Radon measures

D.H.Fremlin

University of Essex, Colchester, England

In these lectures I discuss some relationships between topological properties of a Hausdorff space and the properties of the Radon measures it carries, concentrating on ideas involving cardinal functions. Essentially all this material may be found in my treatise 'Measure Theory'.

1. Measure spaces and algebras I begin with the basic measure theory we shall need.

1A Definitions (a) A **measure space** is a triple (X,Σ,μ) where X is a set, Σ is a σ-algebra of subsets of X, and $\mu:\Sigma\to[0,\infty]$ is countably additive (Fremlin 00, 112A). In this case, the **null ideal** of μ is

$$\mathcal{N}(\mu)=\{E:\exists\,F\in\Sigma,\,E\subseteq F,\,\mu F=0\};$$

(Fremlin 00, 112D); this is a σ-ideal of subsets of X. μ is **totally finite** if $\mu X<\infty$ (Fremlin 01, 211C), **complete** if $\mathcal{N}(\mu)\subseteq\Sigma$ (Fremlin 01, 211A) and **atomless** if for every non-negligible $E\in\Sigma$ there is a measurable $F\subseteq E$ such that neither F nor $E\setminus F$ is negligible (Fremlin 01, 211J).

(b) (See Fremlin 07?, §511.) If X is a set and $\mathcal{I}$ is an ideal of subsets of X, then the **additivity** of $\mathcal{I}$ is

$$\operatorname{add}\mathcal{I}=\min\{\#(\mathcal{A}):\mathcal{A}\subseteq\mathcal{I},\,\bigcup\mathcal{A}\notin\mathcal{I}\}$$

(if $\bigcup\mathcal{I}\in\mathcal{I}$, write $\operatorname{add}\mathcal{I}=\infty$); the **cofinality** of $\mathcal{I}$ is

$$\operatorname{cf}\mathcal{I}=\min\{\#(\mathcal{A}):\mathcal{A}\subseteq\mathcal{I},\,\forall\,I\in\mathcal{I}\ \exists\,A\in\mathcal{A},\,I\subseteq A\};$$

the **uniformity** of $\mathcal{I}$ is

$$\operatorname{non}\mathcal{I}=\min\{\#(A):A\subseteq X,\,A\notin\mathcal{I}\}$$

(if $X\in\mathcal{I}$, write $\operatorname{add}\mathcal{I}=\infty$); the **covering number** of $\mathcal{I}$ is

$$\operatorname{cov}\mathcal{I}=\min\{\#(\mathcal{A}):\mathcal{A}\subseteq\mathcal{I},\,\bigcup\mathcal{A}=X\}$$

(if $\bigcup\mathcal{I}\ne X$, write $\operatorname{cov}\mathcal{I}=\infty$); and the **shrinking number** of $\mathcal{I}$ is

$$\operatorname{shr}\mathcal{I}=\min\{\kappa:\forall\,A\in\mathcal{P}X\setminus\mathcal{I}\ \exists\,B\in[A]^{\le\kappa}\setminus\mathcal{I}\}$$

(if $X\in\mathcal{I}$ set $\operatorname{shr}\mathcal{I}=0$).

$\mathcal{I}$ is a σ-ideal iff $\operatorname{add}\mathcal{I}\ge\omega_1$. Provided that $\bigcup\mathcal{I}=X\notin\mathcal{I}$, we have

$$\operatorname{add}\mathcal{I}\le\operatorname{cov}\mathcal{I}\le\operatorname{cf}\mathcal{I},\quad \operatorname{add}\mathcal{I}\le\operatorname{non}\mathcal{I}\le\operatorname{shr}\mathcal{I}\le\operatorname{cf}\mathcal{I}$$

(Fremlin 07?, §511).

Typeset by $\mathcal{AMS}$-TEX

Remark If (X, Σ, μ) is a complete totally finite measure space and κ is a cardinal, then $\kappa < \operatorname{add}\mathcal{N}(\mu)$ iff $\mu(\bigcup_{\xi<\kappa} E_\xi)$ is defined and equal to $\sum_{\xi<\kappa} \mu E_\xi$ for every disjoint family $\langle E_\xi\rangle_{\xi<\kappa}$ in Σ (Fremlin 07?, §511).

(c) If (X, Σ, μ) and (Y, T, ν) are measure spaces, a function $f : X \to Y$ is **inverse-measure-preserving** if $\mu f^{-1}[F]$ is defined and equal to νF for every $F \in \mathrm{T}$ (Fremlin 01, 235G).

1B Measure algebras (a) A **measure algebra** is a pair $(\mathfrak{A}, \bar\mu)$ where $\mathfrak{A}$ is a Boolean algebra and $\bar\mu : \mathfrak{A} \to [0, \infty]$ is a functional and

$\mathfrak{A}$ is Dedekind σ-complete, that is, any sequence in $\mathfrak{A}$ with an upper (resp. lower) bound has a least upper (resp. greatest lower) bound;

$\bar\mu$ is countably additive, that is, $\bar\mu 0 = 0$ and $\bar\mu(\sup_{n\in\mathbb{N}} a_n) = \sum_{n=0}^{\infty} \bar\mu a_n$ for any disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$;

$\bar\mu$ is strictly positive, that is, $\bar\mu a > 0$ for every non-zero $a \in \mathfrak{A}$.

(Fremlin 02, 321A).

(b) If $\bar\mu$ is finite-valued, that is, $\bar\mu 1 < \infty$, we say that $(\mathfrak{A}, \bar\mu)$ is **totally finite**. In this case $\mathfrak{A}$ is ccc (that is, every disjoint family in $\mathfrak{A}$ is countable) and Dedekind complete (that is, every non-empty subset of $\mathfrak{A}$ with an upper bound has a least upper bound; Fremlin 02, 322G and 322B). If $\bar\mu 1 = 1$ I will say that $(\mathfrak{A}, \bar\mu)$ is a **probability algebra**.

(c) For any measure space (X, Σ, μ), $\Sigma \cap \mathcal{N}(\mu) = \mu^{-1}[\{0\}]$ is an ideal of the Boolean algebra Σ, so we can form the Boolean quotient algebra $\mathfrak{A} = \Sigma/\Sigma \cap \mathcal{N}(\mu)$; we have a functional $\bar\mu : \mathfrak{A} \to [0, \infty]$ defined by saying that $\bar\mu(E^\bullet) = \mu E$ for every $E \in \Sigma$. Now $(\mathfrak{A}, \bar\mu)$ is a measure algebra (Fremlin 02, 321H).

