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Covering $\mathbb{R}$ by translates

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1. Definitions Let (X, ρ) be a metric space. A set $A \subseteq X$ is **uniformly nwd** if for every $\epsilon > 0$ there is a $\delta > 0$ such that $B(x, \epsilon) \setminus A$ includes a ball of radius δ for every $x \in X$. A set $A \subseteq X$ is **thin** if $\bigcup_{n \in \mathbb{N}} f_n[A] \neq X$ for any sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ of lipeomorphisms of X.

2. Proposition Let (X, ρ) be a metric space.
(a) The union of two uniformly nwd sets is uniformly nwd.
(b) A totally bounded nowhere dense set is uniformly nwd.

3. Theorem If $(X, \| \|)$ is a separable Banach space and $\langle A_n \rangle_{n \in \mathbb{N}}$ is a sequence of uniformly nwd subsets of X, then there is a comeager set $H \subseteq X$ such that $H \cup \bigcup_{n \in \mathbb{N}} A_n$ is thin.

proof We may suppose that $\langle A_n \rangle_{n \in \mathbb{N}}$ is non-decreasing. Choose $\langle \alpha_n \rangle_{n \in \mathbb{N}}$, $\langle \beta_n \rangle_{n \in \mathbb{N}}$ inductively, as follows. $\alpha_0 = 1$. Given $\alpha_n > 0$, let $\beta_n > 0$ be such that $B(x, \frac{\alpha_n}{n+1}) \setminus A_n$ includes a ball of radius $(n+1)\beta_n$ for every $x \in X$; now take $\alpha_{n+1} \in]0, 2^{-n}[$ such that $\alpha_{n+1}(1 + \sum_{i=0}^{n}(i+1)) < \beta_n$. Let $\langle q_n \rangle_{n \in \mathbb{N}}$ run over a dense subset of X and set

$$H = \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} B(q_m, \alpha_m).$$

Suppose $\langle f_n \rangle_{n \in \mathbb{N}}$ is any sequence of lipeomorphisms of X. Then $X \neq \bigcup_{n \in \mathbb{N}} f_n[H \cup \bigcup_{i \in \mathbb{N}} A_i]$. **P** Since we can fill in gaps with the identity map, it is enough to consider the case in which f_n and f_n^{-1} are Lipschitz-$(n+1)$ for every n. Next, it is enough to consider the case in which for every n there is an $m > n$ such that $f_m = f_n$, in which case (because $\langle A_i \rangle_{i \in \mathbb{N}}$ is non-decreasing) $\bigcup_{n \in \mathbb{N}} f_n[H \cup \bigcup_{i \in \mathbb{N}} A_i] = \bigcup_{n \in \mathbb{N}} f_n[H \cup A_n]$.

Choose $\langle x_n \rangle_{n \in \mathbb{N}}$ inductively, as follows. Start with $x_0 = 0$. Given x_n, let y_n be such that $B(y_n, (n+1)\beta_n) \subseteq B(f_n^{-1}(x_n), \frac{\alpha_n}{n+1}) \setminus A_n$. In this case, $f_n[B(y_n, (n+1)\beta_n)] \subseteq B(x_n, \alpha_n)$, because f_n is Lipschitz-$(n+1)$; and of course $f_n[B(y_n, (n+1)\beta_n)] \cap f_n[A_n]$ is empty. Next, $B(f_n(y_n), \beta_n) \subseteq f_n[B(y_n, (n+1)\beta_n)]$ because f_n^{-1} is Lipschitz-$(n+1)$. So $B(f_n(y_n), \beta_n) \subseteq B(x_n, \alpha_n) \setminus f_n[A_n]$. At this point, observe that $\bigcup_{i \leq n} f_i[B(q_{n+1}, \alpha_{n+1}] \subseteq \bigcup_{i \leq n} B(f_i(q_{n+1}), (i+1)\alpha_{n+1})$ is covered by sets with diameters summing to at most $2\alpha_{n+1} \sum_{i=0}^{n}(i+1)$; because we are in a normed space, and $\beta_n > \alpha_{n+1}(1 + \sum_{i=0}^{n} i + 1)$, we can find an x_{n+1} such that $B(x_{n+1}, \alpha_{n+1}) \subseteq B(f_n(y_n), \beta_n) \setminus \bigcup_{i \leq n} f_i[B(q_{n+1}, \alpha_{n+1}]$. Continue.

As $B(x_{n+1}, \alpha_{n+1}) \subseteq B(x_n, \alpha_n)$ for every n and $\lim_{n \to \infty} \alpha_n = 0$, $z = \lim_{n \to \infty} x_n$ is defined in X and belongs to $B(x_n, \alpha_n)$ for every n.

As $B(x_{n+1}, \alpha_{n+1}) \subseteq B(f_n(y_n), \beta_n)$ is disjoint from $f_n[A_n]$ for each n, $z \notin \bigcup_{n \in \mathbb{N}} f_n[A_n]$. As $B(x_{n+1}, \alpha_{n+1})$ is disjoint from $f_i[B(q_{n+1}, \alpha_{n+1})]$ whenever $i \leq n$, $f_i[H] \subseteq \bigcup_{n > i} f_i[B(q_{n+1}, \alpha_{n+1})]$ ▮

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX does not contain z for any $i \in \mathbb{N}$. So z witnesses that $\bigcup_{n\in\mathbb{N}} f_n[H \cup A_n] \neq X$, which is what we needed to know. **Q**

Remark Of course this works in a lot of spaces besides Banach spaces, but we need something like 'for every $\beta > 0$, $n \in \mathbb{N}$ there is a $\gamma > 0$ such that no ball of radius β can be covered by n balls of radius γ'.

