Sequences with small difference sets

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1. Construction Let $M \ge 9$ be an integer. Suppose that for each $k \ge 1$ we have a set $D_k \subseteq \mathbb{N} \times \mathbb{N}$ such that

$\|z\|_1 \ge Mk$ for every $z \in D_k$,

$\|w - z\|_1 \ge M^k$ for all distinct w, $z \in D_k$.

Set $D = \bigcup_{k\in\mathbb{N}} D_k$ and consider the directed graph Γ with vertex set $V = (\mathbb{N} \times \mathbb{N}) \setminus D$ and edges (w, z) where w, $z \in V$ and z is either $w + (0, 1)$ or $w + (1, 0)$. (Thus paths in Γ run upwards and rightwards.) For $n \in \mathbb{N}$ let $C_n \subseteq \mathbb{N} \times \mathbb{N}$ be the diagonal $\{z : \|z\|_1 = n\}$. For $z \in V$, $n \ge \|z\|_1$ let $\Delta(z, n)$ be the set of those $w \in C_n$ such that there is a path in Γ from z to w and there is an infinite path in Γ starting at w.

2. Lemma In the context of §1, $\#(\Delta(\mathbf{0}, n)) \ge \frac{M-8}{M+8}(n - 1)$ for every $n \ge 1$.

proof Let Γ' be the subgraph of Γ obtained by deleting the vertices in $\Delta = \Delta(\mathbf{0}, n)$. Let R be the set of all those points in $V' = V \setminus \Delta$ accessible from $\mathbf{0}$ in Γ'. Then R has no infinite path (because any infinite path in R would be part of an infinite path in Γ starting at $\mathbf{0}$ which would have to cross C_n in a point belonging to Δ). By König's lemma (Erdős Hajnal Máté & Rado 84, 10.3), R is finite; of course $\mathbf{0} \in R$. Set $\tilde{R} = R + [0, 1]^2 \subseteq \mathbb{R}^2$. Then $\tilde{R}$ is connected. The boundary $\partial\tilde{R}$ of $\tilde{R}$ can itself be regarded as a directed graph $\tilde{\Gamma}_R$ on $\mathbb{N}\times\mathbb{N}$ in which the edges are those links from one vertex to a neighbour in which $\tilde{R}$ is to the left of the edge and its complement is to the right. This is a finite directed graph in which every vertex has equal in-degree and out-degree. We therefore have a circuit P in $\tilde{\Gamma}_R$ starting with the edge $(0, 0) \to (1, 0)$ and ending with the edge $(0, 1) \to (0, 0)$. Let r, l, u, d respectively be the numbers of edges in P in which we move right, left, up or down, and let D', Δ' be the sets of points of D, Δ respectively through which P passes.

If e is a leftwards segment of P, then the end point z of e belongs to $D' \cup \Delta'$. **P** Because the square below e lies in $\tilde{R}$, $z - (1, 0) \in R$; because the square above e does not lie in $\tilde{R}$, $z \notin R$. Since $z - (1, 0)$ is accessible from $\mathbf{0}$ in Γ', the only way in which z can fail to belong to R is that it lies in the set $D \cup \Delta$ of excised vertices; as it lies on P it belongs to $D' \cup \Delta'$. **Q** Similarly, if e is an upwards segment of P, then the starting point z of e belongs to $D' \cup \Delta'$. Since P repeats no edges, $\#(D' \cup \Delta') \ge \max(l, u)$. Next, P is closed, so $r = l$ and $u = d$ and $\#(D' \cup \Delta') \ge \frac{1}{4}p$ where p is the length of P.

Set $m = \max\{\|z\|_1 : z \text{ lies on } P\}$. Then $r + u = l + d \ge m$, so $p \ge 2m$. For each $k \ge 1$, the number of points of D_k lying on P is at most $\lceil M^{-k}p\rceil$, because any two points must be separated by at least M^k steps along P. Next, if $Mk > m$, no point of D_k can lie on P. So

$$\#(D') \le \sum_{1\le k\le\lfloor m/M\rfloor} \lceil M^{-k}p\rceil \le \frac{1}{2M}p + \sum_{k=1}^{\infty} M^{-k}p = p\Big(\frac{1}{2M} + \frac{1}{M-1}\Big) \le \frac{2}{M}p.$$