1C Cardinal functions of Boolean algebras Let $\mathfrak{A}$ be a Boolean algebra.

(a) The **Maharam type** $\tau(\mathfrak{A})$ of $\mathfrak{A}$ is the least cardinal of any subset A of $\mathfrak{A}$ such that $\mathfrak{A}$ is the order-closed subalgebra of itself generated by A (Fremlin 02, 331F).

(b) The **π-weight** $\pi(\mathfrak{A})$ is the least cardinal of any subset A of $\mathfrak{A}$ such that every non-zero member of $\mathfrak{A}$ includes a non-zero member of A.

(c) The **weak distributivity** $\operatorname{wdistr}(\mathfrak{A})$ is the least cardinal of any family $\mathcal{A}$ of maximal antichains in $\mathfrak{A}$ such that for every maximal antichain B in $\mathfrak{A}$ there are $A \in \mathcal{A}$, $b \in B$ such that $\{a : a \in A,\ a \cap b \neq 0\}$ is infinite; or ∞ if there is no such family $\mathcal{A}$. (Fremlin 07?, §511.)

1D Standard measure algebras For any set I, let ν_I be the usual (completed) measure on $\{0,1\}^I$ (Fremlin 01, 254E), T_I its domain and $(\mathfrak{B}_I, \bar\nu_I)$ its measure algebra. Then $\tau(\mathfrak{B}_I) = \#(I)$. If I is infinite, $\mathfrak{B}_I$ is homogeneous in the sense that it is isomorphic, as Boolean algebra, to all its non-zero principal ideals; in fact $(\mathfrak{B}_I, \bar\nu_I)$ is homogeneous in the sense that if $b \in \mathfrak{B}_I \setminus \{0\}$ and $\mathfrak{D}_b$ is the principal ideal generated by b, there is a Boolean isomorphism $\pi : \mathfrak{B} \to \mathfrak{D}_b$ such that $\bar\nu_I(\pi a) = \bar\nu_I b \cdot \bar\nu_I a$ for every $a \in \mathfrak{B}_I$ (Fremlin 02, 331K-331L).

1E Maharam's theorem If $(\mathfrak{A}, \bar{\mu})$ is a totally finite measure algebra, there is a partition $\langle a_i \rangle_{i\in I}$ of unity in $\mathfrak{A}$ such that each principal ideal $\mathfrak{A}_{a_i}$, with the restricted measure $\bar{\mu}\restriction\mathfrak{A}_{a_i}$, is isomorphic, up to a scalar multiple of the measure, to $(\mathfrak{B}_{\kappa_i}, \bar{\nu}_{\kappa_i})$ (MAHARAM 42, or FREMLIN 02, 332B.) Now $\tau(\mathfrak{A}) = \sup_{i\in I} \kappa_i$ if the right-hand-side is infinite (FREMLIN 07?, §513).

1F Associated facts (a) If $(\mathfrak{A}, \bar{\mu})$ is a probability algebra and $\kappa \geq \max(\omega, \tau(\mathfrak{A}))$ then $(\mathfrak{A}, \bar{\mu})$ embeds in $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ in the sense that it is isomorphic, as measure algebra, to $(\mathfrak{C}, \bar{\nu}_\kappa\restriction\mathfrak{C})$ for some subalgebra $\mathfrak{C}$ of $\mathfrak{B}_\kappa$ (FREMLIN 02, 332N).

(b) If $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ are probability algebras and each embeds into the other then they are isomorphic (FREMLIN 02, 332Q).

2. Radon measures

2A Definition Let $(X, \mathfrak{T})$ be a Hausdorff space. A totally finite measure μ on X, with domain Σ, is a **Radon measure** if

$\mathfrak{T} \subseteq \Sigma$ (so that all Borel sets are measurable);

μ is inner regular with respect to the compact sets, that is, $\mu E = \sup\{\mu K : K \subseteq E,\ K \text{ is compact}\}$ for every $E \in \Sigma$;

μ is complete.

(FREMLIN 03, 411H).

For any set I, ν_I is a Radon measure on $\{0,1\}^I$ (FREMLIN 03, 416U).

2B THE PROBLEM For a given Hausdorff space X, describe the possible types of Radon measure on X in terms of topological properties of X.

For a random example of what can be said, see 4J below.

2C Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a non-empty totally finite Radon measure space, and (Y, T, ν) a complete measure space. Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be the measure algebras of (X, Σ, μ) and (Y, T, ν) respectively. If $\pi : \mathfrak{A} \to \mathfrak{B}$ is a measure-preserving Boolean homomorphism, there is a function $g : Y \to X$ which represents π in the sense that $g^{-1}[E] \in \mathrm{T}$ and $(g^{-1}[E])^\bullet = \pi(E^\bullet)$ for every $E \in \Sigma$.

skeleton of proof Let W, Z be the Stone spaces of the algebras $\mathfrak{A}$, $\mathfrak{B}$ respectively; for $a \in \mathfrak{A}$, let $\widehat{a}$ be the open-and-closed subset of W corresponding to a. Then we have a continuous $\hat{g} : Z \to W$ such that $\hat{g}^{-1}[\widehat{a}] = \widehat{\pi a}$ for every $a \in \mathfrak{A}$ (FREMLIN 02, 312P). Let $\theta : \mathfrak{B} \to \mathrm{T}$ be a lifting (that is, a Boolean homomorphism such that $(\theta b)^\bullet = b$ for every $b \in \mathfrak{B}$; see FREMLIN 02, 341K). Then we have a function $h : Y \to Z$ defined by saying that $h^{-1}[\widehat{b}] = \theta b$ for every $b \in \mathfrak{B}$. Set $W_0 = \bigcup\{\widehat{K^\bullet} : K \subseteq X \text{ is compact}\}$, and define $f : W_0 \to X$ by saying that $f^{-1}[K] = \widehat{K^\bullet}$ for every compact $K \subseteq X$ (FREMLIN 03, 416V). Take $g : Y \to X$ extending $f\hat{g}h$.

(For details see FREMLIN 02, 343B and FREMLIN 03, 416W.)

2D Corollary If $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ are totally finite Radon measure spaces with isomorphic measure algebras $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$, then

$$\mathrm{non}\,\mathcal{N}(\mu) = \mathrm{non}\,\mathcal{N}(\nu), \quad \mathrm{cov}\,\mathcal{N}(\mu) = \mathrm{cov}\,\mathcal{N}(\nu).$$

proof The case in which $\mu X = \nu Y = 0$ is trivial. Otherwise, let $\pi : \mathfrak{A} \to \mathfrak{B}$ be a measure-preserving isomorphism represented by $g : Y \to X$.