4. Corollary There is a Borel subset E of $\mathbb{R}$, neither meager nor Lebesgue negligible, such that no sequence of translates of E covers $\mathbb{R}$.

5. Remark A striking improvement on Corollary 4 is given by KUN P04: there is a non-decreasing sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ of Borel subsets of $\mathbb{R}$, with union $\mathbb{R}$, such that $E_n + A \neq \mathbb{R}$ for any $n \in \mathbb{N}$ and any countable set $A \subseteq \mathbb{R}$. [Set $E_n = \{x : \liminf_{k\to\infty} \frac{1}{k} p_k(x) \notin \,]0, \frac{1}{k}[\}$, where $p_k(x)$ is the number of zeroes in the first k digits of the binary expansion of the fractional part of x.]

6. Rearranging dissections The original question to which these ideas was addressed is the following. Given a set $E \subseteq \mathbb{R}$, under what circumstances will there be a partition $\langle E_n\rangle_{n\in\mathbb{N}}$ of E and a sequence $\langle t_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $\langle E_n + t_n\rangle_{n\in\mathbb{N}}$ is a partition of $\mathbb{R}$? Of course a necessary condition is that there should be a countable set $A \subseteq \mathbb{R}$ such that $E + A = \mathbb{R}$. In fact this is sufficient, as the following result shows.

7. Theorem Let X be a set, E and F two subsets of X, and G a group acting on X. For a subgroup H of G, let $\mathrm{Orb}(H)$ be the set of orbits of H (that is, the set of equivalence classes under the relation $\{(x, h{\bullet}x) : x \in X,\ h \in H\}$), and $\mathrm{Orb}'(H)$ the trace of $\mathrm{Orb}(H)$ on $E \cup F$. Suppose that

(i) there is a countable set $A \subseteq G$ such that $E \subseteq \bigcup_{g\in A} g{\bullet}F$ and $F \subseteq \bigcup_{g\in A} g{\bullet}E$;

(ii) for every countable subgroup H of G there is a countable subgroup H' of G, including H, such that $\mathrm{Orb}'(H) \cap \mathrm{Orb}'(H') = \emptyset$.

Then there are a partition $\langle E_n\rangle_{n\in\mathbb{N}}$ of E and a sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ in G such that $\langle g_n{\bullet}E_n\rangle_{n\in\mathbb{N}}$ is a partition of F.

proof Choose a non-decreasing sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of countable subgroups of G such that

$E \subseteq \bigcup_{g\in H_0} g{\bullet}F$ and $F \subseteq \bigcup_{g\in H_0} g{\bullet}E$ (using hypothesis (i)),

for every $n \in \mathbb{N}$, $\mathrm{Orb}'(H_{n+1}) \cap \mathrm{Orb}'(H_n) = \emptyset$ (using hypothesis (ii)).

Set $H = \bigcup_{n\in\mathbb{N}} H_n$, so that H is a countable subgroup of G. Now every orbit of H which meets $E \cup F$ meets E in an infinite set. **P** Let D be an orbit of H. Fixing $z \in D \cap (E \cup F)$, let D_n be the orbit of H_n containing z for each n. Then the sets $D_n \cap (E \cup F)$ are all different; say $z_n \in D_{n+1} \cap (E \cup F) \setminus D_n$ for each n. If $z_n \in E$, set $x_n = z_n$; otherwise, there is an $h_n \in H_0$ such that $z_n \in h_n{\bullet}E$; set $x_n = h_n^{-1}{\bullet}z_n$. In either case, we have $x_n \in D_{n+1} \cap E \setminus D_n$ for each n, so $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence of distinct elements of $D \cap E$. **Q**

Similarly, every orbit of H which meets $E \cup F$ meets F in an infinite set. Let $\mathcal{D}$ be the set of orbits of H meeting $E \cup F$. For each $D \in \mathcal{D}$, $D \cap E$ and $D \cap F$ are countably infinite, so there is a bijection $\phi_D : D \cap E \to D \cap F$. Set $\phi = \bigcup_{D\in\mathcal{D}} \phi_D$, so that $\phi : E \to F$ is a bijection. Let $\langle g_n\rangle_{n\in\mathbb{N}}$ be an enumeration of H, and set

$$E_n = \{x : x \in E,\ n = \min\{j : \phi x = g_j \bullet x\}\}.$$

Then $\langle E_n \rangle_{n\in\mathbb{N}}$ is a partition of E and $\langle g_n \bullet E_n \rangle_{n\in\mathbb{N}}$ is a partition of F.

8. Corollary If $E \subseteq \mathbb{R}$ is such that there is a countable set $A \subseteq \mathbb{R}$ with $E + A = \mathbb{R}$, then there is a partition $\langle E_n \rangle_{n\in\mathbb{N}}$ of E and a sequence $\langle t_n \rangle_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $\langle E_n + t_n \rangle_{n\in\mathbb{N}}$ is a partition of $\mathbb{R}$.

proof Apply Theorem 7 with $F = \mathbb{R}$ and $\bullet$ the (left) action of $\mathbb{R}$ on itself. ∎

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Reference

Kun G. [p04] ‘A surprising covering of the real line’, preprint, January 2004.