and $\langle c_\xi\rangle_{\xi<\kappa}$ as in Corollary 4. For each $\xi < \kappa$ let J^*_ξ be the minimal subset of $\kappa \times \mathbb{N}$ such that c_ξ belongs to the closed subalgebra generated by $\{e_j : j \in J\}$, and set $L_\xi = \{\eta : (\eta, n) \in J\}$. As in Fremlin 06?, 534Je, $(\xi, 0) \notin J^*_\xi$. By Hajnal's Free Set Theorem there is a $\Gamma_0 \in [\kappa]^\kappa$ such that $\xi \notin L_\eta$ for all distinct ξ, $\eta \in \Gamma_0$. If I, $J \subseteq C$ are disjoint finite sets and $c = \inf_{\xi\in I\cup J} c_\xi$, then c is stochastically independent of the $e_{\xi 0}$, for $\xi \in I \cup J$, and

$$\begin{aligned}\bar{\mu}(\inf_{\eta\in J} a_{\alpha(\eta)} \setminus \sup_{\xi\in I} a_{\alpha(\xi)}) &\ge \bar{\mu}(\inf_{\xi\in I}(e_{\xi 0} \setminus a_{\alpha(\xi)}) \cap \inf_{\eta\in J} a_{\alpha(\eta)} \setminus e_{\eta 0}))\\ &\ge \bar{\mu}(\inf_{\xi\in I}(c_\xi \cap e_{\xi 0}) \cap \inf_{\eta\in J} c_\eta \setminus e_{\eta 0}))\\ &= \bar{\mu}(c \cap \inf_{\xi\in I} e_{\xi 0} \setminus \sup_{\eta\in J} e_{\eta 0}) = 2^{-\#(I\cup J)}\bar{\mu}c.\end{aligned}$$

Now, because (κ, λ) is a measure-precaliber pair of $(\mathfrak{A}, \bar{\mu})$, there is a $\Gamma \in [\Gamma_0]^\lambda$ such that $\{c_\xi : \xi \in \Gamma\}$ is centered, in which case $\langle a_{\alpha(\xi)}\rangle_{\xi\in\Gamma}$ is Boolean-independent.

(b) For the general case, we have only to note that, for a totally finite measure algebra $(\mathfrak{A}, \bar{\mu})$, (κ, λ) is an (independence-)measure-precaliber pair of $(\mathfrak{A}, \bar{\mu})$ iff it is an (independence-)measure-precaliber pair of every Maharam homogeneous principal ideal of $(\mathfrak{A}, \bar{\mu})$ (cf. Fremlin 06?, 524Ha).

6 Proposition If $\omega \le \lambda \le \kappa$ and (κ, λ) is a measure-independence-precaliber pair of every probability algebra then it is a measure-precaliber pair of every probability algebra.

proof Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\langle a_\xi\rangle_{\xi<\lambda}$ a family in $\mathfrak{A}$ such that $\inf_{\xi<\kappa} \bar{\mu} a_\xi = \delta > 0$. Let $(\mathfrak{C}, \bar{\nu})$ be the probability algebra free product of $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$, and $\langle e_\xi\rangle_{\xi<\kappa}$ a stochastically independent family in $\mathfrak{B}_\kappa$ of elements of measure $\frac{1}{2}$. Set $c_\xi = a_\xi \otimes e_\xi$ for each ξ; then $\bar{\nu}(c_\eta \triangle c_\xi) = \bar{\mu}(a_\eta \cup a_\xi) \ge \delta$ whenever $\xi < \eta$. There is therefore a $\Gamma \in [\kappa]^\lambda$ such that $\langle c_\xi\rangle_{\xi\in\Gamma}$ is Boolean-independent, in which case $\{a_\xi : \xi \in \Gamma\}$ must be centered.

7 Lemma Let $n \geq 1$ be an integer, κ a regular uncountable cardinal, $X \subseteq (\{0,1\}^n)^\kappa$ a closed set, and $f_\xi : \{0,1\}^n \to \{0,1\}$ a function. Set $f(x) = \langle f_\xi(x(\xi))\rangle_{\xi<\omega_1}$ for $x \in X$. If $f[X] = \{0,1\}^\kappa$, there are $i < n$ and $C \in [\kappa]^\kappa$ such that $\pi_{iC}[X] = \{0,1\}^C$, where $\pi_{iC}(x)(\xi) = x(\xi)(i)$ for $x \in X$, $\xi \in C$, $i < n$.

proof Induce on n. If $n = 1$ then every f_ξ has to be surjective, therefore bijective, and f is a bijection, so we can take $C = \kappa$, $i = 0$. For the inductive step to $n+1$, identify $(\{0,1\}^{n+1})^\kappa$ with $(\{0,1\}^n)^\kappa \times \{0,1\}^\kappa$, so that we have $X \subseteq (\{0,1\}^n)^\kappa \times \{0,1\}^\kappa$ and $f_\xi : \{0,1\}^n \times \{0,1\} \to \{0,1\}$ such that $f[X] = \{0,1\}^\kappa$, where $f(x,y)(\xi) = f_\xi(x(\xi), y(\xi))$ for $\xi < \kappa$, $x \in (\{0,1\}^n)^\kappa$, $y \in \{0,1\}^\kappa$. Let $X' \subseteq X$ be a minimal closed set such that $f[X'] = \{0,1\}^\kappa$.