If $B \subseteq Y$ is not negligible then $B \subseteq g^{-1}[g[B]]$ so $g[B]$ is also not negligible and $\mathrm{non}\,\mathcal{N}(\mu) \le \#(g[B]) \le \#(B)$; as B is arbitrary, $\mathrm{non}\,\mathcal{N}(\mu) \le \mathrm{non}\,\mathcal{N}(\nu)$.

If $\mathcal{A} \subseteq \mathcal{N}(\mu)$ covers X then $\{g^{-1}[A] : A \in \mathcal{A}\} \subseteq \mathcal{N}(\nu)$ covers Y; hence $\mathrm{cov}\,\mathcal{N}(\nu) \le \mathrm{cov}\,\mathcal{N}(\mu)$.

2E Elementary facts Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space.

(a) μ is inner regular with respect to the family of compact sets K which are **self-supporting**, that is, $\mu(K \cap G) > 0$ whenever G is open and $K \cap G \neq \emptyset$. **P** If $K \subseteq X$ is compact and $\mathcal{G} = \{G : G \in \mathfrak{T},\ \mu(K \cap G) = 0\}$, then $\mu(K \cap \bigcup \mathcal{G}) = 0$ and $K \setminus \bigcup \mathcal{G}$ is self-supporting. **Q**

(b) If μ is atomless and non-zero, there is a function $f : E \to \{0,1\}^\omega$ such that $\mu f^{-1}[F] = \mu X \cdot \nu_\omega F$ for every $F \in \mathrm{T}_\omega$ (Fremlin 02, 343C). In this case, $\mathrm{T}_\omega = \{F : F \subseteq \{0,1\}^\omega,\ f^{-1}[F] \in \Sigma\}$ (Fremlin 03, 418I).

2F Theorem (Fremlin 84, Fremlin 91) If $(X, \mathfrak{T}, \Sigma, \mu)$ is an atomless totally finite Radon measure space with Maharam type $\kappa \ge \omega$, then

$$\mathrm{cf}\,\mathcal{N}(\mu) = \max(\mathrm{cf}\,\mathcal{N}(\nu_\omega), \mathrm{cf}[\kappa]^{\le\omega}), \quad \mathrm{add}\,\mathcal{N}(\mu) = \min(\mathrm{add}\,\mathcal{N}(\nu_\omega), \mathrm{add}[\kappa]^{\le\omega}).$$

Remark Here $[\kappa]^{\le\omega}$ is the σ-ideal of countable subsets of κ. Of course $\mathrm{add}[\kappa]^{\le\omega}$ is ∞ if κ is countable, ω_1 otherwise, so I am saying that $\mathrm{add}\,\mathcal{N}(\mu) = \omega_1$ if $\kappa > \omega$.

sketch of proof All the proofs I know depend on non-trivial facts about measure algebras. A possible route is the following, relating the π-weight of a measure algebra to the cofinality of a null ideal, and the weak distributivity of the algebra to the additivity of the ideal. Full details may be found in Fremlin 07?, chap. 52.

(a) $\mathrm{cf}[\kappa]^{\le\omega} \le \mathrm{cf}\,\mathcal{N}(\mu)$, $\mathrm{add}\,\mathcal{N}(\mu) \le \mathrm{add}[\kappa]^{\le\omega}$. **P** There is a family $\langle E_\xi \rangle_{\xi<\kappa}$ of negligible sets such that $\{\xi : E_\xi \subseteq E\}$ is countable for every $E \in \mathcal{N}(\mu)$ (Fremlin 07?, §523). **Q**

(b) $\mathrm{cf}\,\mathcal{N}(\nu_\omega) \le \mathrm{cf}\,\mathcal{N}(\mu)$, $\mathrm{add}\,\mathcal{N}(\mu) \le \mathrm{add}\,\mathcal{N}(\nu_\omega)$. **P** Let $f : E \to \{0,1\}^\omega$ be as in 2E(b). If $\mathcal{A} \subseteq \mathcal{N}(\nu_\omega)$ and $\bigcup \mathcal{A} \notin \mathcal{N}(\nu_\omega)$, then $\{f^{-1}[A] : A \in \mathcal{A}\} \subseteq \mathcal{N}(\mu)$ has non-negligible union. If $\mathcal{C}$ is a cofinal subset of $\mathcal{N}(\mu)$, then $\{\{0,1\}^\omega \setminus f[X \setminus C] : C \in \mathcal{C}\}$ is cofinal with $\mathcal{N}(\nu_\omega)$. **Q**

(c) $\mathrm{cf}\,\mathcal{N}(\mu) \le \pi(\mathfrak{B}_\kappa)$. **P** It is enough to consider the case in which $\mu X = 1$ and every non-empty open set has non-zero measure. In this case the product Radon measure μ_ω on $X^{\mathbb{N}}$ (Fremlin 03, 417Q) has measure algebra $(\mathfrak{A}_\omega, \bar{\mu}_\omega)$ isomorphic to $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ (Fremlin 02, 334F). Let $D \subseteq \mathfrak{A}_\omega \setminus \{0\}$ be an order-dense set. For each $d \in D$ let $L_d \subseteq X^{\mathbb{N}}$ be a non-negligible self-supporting compact set such that $L_d^\bullet \subseteq d$. Set $\phi_n(w) = w(n)$ for $w \in X^{\mathbb{N}}$ and $n \in \mathbb{N}$, and $E_d = X \setminus \bigcup_{n\in\mathbb{N}} \phi_n[L_d]$; as $L_d \subseteq (X \setminus E_d)^{\mathbb{N}}$, $E_d \in \mathcal{N}(\mu)$. If $E \in \mathcal{N}(\mu)$, there is a sequence $\langle K_n \rangle_{n\in\mathbb{N}}$ of compact sets in X such that $K_n \subseteq X \setminus E$ and $\mu K_n \ge 1 - 2^{-n}$ for each n. There is a $d \in D$ such that $d \subseteq (\prod_{n\in\mathbb{N}} K_n)^\bullet$, and now $L_d \setminus \prod_{n\in\mathbb{N}} K_n$ is negligible;

because L_d is self-supporting and $\prod_{n\in\mathbb{N}} K_n$ is closed, $L_d \subseteq \prod_{n\in\mathbb{N}} K_n$, so that $E \subseteq E_d$. Thus $\{E_d : d \in D\}$ is cofinal with $\mathcal{N}(\mu)$; as D is arbitrary, $\mathrm{cf}\,\mathcal{N}(\mu) \le \pi(\mathfrak{A}_\mu) = \pi(\mathfrak{B}_\kappa)$. **Q**