As $\frac{2}{M} < \frac{1}{4}$, $\Delta' \ne \emptyset$ and P reaches C_n. Let z_1, z_2 be the lowest and highest points of C_n lying on P. Since P passes through $\mathbf{0}$, z_1 and z_2, $p \ge \|z_1\|_1 + \|z_1 - z_2\|_1 + \|z_2\|_1 = 2n + \|z_1 - z_2\|_1$. Also, every element of Δ' must lie between z_1 and z_2 on P, so

$$\#(\Delta') \le 1 + \frac{1}{2}\|z_1 - z_2\|_1 \le 1 + \frac{1}{2}p - n.$$

We therefore have

$$\frac{1}{4}p \le \frac{2}{M}p + 1 + \frac{1}{2}p - n, \quad n \le 1 + (\frac{1}{4} + \frac{2}{M})p$$

and

$$\begin{aligned}\#(\Delta) \ge \#(\Delta') &\ge \frac{1}{4}p - \frac{2}{M}p \\ &\ge \frac{\frac{1}{4} - \frac{2}{M}}{\frac{1}{4} + \frac{2}{M}}(n-1) = \frac{M-8}{M+8}(n-1),\end{aligned}$$

as claimed.

3. Lemma Suppose that $n_1 > 0$, $z_0 \in \Delta(\mathbf{0}, n_1)$ and $n_2 \ge Mn_1$. Then $\#(\Delta(z_0, n_2)) \ge (1 - \frac{31}{M})n_2$.

proof This time, let R be the set of points in V accessible from z_0 by a path in Γ, and $\tilde{R} = R + [0,1]^2$; again let P be a path along the boundary of $\tilde{R}$, starting with $z_0 \to z_0 + (0,1)$, keeping $\tilde{R}$ to the left. Because $z_0 \in \Delta(\mathbf{0}, n_1)$, R is infinite, $\tilde{R}$ is unbounded and P is infinite; accordingly there is a first point (along P) where it meets the diagonal C_{n_2}; suppose that this is $w_1 = (j_1, k_1)$, and let P' be the section of P between z_0 and w_1. Let r, l, u, d be the numbers of unit segments in P' in which we move right, left, up or down respectively, so that if $z_0 = (j_0, k_0)$ we have $r - l = j_1 - j_0 \ge 0$ and $u - d = k_1 - k_0 \ge 0$, while the length $p = r + l + u + d$ of P' is at least $n_2 - n_1$.

As before, we see that every leftwards segment of P' ends at a point of D and every upwards segment of P' starts at a point of D. Let D' be the set of points of D lying on P'. Because $\|z\|_1 \le n_2$ for every z on P', our estimate becomes

$$\max(l, u) \le \#(D') \le \frac{1}{M-1}p + \frac{1}{M}n_2.$$

Now

$$r \le l + n_2 - n_1, \quad d \le u,$$

so

$$p = r + d + l + u \le 2l + 2u + n_2 - n_1 \le \frac{4}{M-1}p + \frac{4}{M}n_2 + n_2,$$

$$(1 - \frac{4}{M-1})p \le (1 + \frac{4}{M})n_2,$$

$$k_1 \le u + n_1 \le \frac{1}{M-1}p + \frac{2}{M}n_2 \le \frac{3M-6}{M(M-5)}n_2 \le \frac{6}{M}n_2.$$

The edge of P ending at w_1 must either have gone upwards, in which case $z_1 = (j_1 - 1, k_1 - 1) \in R$, or have gone rightwards, in which case $z_1 = (j_1 - 1, k_1) \in R$. In the same way, there is a point (j_2, k_2) with $j_2 + k_2 = n_2$ and $j_2 \le \frac{6}{M} n_2$ and $z_2 \in R$, where z_2 is one of $(j_2 - 1, k_2 - 1)$, $(j_2, k_2 - 1)$. Now consider the points $(i, n_2 - i) \in \Delta(\mathbf{0}, n_2)$ with $j_2 + 1 \le i \le n_2 - k_1 - 1$. The number of these is at least

$$\frac{M-8}{M+8}(n_2 - 1) - \frac{12}{M} n_2 - 2 \ge n_2(1 - \frac{16}{M} - \frac{3}{M} - \frac{12}{M}) = n_2(1 - \frac{31}{M})$$

because $n_2 \ge M$. On the other hand, all of these are accessible from $\mathbf{0}$ by a path in Γ which must intersect one of the paths $z_0 \to z_1$, $z_0 \to z_2$ in Γ; so they are all accessible from z_0; and they are all the starting points of infinite paths. So they all belong to $\Delta(z_0, n_2)$, and $\#(\Delta(z_0, n_2)) \ge (1 - \frac{31}{M}) n_2$, as claimed.