Set $Y = \{y : (x,y) \in X'\}$, so that Y is a closed subset of $\{0,1\}^\kappa$. Set

$$A = \{\xi : \xi < \kappa,\ \forall\, y \in Y\ \forall\, I \in [\xi]^{<\omega}\ \exists\, y',\, y'' \in Y,$$
$$y'{\restriction}I = y''{\restriction}I = y{\restriction}I,\ y'(\xi) = 0,\ y''(\xi) = 1\}.$$

If $I \in [A]^{<\omega}$ and $u \in \{0,1\}^I$ there is a $y \in Y$ such that $y{\restriction}I = u$ (induce on $\max I$), so $\{y{\restriction}A : y \in Y\} = \{0,1\}^A$ and if $\#(A) = \kappa$ we can take $C = A$, $i = n$ and stop. Otherwise, for $\xi \in \kappa \setminus A$ take $y_\xi \in Y$, $I_\xi \in [\xi]^{<\omega}$ and $v(\xi) \in \{0,1\}$ such that $y(\xi) = v(\xi)$ whenever $y \in Y$ and $y{\restriction}I_\xi = y_\xi{\restriction}I_\xi$. Let $J \in [\kappa]^{<\omega}$, $u \in \{0,1\}^J$ be such that $B = \{\xi : \xi \in \kappa \setminus A$, $I_\xi = J$, $y_\xi{\restriction}I_\xi = u\}$ has cardinal κ. For $\xi \in B$, $s \in \{0,1\}^n$ set $g_\xi(s) = f_\xi(s, v(\xi))$.

Now consider $\{(x,y) : (x,y) \in X',\ y{\restriction}J = u\}$. This is a non-empty open subset of X' and $f{\restriction}X'$ is irreducible, so there is an open cylinder set $V \subseteq \{0,1\}^\kappa$ such that $y{\restriction}J = u$ whenever $(x,y) \in X'$ and $f(x,y) \in V$. Let $K \in [\kappa]^{<\omega}$ be such that V is determined by coordinates in K. Set $D = B \setminus K$, $Z = \{x{\restriction}D : (x,y) \in X'\}$, $g(z) = \langle g_\xi(z(\xi))\rangle_{\xi\in D}$ for $z \in Z$. If $w \in \{0,1\}^D$, there is a $w' \in V$ such that $w'{\restriction}D = w$; there is an element $(x,y) \in X'$ such that $f(x,y) = w'$; now $y{\restriction}J = u$ so $y{\restriction}B = v{\restriction}B$ and

$$g_\xi(x(\xi)) = f(x(\xi), y(\xi)) = w'(\xi) = w(\xi)$$

for every $\xi \in D$, that is, $g(x{\restriction}D) = w$. Thus $g[Z] = \{0,1\}^D$, while $Z \subseteq (\{0,1\}^n)^\kappa$.

By the inductive hypothesis, there are $C \in [D]^\kappa$ and an $i < n$ such that $\pi_{iC}[Z] = \{0,1\}^C$; now, re-interpreting the formula π_{iC} appropriately, we have $\pi_{iC}[X] = \{0,1\}^C$ and the induction continues.

8 Proposition Let M be a set, κ a regular uncountable cardinal and X a closed subset of $\{0,1\}^M$. If there is a continuous surjection $h : X \to \{0,1\}^\kappa$, then there is a $C \in [M]^\kappa$ such that $x \mapsto x{\restriction}C : X \to \{0,1\}^C$ is surjective.

proof Shrinking X if necessary, we may suppose that h is irreducible. For each $\xi < \kappa$, $\{x : x \in X,\ h(x)(\xi) = 1\}$ is an open-and-closed set in X, so there is a finite set $I_\xi \subseteq \kappa$ such that $h(x)(\xi) = h(x')(\xi)$ whenever $x,\ x' \in X$ and $x{\restriction}I_\xi = x'{\restriction}I_\xi$. Let $A \in [\kappa]^\kappa$ be such that $\langle I_\xi\rangle_{\xi\in A}$ is a constant-size Δ-system with root I say. Let u be any member of $\{x{\restriction}I : x \in X\}$; then there is a cylinder set $V \subseteq \{0,1\}^\kappa$ such that $x{\restriction}I = u$ whenever $x \in X$ and $h(x) \in V$; shrinking A slightly if necessary, we can suppose that V is determined by coordinates in $\kappa \setminus A$. For $\xi \in A$, set $J_\xi = I_\xi \setminus I$ and define $f_\xi : \{0,1\}^{J_\xi} \to \{0,1\}$ by saying that $f_\xi(v) = h(x)(\xi)$ whenever $x \in X$, $x{\restriction}I = u$ and $x{\restriction}J_\xi = v$. Now observe that for any

$w \in \{0,1\}^A$ we have a $w' \in V$ extending w, so that there is an $x \in X$ with $h(x) = w'$, $x{\restriction}I = u$ and $f_\xi(x{\restriction}J_\xi) = w(\xi)$ for every $\xi \in A$.