(d) $\pi(\mathfrak{B}_\kappa) \le \max(\mathrm{cf}[\kappa]^{\le\omega}, \pi(\mathfrak{B}_\omega))$. **P** If $\mathcal{I} \subseteq [\kappa]^\omega$ is cofinal, and for $I \in \mathcal{I}$ we set $\mathfrak{C}_I = \{E^\bullet : E \in \mathrm{T}_\kappa$ is determined by coordinates in $I\}$, then every $\mathfrak{C}_I$ is isomorphic to $\mathfrak{B}_\omega$ and $\mathfrak{B}_\kappa = \bigcup_{I\in\mathcal{I}} \mathfrak{C}_I$. **Q**

(e) (Cichoń Kamburelis & Pawlikowski 85) $\pi(\mathfrak{B}_\omega) \le \mathrm{cf}\,\mathcal{N}(\nu_\omega)$. **P** Let $\mathcal{A}$ be a cofinal subset of $\mathcal{N}(\nu_\omega)$. Let $+$ be the group operation on $\{0,1\}^\omega \cong \mathbb{Z}_2^{\mathbb{N}}$, and set $Q = \{q : q \in \{0,1\}^\omega,\ \lim_{n\to\infty} q(n) = 0\}$. For $A \in \mathcal{A}$, $A + Q$ is negligible; let $K_A \subseteq \{0,1\}^\omega$ be a non-negligible self-supporting compact set. Let $\mathcal{V}$ be the countable set of open-and-closed subsets of $\{0,1\}^\omega$ and set $D = \{((K_A \cap V) + q)^\bullet : A \in \mathcal{A},\ V \in \mathcal{V},\ q \in Q\} \subseteq \mathfrak{B}_\omega$.

If $b \in \mathfrak{B}_\omega$ is non-zero, let $K \subseteq \{0,1\}^\omega$ be a non-negligible compact set such that $K^\bullet \subseteq b$. By the zero-one law (Fremlin 01, 254S), $K + Q$ is conegligible and includes K_A for some $A \in \mathcal{A}$. By Baire's theorem, there are $q \in Q$ and $V \in \mathcal{V}$ such that $K + q \supseteq K_A \cap V \ne \emptyset$. Now $K_A \cap V$ and $K_A \cap V + q$ are non-negligible and $0 \ne ((K_A \cap V) + q)^\bullet \subseteq b$. As b is arbitrary, D is order-dense and $\pi(\mathfrak{B}_\omega) \le \#(D) \le \#(\mathcal{A})$; as $\mathcal{A}$ is arbitrary, $\pi(\mathfrak{B}_\omega) \le \mathrm{cf}\,\mathcal{N}(\nu_\omega)$. **Q**

(f) $\mathrm{add}\,\mathcal{N}(\mu) \ge \mathrm{wdistr}(\mathfrak{A})$. **P** Let $\langle E_\xi\rangle_{\xi<\kappa}$ be a family in $\mathcal{N}(\mu)$ where $\kappa < \mathrm{wdistr}(\mathfrak{A})$. For each $\xi < \kappa$ let $A_\xi \subseteq \mathfrak{A}$ be a maximal antichain consisting of sets of the form $K^\bullet$ where $K \subseteq X \setminus E_\xi$ is a compact set. Let $B \subseteq \mathfrak{A}$ be a maximal antichain such that $\{a : a \in A_\xi,\ a \cap b \ne 0\}$ is finite for every $b \in B$. Given $\epsilon > 0$, we have a $c \in \mathfrak{A}$, a finite union of members of B, such that $\bar\mu(1 \setminus c) < \epsilon$; by 2E, there is a self-supporting $K \subseteq X$ such that $\mu(X \setminus K) < \epsilon$ and $K^\bullet \subseteq c$. For each $\xi < \kappa$, there are $a_0, \dots, a_n \in A_\xi$ such that $c \subseteq \sup_{i\le n} a_i$, so there are compact sets $K_0, \dots, K_n \subseteq X \setminus E_\xi$ such that $K \setminus \bigcup_{i\le n} K_i$ is negligible. As K is self-supporting, $K \subseteq \bigcup_{i\le n} K_i \subseteq X \setminus E_\xi$. As ξ is arbitrary, $\bigcup_{\xi<\kappa} E_\xi \subseteq X \setminus K$, which has measure at most ϵ. As ϵ is arbitrary, $\bigcup_{\xi<\kappa} E_\xi$ is negligible. **Q**

(g) Let $\mathcal{S}$ be the family of subsets S of $\mathbb{N}\times\mathbb{N}$ such that $\#(S[\{n\}]) \le 2^n$ for every $n \in \mathbb{N}$. Write $\mathfrak{l}$ for the 'localization number', that is, the smallest cardinal of any set $F \subseteq \mathbb{N}^{\mathbb{N}}$ such that for every $S \in \mathcal{S}$ there is an $f \in F$ such that $\{n : (n, f(n)) \notin S\}$ is infinite. Then $\mathrm{wdistr}(\mathfrak{B}_\omega) \ge \mathfrak{l}$. **P** Let $\mathcal{E}$ be the countable algebra of open-and-closed subsets of $\{0,1\}^\omega$ and for $n \in \mathbb{N}$ let $\mathcal{E}_n = \{V : V \in \mathcal{E},\ \nu_\omega V \le 4^{-n}\}$. Suppose that $\kappa < \mathfrak{l}$ and that $\langle A_\xi\rangle_{\xi<\kappa}$ is a family of maximal antichains in $\mathfrak{B}_\omega$. For $\xi < \kappa$, $n \in \mathbb{N}$ let $K_{\xi n} \subseteq \{0,1\}^\omega$ be a set of measure greater than $1 - 8^{-n}$ such that $K_{\xi n}^\bullet$ meets only finitely many members of A_ξ. Express $\{0,1\}^\omega \setminus K_{\xi n}$ as $\bigcup_{i\ge n} V_{\xi ni}$ where $\nu_\omega V_{\xi ni} < 8^{-i}$ for each i and each $V_{\xi ni}$ belongs to $\mathcal{E}$. Set $f_\xi(n) = \bigcup_{m\le n} V_{\xi mn} \in \mathcal{E}_n$ for each n. Because $\kappa < \mathfrak{l}$ there is an $S \subseteq \{(n, V) : n \in \mathbb{N},\ V \in \mathcal{E}_n\}$ such that $\#(S[\{n\}]) \le 2^n$ for every n and $\{n : (n, f_\xi(n)) \notin S\}$ is finite for every $\xi < \kappa$. Set $H_n = \bigcup_{i\ge n} \bigcup S[\{i\}]$ for $n \in \mathbb{N}$. Then $\nu_\omega H_n \le \sum_{i=n}^\infty 2^i \cdot 4^{-i} = 2^{-n+1}$ for each n. Set $c_n = 1 \setminus H_n^\bullet$, $b_0 = c_0$, $b_n = c_n \setminus c_{n-1}$ for $n \ge 1$; then $B = \{b_n : n \in \mathbb{N}\}$ is a maximal antichain in $\mathfrak{B}_\omega$. If $\xi < \kappa$ and $m \in \mathbb{N}$, there is an $n \ge m$ such that $(i, f_\xi(i)) \in S$ for every $i \ge n$. So