4. Theorem Suppose that α, $\beta \in [0, 1[$ are such that 1, α, β are linearly independent over $\mathbb{Q}$. Give $[0, 1[$ its group operation $+_1$ corresponding to its identification with $\mathbb{R}/\mathbb{Z}$. Let $\epsilon > 0$. Then there is an isolated sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in $[0, 1[$ such that $\#(\{x_{n+1} -_1 x_n : n \in \mathbb{N}\}) = \{\alpha, \beta\}$ and $\overline{\{x_n : n \in \mathbb{N}\}}$ has Lebesgue measure at least $1 - \epsilon$.

proof Let $M \ge 9$ be such that $\frac{31}{M} \le \epsilon$. For $z = (i, j) \in \mathbb{N} \times \mathbb{N}$ set $c(z) = {<}i\alpha + j\beta{>}$, the fractional part of $i\alpha + j\beta$; if we interpret $i\alpha$ and $j\beta$ in the abelian group $[0, 1[\cong \mathbb{R}/\mathbb{Z}$, this is just $i\alpha +_1 j\beta$. Note that because 1, α and β are linearly independent over $\mathbb{Q}$, all the $c(z)$ are different. For each $k \in \mathbb{N}$ let $\eta_k > 0$ be such that $|i\alpha + j\beta - n| > 2\eta_k$ whenever i, j, $n \in \mathbb{Z}$ and $0 < |i| + |j| \le M^k$; then $|c(z) - c(z')| > 2\eta_k$ whenever z, $z' \in \mathbb{N} \times \mathbb{N}$ and $0 < \|z - z'\|_1 \le M^k$. Enumerate $\mathbb{N} \times \mathbb{N}$ as $\langle (i_k, j_k) \rangle_{k \in \mathbb{N}}$, and set

$$D_k = \{(i, j) : i,\ j \in \mathbb{N},\ i + j \ge Mk,\ |c(i, j) - c(i_k, j_k)| < \eta_k\}.$$

Then M and $\langle D_k \rangle_{k \in \mathbb{N}}$ satisfy the conditions of §1.

Choose $\langle z_k \rangle_{k \in \mathbb{N}}$ in V inductively so that $z_0 = (0, 0)$ and, for each k, $z_{k+1} \in \Delta(z_k, M^k)$ is such that

$$\min_{j \le k} |c(z_{k+1}) - c(z_j)| = \max_{z \in \Delta(z_k, M^k)} \min_{j \le k} |c(z) - c(z_j)|.$$

Because each z_{k+1} is accessible from z_k in Γ, we have a path P in Γ starting at $\mathbf{0}$ and passing through every z_k. Enumerate the points of P in order as $\langle w_n \rangle_{n \in \mathbb{N}}$, and set $x_n = c(w_n)$ for each n. Of course this ensures that $x_{n+1} -_1 x_n \in \{\alpha, \beta\}$ for every n, and also that all the x_n are different.

For every n, there is a $k \in \mathbb{N}$ such that $w_n = (i_k, j_k)$. Now no w_m can belong to D_k, so there can be only finitely many m such that $|x_m - x_n| < \eta_k$. Thus $\langle x_n \rangle_{n \in \mathbb{N}}$ is isolated.