We can therefore apply Lemma 7 to $\{x{\restriction}\bigcup_{\xi\in A} J_\xi : x \in X\}$, identifying $\bigcup_{\xi\in A} J_\xi$ with $n \times A$ where n is the common value of $\#(J_\xi)$ for $\xi \in A$, to see that there is an uncountable $C \subseteq \bigcup_{\xi\in A} J_\xi$ such that $\{x{\restriction}C : x \in X\} = \{0,1\}^\kappa$.

9 Corollary If ω_1 has Haydon's property then ω_1 is a measure-independence-precaliber of every probability algebra.

proof Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\langle a_\xi\rangle_{\xi<\omega_1}$ a family in $\mathfrak{A}$ such that $\inf_{\eta<\xi<\omega_1} \bar\mu(a_\eta, a_\xi) > 0$. Then we have a Radon probability measure ν on $\{0,1\}^{\omega_1}$ defined by saying that

$$\nu\{x : x{\restriction}I = u\} = \bar\mu(\inf_{\xi\in I,u(\xi)=1} a_\xi \setminus \sup_{\xi\in I,u(\xi)=0} a_\xi)$$

whenever $I \in [\omega_1]^{<\omega}$ and $u \in \{0,1\}^I$. Let Z be the support of ν. Since ν has Maharam type ω_1, there is a continuous surjection from Z onto $[0,1]^{\omega_1}$; so we can find a closed subset Z' of Z and a continuous surjection $h : Z' \to \{0,1\}^{\omega_1}$. By Proposition 8, there is a $C \in [\omega_1]^{\omega_1}$ such that $\{z{\restriction}C : z \in Z'\} = \{0,1\}^C$. But this means that $\langle a_\xi\rangle_{\xi\in C}$ is Boolean-independent.

10 Proposition Suppose that there is a family $\langle W_\xi\rangle_{\xi<\omega_1}$ in $\mathcal{N}_{\omega_1}$ such that every closed subset of $\{0,1\}^{\omega_1} \setminus \bigcup_{\xi<\omega_1} W_\xi$ is scattered. Then ω_1 is not a measure-independence-precaliber of every probability algebra.

proof As in Fremlin 06?, 534N (following Plebanek 97), there is a zero-dimensional compact Hausdorff space X such that $\omega_1 \in \mathrm{Mah}_{\mathrm{R}}(X)$ but there is no continuous surjection from X onto $\{0,1\}^{\omega_1}$. Let μ be a Maharam homogeneous Radon probability measure on X with Maharam type ω_1 and $(\mathfrak{A}, \bar\mu)$ its measure algebra. Because X is zero-dimensional, there is a family $\langle K_\xi\rangle_{\xi<\omega_1}$ of open-and-closed sets in X such that $\mu(K_\xi \triangle K_\eta) \ge \frac{1}{3}$ whenever $\eta < \xi$; now $\langle K_\xi\rangle_{\xi<\omega_1}$ has no Boolean-independent subfamily of size ω_1, so $\langle K_\xi^\bullet\rangle_{\xi<\omega_1}$ has no Boolean-independent subfamily of size ω_1.

11 Proposition Suppose that θ and κ are cardinals such that $\max(2^{\mathfrak{c}}, \theta^\omega) < \mathrm{cf}\,\kappa \le \kappa \le 2^\theta$. Then κ is an independence-precaliber of every totally finite measure algebra.

proof Džamonja & Plebanek p04, Theorem 6.3.

Remark Note that if we have only

$$\theta^\omega < \mathrm{cf}\,\kappa \le \kappa \le 2^\theta$$

then κ is a measure-precaliber of every measure algebra (Džamonja & Plebanek p04, 4.7; Fremlin p06?, 524V), therefore a measure-independence-precaliber of every probability algebra (Džamonja & Plebanek p04, 6.6). If $\lambda^\omega < \kappa$ for every $\lambda < \kappa$, then any family of size κ in any metric space has a discrete subfamily of size κ, so κ will be an independence-precaliber of every probability algebra. But the result here also covers the case $\kappa = 2^{2^{\mathfrak{c}}} = \omega_{\omega+1}$.

12 Problem From Fremlin 06?, 534L, and Propositions 2, 5 and 6 above, we see that for $\kappa \geq \omega_2$ the following are equiveridical:

κ is a measure-independence-precaliber of every probability algebra;

κ is a measure-precaliber of every probability algebra;

κ has Haydon's property.

By Proposition 10 and the remarks in the notes to §534 of Fremlin 06?, it is possible that

ω_1 is a measure-precaliber of every probability algebra,

ω_1 is not a measure-independence-precaliber of every probability algebra.

By Corollary 9, we see that if ω_1 has Haydon's property then it is a measure-independence-precaliber of every probability algebra. But we do not know whether the converse is true.

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