$$H_m \supseteq \bigcup_{i\ge n} f_\xi(i) \supseteq \bigcup_{i\ge n} V_{\xi ni} = \{0,1\}^\omega \setminus K_{\xi n}$$

and $b_m \subseteq c_m \subseteq K_{\xi n}^\bullet$ meets only finitely many members of A_ξ. As $\langle A_\xi\rangle_{\xi<\kappa}$ is arbitrary, $\mathrm{wdistr}(\mathfrak{B}_\omega) \ge \mathfrak{l}$. **Q**

(h) $\mathfrak{l} \ge \operatorname{add}\mathcal{N}(\nu_\omega)$. (Bartoszyński 84, or Fremlin 07?, §521).

(i) Assembling these,
— (a) and (b) show that

$$\operatorname{cf}\mathcal{N}(\mu) \ge \max(\operatorname{cf}\mathcal{N}(\nu_\omega), \operatorname{cf}[\kappa]^{\le\omega}), \qquad \operatorname{add}\mathcal{N}(\mu) \le \min(\operatorname{add}\mathcal{N}(\nu_\omega), \operatorname{add}[\kappa]^{\le\omega});$$

— (c)-(e) show that $\operatorname{cf}\mathcal{N}(\mu) \le \max(\operatorname{cf}\mathcal{N}(\nu_\omega), \operatorname{cf}[\kappa]^{\le\omega})$;
— (f)-(h) show that if $\kappa = \omega$ then $\operatorname{add}\mathcal{N}(\mu) \ge \operatorname{add}\mathcal{N}(\nu_\omega)$.

2F A problem From 2D and 2F we see that, at least for atomless totally finite Radon measures μ (and atoms in Radon measure spaces are trivial), the cardinals $\operatorname{add}\mathcal{N}(\mu)$, $\operatorname{non}\mathcal{N}(\mu)$, $\operatorname{cov}\mathcal{N}(\mu)$ and $\operatorname{cf}\mathcal{N}(\mu)$ depend only on the measure algebra of μ. I do not know whether this is true of the shrinking number $\operatorname{shr}\mathcal{N}(\mu)$.

In particular, let Z be the Stone space of $\mathfrak{B}_\omega$ and λ its usual Radon measure, defined by setting $\lambda\widehat{b} = \bar\nu_\omega b$ for every $b \in \mathfrak{B}_\omega$ (Fremlin 03, 411P). Then the measure algebra of λ is isomorphic to $(\mathfrak{B}_\omega, \bar\nu_\omega)$ (Fremlin 02, 321K), and $\mathcal{N}(\lambda)$ is the ideal of nowhere dense subsets of Z (Fremlin 02, 322Q). Now $\operatorname{shr}\mathcal{N}(\nu_\omega) \le \operatorname{shr}\mathcal{N}(\lambda)$. **P** There are inverse-measure-preserving functions $f : Z \to \{0,1\}^\omega$ and $g : \{0,1\}^\omega \to Z$ such that $f(g(x)) = x$ for every $x \in \{0,1\}^\omega$ (Fremlin 03, 453M). If $A \subseteq \{0,1\}^\omega$ is non-negligible, so is $g[A] \subseteq Z$, so there is a non-negligible $B \subseteq g[A]$ such that $\#(B) \le \operatorname{shr}\mathcal{N}(\lambda)$; now $f[B] \subseteq A$ is non-negligible and $\#(f[B]) \le \operatorname{shr}\mathcal{N}(\lambda)$. **Q** But is it consistent to suppose that $\operatorname{shr}\mathcal{N}(\lambda) > \operatorname{shr}\mathcal{N}(\nu_\omega)$? In random real models over a ground model in which the continuum hypothesis is true, $\operatorname{shr}\mathcal{N}(\nu_\omega) = \omega_1$; what happens to $\operatorname{shr}\mathcal{N}(\lambda)$?

3. Maharam types of Radon measures

The results in §2 show that, at least for the questions considered here, properties of a Radon measure can often be inferred from its Maharam type. It therefore becomes natural to ask which cardinals can appear as Maharam types of Radon measures on which topological spaces. Since there seems to be no simple answer we come to another definition.

3A Definition If X is a Hausdorff space, write $\mathrm{Mah_R}(X)$ for the set of those infinite κ such that $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ is isomorphic to the measure algebra of some Radon probability measure on X.

It is easy to see (using the homogeneity of the algebras $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$) that $\mathrm{Mah_R}(X) = \bigcup\{\mathrm{Mah_R}(K) : K \subseteq X \text{ is compact}\}$.

3B Lemma (Fremlin 03, 416O) Let X be a Hausdorff space and ν a functional such that

(†) $\operatorname{dom}\nu$ is a subalgebra of $\mathcal{P}X$ and ν is a (finitely) additive functional;

(‡) $\nu E = \sup\{\nu K : K \in \operatorname{dom}\nu,\ K \subseteq E,\ K \text{ is compact}\}$ for every $E \in \operatorname{dom}\nu$.

Then there is a Radon measure μ on X, extending ν, such that

(*) for every $E \in \operatorname{dom}\mu$ and $\epsilon > 0$ there is an $F \in \operatorname{dom}\nu$ such that $\mu(E \triangle F) \le \epsilon$.

sketch of proof Let μ be a maximal extension of ν satisfying (†), (‡) and (*); write Σ for the domain of μ.