? Suppose, if possible, that $F = \overline{\{x_n : n \in \mathbb{N}\}}$ has measure less than $1 - \epsilon$. Then there is a $\delta > 0$ such that $F' = \{x : \rho(x, F) \le \delta\}$ also has measure less than $1 - \epsilon$. Because $\langle {<}n(\alpha - \beta){>} \rangle_{n \in \mathbb{N}}$ is well-distributed in $[0, 1[$, there is an $m^* \ge 1$ such that

$$\#(\{i : i \le m,\ c(i, m - i) \in F'\}) < (1 - \epsilon) m$$

whenever $m \ge m^*$. Now let k^* be such that $M^{k^*} \ge m^*$. For every $k \ge k^*$, there must be a $z \in \Delta(z_k, M^k)$ such that $c(z) \notin F'$. If $j \le k$, then $z_j = w_n$ for some n, and $c(z_j) = x_n \in F$; so $|c(z) - c(z_j)| > \delta$. This is true for every $j \le k$, so we could have chosen z_{k+1} such that $|c(z_{k+1}) - c(z_j)| > \delta$ for every $j \le k$, and we therefore did, for every $k \ge k^*$. But this is impossible, because $c(z_k) \in [0,1[$ for every k. **X** So the measure of F is at least $1-\epsilon$, as required.

5. Theorem Let $\epsilon > 0$. Then there is an isolated sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ in $[0,1]$ such that $\#(\{x_{n+1} - x_n : n \in \mathbb{N}\}) = 3$ and $\overline{\{x_n : n \in \mathbb{N}\}}$ has Lebesgue measure at least $1-\epsilon$.

proof Take α, $\beta \in \left]0, \frac{1}{2}\epsilon\right[$ such that 1, α, β are linearly independent over $\mathbb{Q}$. By Theorem 4, there is an isolated sequence $\langle x'_l \rangle_{l\in\mathbb{N}}$ in $[0,1[$ such that $\{x'_{l+1} -_1 x'_l : l \in \mathbb{N}\} = \{\alpha, \beta\}$ and $F' = \overline{\{x'_l : l \in \mathbb{N}\}}$ has measure at least $1 - \frac{1}{2}\epsilon$; we can of course suppose that, as in the construction of Theorem 4, $x_0 = 0$. Define $\langle y_n \rangle_{n\in\mathbb{N}}$ by saying that $y_0 = 0$ and

— if $y_n = x'_l$ and $x'_{l+1} = x'_l + \alpha$ or $x'_{l+1} = x'_l + \beta$ then $y_{n+1} = x'_{l+1}$,

— otherwise, $y_{n+1} = x'_{l+1} + 1$ and $y_{n+2} = x'_{l+1}$.

The point here is that $<y_n>$ will always be x'_l for some l, and that every x'_l will appear as $<y_n>$ either once or twice. Of course this ensures that $y_{n+1} - y_n \in \{\alpha, \beta, -1\}$ for every n, and also that $\langle y_n \rangle_{n\in\mathbb{N}}$ is isolated. Now $\overline{\{y_n : n \in \mathbb{N}\}}$ has measure at least $1 - \frac{1}{2}\epsilon$, while $y_n \in [0, 1 + \frac{1}{2}\epsilon]$ for every n. So if we set $x_n = y_n/(1 + \frac{1}{2}\epsilon)$ for every n, we get an appropriate sequence.

6. Remarks (a) The original question, due to T.Keleti (Lhota, February 2004), was

is there a bounded isolated sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $\{x_{n+1} - x_n : n \in \mathbb{N}\}$ is finite?

This is already answered, in effect, by Lemma 2. For any such sequence, the set $F = \overline{\{x_n : n \in \mathbb{N}\}}$ must be nowhere dense. The additional refinement provided by Lemma 3 concerns the Lebesgue measure of F, and shows that it can be large in the sense that it is nearly equal to the diameter of F. We do not know whether it can be 0.

(b) We note that any solution to (a) above must have $\#(\{x_{n+1} - x_n : n \in \mathbb{N}\}) \geq 3$. **P** If $\langle x_n \rangle_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$ such that $\{x_{n+1} - x_n : n \in \mathbb{N}\} = \{\alpha, \beta\}$, then if $\alpha = \beta$ they are both 0 and the sequence is constant; if $\beta \neq 0$ and α/β is rational, then $\{x_n : n \in \mathbb{N}\}$ is finite; and if $\beta \neq 0$ and α/β is irrational, then $\{e^{2\pi i x_n/\beta} : n \in \mathbb{N}\}$ is dense in S^1, by Weyl's theorem again, so $\{x_n : n \in \mathbb{N}\}$ is somewhere dense. **Q**

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Reference

Erdős P., Hajnal A., Máté A. & Rado R. [84] *Combinatorial Set Theory: Partition Relations for Cardinals.* Akadémiai Kiadó, 1984 (Disquisitiones Math. Hung. 13).