(a) If $G\subseteq X$ is open, set

$$\begin{aligned}\mu_1((E\cap G)\cup(F\setminus G))&=\sup_{H\in\Sigma,H\subseteq G}\mu(E\cap H)+\inf_{H\in\Sigma,H\subseteq G}\mu(F\setminus H)\\&=\lim_{H\in\Sigma,H\subseteq G,H\uparrow}\mu((E\cap H)\cup(F\setminus H))\end{aligned}$$

for E, $F\in\Sigma$; show that μ_1 satisfies (†), (‡) and (*), so $\mu_1=\mu$ and $G\in\Sigma$.

(b) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ with empty intersection, then $\lim_{n\to\infty}\mu E_n=0$. **P** Given $\epsilon>0$, we can find compact sets $K_n\subseteq E_n$ such that $\mu(E_n\setminus K_n)\le 2^{-n}\epsilon$ for each n. There is some $m\in\mathbb{N}$ such that $\bigcap_{i\le m}K_i=\emptyset$, so $\inf_{n\in\mathbb{N}}\mu E_n\le\mu E_m\le 2\epsilon$. **Q**

(c) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in Σ with union G, define μ_1 by the formula in (a); show that μ_1 satisfies (†), (‡) and (*). Hence Σ is a σ-algebra and (using (b) again) μ is a measure. It is now easy to check that μ is complete, so is a Radon measure.

3C Theorem If X is a Hausdorff space and $\omega\le\kappa\le\lambda\le\mathrm{Mah}_{\mathrm{R}}(X)$ then $\kappa\in\mathrm{Mah}_{\mathrm{R}}(X)$.

sketch of proof Let μ_0 be a Radon probability measure on X with measure algebra isomorphic to $(\mathfrak{B}_\lambda,\nu_\lambda)$; let $\langle E_\xi\rangle_{\xi<\lambda}$ be a stochastically independent family of measurable sets all with measure $\frac{1}{2}$. Let $\mathrm{T}\subseteq\operatorname{dom}\mu_0$ be a subalgebra of $\mathcal{P}X$, of size κ, containing E_ξ for every $\xi<\kappa$, and such that for every $E\in\mathrm{T}$ and $n\in\mathbb{N}$ there is a compact $K\in\mathrm{T}$ such that $K\subseteq E$ and $\mu_0(E\setminus K)\le 2^{-n}$. Apply Lemma 3B with $\nu=\mu_0\restriction\mathrm{T}$.

3D Theorem Let X and Y be compact Hausdorff spaces and $f:X\to Y$ a continuous surjection. Then $\mathrm{Mah}_{\mathrm{R}}(Y)\subseteq\mathrm{Mah}_{\mathrm{R}}(X)$.

sketch of proof Let λ be a Radon probability measure on Y with measure algebra isomorphic to $(\mathfrak{B}_\kappa,\nu_\kappa)$. Set $\nu f^{-1}[F]=\lambda F$ for $F\in\operatorname{dom}\lambda$ and apply 3B.

3E Theorem If X is a compact Hausdorff space, then the following are equiveridical:
(i) X is not scattered, that is, there is a non-empty subset of X with no isolated point;
(ii) there is a continuous surjection from X onto $[0,1]$;
(iii) $\omega\in\mathrm{Mah}_{\mathrm{R}}(X)$.

scheme of proof (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(i); see Fremlin 07?, chap. 53.

3F Theorem For any Hausdorff space, $\sup\mathrm{Mah}_{\mathrm{R}}(X)$ is at most the weight $w(X)$ of X.

proof If μ is a totally finite Radon measure on X with measure algebra $\mathfrak{A}$, and $\mathcal{U}$ is a base for the topology of X, then $\{U^\bullet:U\in\mathcal{U}\}$ generates $\mathfrak{A}$. **P** For any open $G\subseteq X$,

$$G^\bullet=\sup\{K^\bullet:K\subseteq G\text{ is compact}\}=\sup\{U^\bullet:U\in\mathcal{U},\,U\subseteq G\};$$

for any $E\in\operatorname{dom}\mu$, $E^\bullet=\sup\{K^\bullet:K\subseteq E\text{ is compact}\}$. **Q**

3G Theorem Let $\langle X_i\rangle_{i\in I}$ be a family of non-empty Hausdorff spaces with product X. Then an infinite cardinal κ belongs to $\mathrm{Mah}_{\mathrm{R}}(X)$ iff *either* $\kappa\le\#(\{i:i\in I,\,\#(X_i)\ge 2\})$ *or* κ is expressible as $\sup_{i\in I}\kappa_i$ where $\kappa_i\in\mathrm{Mah}_{\mathrm{R}}(X_i)$ for every $i\in I$.

proof Fremlin 07?, chap. 53.

4. Precalibers When we come to the problem of determining $\mathrm{Mah}_{\mathrm{R}}(X)$ for given topological spaces X, the most interesting results are associated with the following idea.

4A Definition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. A cardinal κ is a **measure-precaliber** of $(\mathfrak{A}, \bar{\mu})$ if whenever $\langle a_\xi \rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}$ such that $\inf_{\xi<\kappa} \bar{\mu} a_\xi > 0$ then there is a $\Gamma \in [\kappa]^\kappa$ such that $\{a_\xi : \xi \in \Gamma\}$ is centered, that is, $\inf_{\xi\in I} a_\xi \neq 0$ for every finite $I \subseteq \Gamma$.

4B Lemma (Fremlin 07?, §524) For an infinite cardinal κ, the following are equiveridical:

(i) κ is a measure-precaliber of $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$;

(ii) κ is a measure-precaliber of every totally finite measure algebra.

sketch of proof If κ is a measure-precaliber of $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$, $(\mathfrak{A}, \bar{\mu})$ is a probability algebra, and $\langle a_\xi \rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}$ such that $\inf_{\xi<\kappa} \bar{\mu} a_\xi > 0$, let $\mathfrak{C}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $\{a_\xi : \xi < \kappa\}$. Then $\tau(\mathfrak{C}) \leq \kappa$ so $(\mathfrak{C}, \bar{\mu}\restriction\mathfrak{C})$ can be embedded in $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$; this makes $\langle a_\xi \rangle_{\xi<\kappa}$ correspond to a family in $\mathfrak{B}_\kappa$ which must have a large centered subfamily.

4C Proposition (a) ω is a measure-precaliber of every totally finite measure algebra.

(b) If κ is a regular cardinal greater than $\mathrm{non}\,\mathcal{N}(\mu_\omega)$ and $\mathrm{cf}[\lambda]^{\leq\omega} < \kappa$ for every $\lambda < \kappa$, then κ is a measure-precaliber of every totally finite measure algebra.

(c) If $\kappa < \mathfrak{m}$ (that is, if $\mathrm{MA}(\kappa)$ is true), then κ is a measure-precaliber of every totally finite measure algebra.

(d) If κ is either ω or a strong limit cardinal of countable cofinality, and $2^\kappa = \kappa^+$, then κ^+ is not a measure-precaliber of $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$.

sketches of proofs (a) If $(\mathfrak{A}, \bar{\mu})$ is a totally finite measure algebra and $\bar{\mu} a_n \geq \delta$ for each n, choose a strictly increasing sequence $\langle n_k \rangle_{k\in\mathbb{N}}$ such that $\bar{\mu}(\sup_{n_k\leq n<n_{k+1}} a_n) \geq \bar{\mu}(\sup_{n\geq n_k} a_n) - 2^{-k-1}\delta$ for each k. Then $\langle b_k \rangle_{k\in\mathbb{N}}$ is centered, where $b_k = \sup_{n_k\leq n<n_{k+1}} a_n$ for each k. If A is a maximal centered family in $\mathfrak{A}$ containing every b_k, then for every k there is an $n \in [n_k, n_{k+1}[$ such that $a_n \in A$.

(b) Let $\langle a_\xi \rangle_{\xi<\kappa}$ be a family of non-zero elements of $\mathfrak{B}_\kappa$. For each $\xi < \kappa$, there is an $E_\xi \in \mathrm{T}_\kappa$ such that $E_\xi^\bullet = a_\xi$ and E_ξ is determined by coordinates in a countable subset I_ξ of κ (Fremlin 01, 254O). Because $\kappa > \omega$, $\kappa > \mathfrak{c}$; because $\kappa \geq \omega_2$ is regular, there are a $\Gamma_0 \in [\kappa]^\kappa$ and a $\zeta < \kappa$ such that $I_\xi \cap I_\eta \subseteq \zeta$ for all distinct ξ, $\eta \in \Gamma_0$. Because $\mathrm{cf}[\zeta]^{\leq\omega} < \kappa$ and κ is regular, there is a $J \in [\zeta]^{\leq\omega}$ such that $\Gamma_1 = \{\xi : \xi \in \Gamma_0,\ I_\xi \cap \zeta \subseteq J\}$ has cardinal κ. Identify $\{0,1\}^\kappa$ with $\{0,1\}^J \times \{0,1\}^{\kappa\setminus J}$ and set $F_\xi = \{w : w \in \{0,1\}^J,\ \nu_{\kappa\setminus J} E_\xi[\{w\}] > 0\}$ for $\xi < \kappa$. Then $\nu_J F_\xi > 0$. Let $\theta : \mathfrak{B}_J \to \mathrm{T}_J$ be a lifting, and set $F'_\xi = \theta F_\xi^\bullet$. Because $\mathrm{non}\,\mathcal{N}_\omega < \kappa$, there is a non-negligible set $A \subseteq \{0,1\}^J$ of cardinal less than κ; now $A' = \{w : w \in \{0,1\}^J, \exists\ w \in A,\ \{\eta : w'(\eta) \neq w(\eta)\}$ is finite$\}$ has full outer measure and $\#(A') < \kappa$. So every F'_ξ meets A'; let $w \in A'$ be such that $\Gamma = \{\xi : \xi \in \Gamma_1,\ w \in F'_\xi\}$ has cardinal κ. If $K \in [\Gamma]^{<\omega}$, then $\theta(\inf_{\xi\in K} F_\xi^\bullet) = \bigcap_{\xi\in K} F'_\xi \neq \emptyset$ so $\nu_J(\bigcap_{\xi\in K} F_\xi) > 0$ and

$$\bar\nu_\kappa(\inf_{\xi\in K}a_\xi)=\nu_\kappa(\bigcap_{\xi\in K}E_\xi)=\int\bigl(\prod_{\xi\in K}\nu_{\kappa\setminus J}E_\xi[\{w\}]\bigr)\nu_J(dw)>0.$$

(c) FREMLIN 07?, §524.

(d) Because κ is a strong limit cardinal of countable cofinality, $\operatorname{non}\mathcal{N}(\nu_\kappa)>\kappa$ (FREMLIN 07?, §522). Enumerate $\{0,1\}^\kappa$ as $\langle x_\xi\rangle_{\xi<\kappa^+}$. For each $\xi<\kappa^+$ let $K_\xi\subseteq\{x_\eta:\eta>\xi\}$ be a compact set of measure at least $\frac{1}{2}$. If $\Gamma\in[\kappa]^\kappa$ then $\bigcap_{\xi\in\Gamma}K_\xi=\emptyset$ so there is a finite set $I\subseteq\Gamma$ such that $\bigcap_{\xi\in I}K_\xi=\emptyset$. Accordingly $\langle K_\xi^\bullet\rangle_{\xi<\kappa^+}$ witnesses that κ^+ is not a measure-precaliber of $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$.

Remarks For any Boolean algebra $\mathfrak{A}$, we can say that a cardinal κ is a **precaliber** of $\mathfrak{A}$ if if whenever $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}\setminus\{0\}$ then there is a $\Gamma\in[\kappa]^\kappa$ such that $\{a_\xi:\xi\in\Gamma\}$ is centered. If $\operatorname{cf}\kappa>\omega$ and $(\mathfrak{A},\bar\mu)$ is a totally finite measure algebra, then κ is a measure-precaliber of $(\mathfrak{A},\bar\mu)$ iff it is a precaliber of $\mathfrak{A}$. The point of (a) and (c) here is that cardinals of countable cofinality can be measure-precalibers without being calibers.

Note that (b) relies essentially on the infinitary combinatorics of families of countable sets; this is a recurrent theme in this topic.

4D Precalibers and covering numbers Adapting the method of 4C(a), we find that if $\omega\le\kappa<\operatorname{cov}\mathcal{N}(\nu_\lambda)$ then κ is a measure-precaliber of $(\mathfrak{B}_\lambda,\bar\nu_\lambda)$; this can be regarded as sharpening 4C(c).

Conversely, if $\operatorname{cov}\mathcal{N}(\nu_\lambda)=\omega_1$ then ω_1 is not a measure-precaliber of $(\mathfrak{B}_\lambda,\bar\nu_\lambda)$ (use the idea of 4C(d)).

4E Problem Is it relatively consistent with ZFC to suppose that every infinite cardinal is a measure-precaliber of every probability algebra? (From 4C(d) we see that this is a large-cardinal problem. See FOREMAN & WOODIN 91.)

4F Haydon's property Let X be a compact Hausdorff space. If there is a continuous surjection from X onto $[0,1]^\kappa$, where $\kappa\ge\omega$, then $\kappa\in\operatorname{Mah}_{\mathrm{R}}(X)$, by 3D. Conversely, we say that an infinite cardinal κ has **Haydon's property** if whenever X is a compact Hausdorff space and $\kappa\in\operatorname{Mah}_{\mathrm{R}}(X)$, then there is a continuous surjection from X onto $[0,1]^\kappa$. Thus 3E tells us that ω has Haydon's property.

We have the following remarkable results.

4G Theorem (a) (HAYDON 77, PLEBANEK 97) If $\kappa\ge\omega_2$, then κ has Haydon's property iff it is a measure-precaliber of $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$.

(b) (FREMLIN 97) If $\mathfrak{m}>\omega_1$ then ω_1 has Haydon's property.

proof See FREMLIN 07?, chap. 53. Here I remark only that the arguments depend on a deep analysis of the measure algebras $\mathfrak{B}_\kappa$, standard infinitary combinatorics (as in the proof of 4C(b)) and straightforward general topology.

4H If we restrict attention to particular classes of topological space, of course the patterns can change. For instance, we have the following.

Theorem (a) (PLEBANEK 95) The following are equiveridical:

(i) $\operatorname{cov}\mathcal{N}(\nu_{\omega_1}) = \omega_1$;
(ii) ω_1 is not a measure-precaliber of $(\mathfrak{B}_{\omega_1}, \bar{\nu}_{\omega_1})$;
(iii) there is a first-countable compact Hausdorff space such that $\omega_1 \in \mathrm{Mah}_{\mathrm{R}}(X)$.

(b) (Kunen & Mill 95) Suppose that $\mathrm{cf}\,\mathcal{N}_\omega = \omega_1$. Then there is a perfectly normal compact Hausdorff space X such that $\omega_1 \in \mathrm{Mah}_{\mathrm{R}}(X)$.

proof Fremlin 07?, chap. 53. (This time we seem to need non-trivial measure-theoretic arguments added to Haydon's inductive limit construction of compact spaces.)

4I Problem Can there be a perfectly normal compact Hausdorff space X such that $\omega_2 \in \mathrm{Mah}_{\mathrm{R}}(X)$?

Note that a perfectly normal compact Hausdorff space has weight at most $\mathfrak{c}$ and π-weight at most ω_1. Rather few examples are known which can be built in ZFC; the standard non-mertrizable one is the 'split interval' or 'double arrow space' $I^{\|}$ (Fremlin 03, 419L), which has weight $\mathfrak{c}$, but as it is a non-scattered compact linearly ordered topological space $\mathrm{Mah}_{\mathrm{R}}(I^{\|}) = \{\omega\}$. The same applies to another class of familiar examples, Souslin lines.

4J Exercise Let X be a non-empty Hausdorff space and μ a totally finite Radon measure on X. Show that $\mathrm{cf}\,\mathcal{N}(\mu) \le t(X)^\omega$, where $t(X)$ is the tightness of X.

solution If $t(X) = 1$ this is trivial. Otherwise, set $\kappa_0 = t(X)^\omega$, $\kappa = \kappa_0^+ > \mathfrak{c} \ge \mathrm{non}\,\mathcal{N}(\nu_\omega$; then $\mathrm{cf}[\lambda]^\omega \le \kappa_0^\omega = \kappa_0 < \kappa$ for every $\lambda < \kappa$. So κ is a measure-precaliber of the measure algebra $(\mathfrak{A}, \bar{\mu})$ of μ, while surely $\kappa \ge \omega_2$; thus κ has Haydon's property.

Let $K \subseteq X$ be compact. As $t(K) < \kappa = t([0,1]^\kappa)$, $[0,1]^\kappa$ is not a continuous image of K and $\kappa \notin \mathrm{Mah}_{\mathrm{R}}(K)$. As K is arbitrary, $\kappa \notin \mathrm{Mah}_{\mathrm{R}}(X)$.

It follows that $\tau(\mathfrak{A}) \le \kappa_0$. So

$$\mathrm{cf}\,\mathcal{N}(\mu) \le \max(\mathrm{cf}\,\mathcal{N}(\nu_\omega), \mathrm{cf}[\kappa_0]^{\le\omega}) \le \kappa_0^\omega = \kappa_0,$$

as required.

References

Bartoszyński T. [84] 'Additivity of measure implies additivity of category', Trans. Amer. Math. Soc. 281 (1984) 209-213.

Cichoń J., Kamburelis T. & Pawlikowski J. [85] 'On dense subsets of the measure algebra', Proc. Amer. Math. Soc. 94 (1985) 142-146.

Foreman M. & Woodin W. [91] 'The generalized continuum hypothesis can fail everywhere', Annals of Math. 133 (1991) 1-35.

Fremlin D.H. [84] 'On the additivity and cofinality of Radon measures', Mathematika 31 (1984) 323-335.

Fremlin D.H. [91] 'The partially ordered sets of measure theory and Tukey's ordering,' Note di Matematica 11 (1991) 177-214.

Fremlin D.H. [97] 'On compact spaces carrying Radon measures of uncountable Maharam type', Fundamenta Math. 154 (1997) 295-304.

Fremlin D.H. [00] *Measure Theory, Vol. 1: The Irreducible Minimum.* Torres Fremlin, 2000.

Fremlin D.H. [01] *Measure Theory, Vol. 2: Broad Foundations.* Torres Fremlin, 2001.

Fremlin D.H. [02] *Measure Theory, Vol. 3: Measure Algebras.* Torres Fremlin, 2002.

Fremlin D.H. [03] *Measure Theory, Vol. 4: Topological Measure Theory.* Torres Fremlin, 2003.

Fremlin D.H. [07?] *Measure Theory, Vol. 5: Set-theoretic measure theory,* in preparation. *Partial drafts available via* `http://www.essex.ac.uk/maths/staff/fremlin/mt.htm`.

Haydon R.G. [77] 'On Banach spaces which contain $\ell^1(\tau)$ and types of measures on compact spaces', Israel J. Math. 28 (1977) 313-324.

Maharam D. [42] 'On homogeneous measure algebras', Proc. Nat. Acad. Sci. U.S.A. 28 (1942) 108-111.

Plebanek G. [97] 'Non-separable Radon measures and small compact spaces', Fundamenta Math. 153 (1997) 25-40.