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Fair rents

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The problem Seven students are sharing a house with seven bedrooms. The total rent is fixed, and they have to agree on (i) the rent for each bedroom (ii) an allocation of the rooms. Is there a division of the rent, and an allocation of the rooms, such that every student is satisfied with what she has, in the sense that (at the given prices) she doesn't want to move to one of the other rooms?

Subject to appropriate conditions, the answer is 'yes'.

The model Suppose we have n bedrooms, where $n \geq 1$, and n students. The set of possible rent divisions can be represented as a set of vectors $\mathbf{p} = (p_1, \ldots, p_n)$ where each p_j is greater than or equal to 0, and $p_1 + \ldots + p_n = 1$; p_j is the proportion of the total rent assigned to room j. The set of possible room allocations can be represented by the set S_n of permutations of $\{1, \ldots, n\}$; $\rho \in S_n$ corresponds to the allocation in which each student i gets the room $\rho(i)$. For $1 \leq i, j \leq n$, let C_{ij} be the set of prices $\mathbf{p}$ under which student i would find room j acceptable, in the sense that no other room would be positively preferred at those prices.

The theorem Let P be the set of all real n-vectors $\mathbf{p}$ such that $p_1 + \ldots + p_n = 1$ and $p_j \geq 0$ for every j. For $1 \leq i, j \leq n$ let C_{ij} be a subset of P. Suppose that

(i) $C_{i1} \cup \ldots \cup C_{in} = P$ for every i;

(ii) every C_{ij} is closed.

Then there are $\mathbf{p} \in P$ and $\rho \in S_n$ such that

(†) $\mathbf{p} \in C_{i,\rho(i)}$ for every i such that $p_{\rho(i)} > 0$.

Interpretation of the clauses (i), (ii) and (†) (i) says just that for any student i, and for any price vector $\mathbf{p}$, there is at least one acceptable room; the student won't abandon the group and go to live elsewhere if given her choice of room at those prices.

(ii) says that if you have a price vector $\mathbf{p}$ which is *not* in C_{ij}, so that student i would definitely resent being given room j at that price, and you vary $\mathbf{p}$ to a nearly identical vector $\mathbf{p}'$, then she will still reject the room at the price $\mathbf{p}'$. As the price vector moves around P, we expect the student's preferred rooms to change; the assumption (ii) is that at the switch points she will accept any of the rooms which she would have accepted at prices close to the switch point.

As for the conclusion, (†) says that anyone who doesn't like the result can walk out without the others having to pay extra. If, for instance, one of the rooms has a dead rat under the floor, there may be no way of making the occupant happy; but we can ensure that at least the paying residents are content with what they've got.

A scrap of notation It will save space if I write

$$C'_{ij} = C_{ij} \cup \{\mathbf{p} : \mathbf{p} \in P,\, p_j = 0\}$$

for $i, j \leq n$. Note that C'_{ij} is closed, because C_{ij} is. Also (†) can now be re-phrased as '$\mathbf{p} \in C'_{i,\rho(i)}$ for every i'.

Strategy of the proof The proof is in two main stages. The first, algebraic-geometric, step is to show (using assumption (i) only) that, for any $\delta > 0$, we can find a permutation $\rho \in S_n$ and price vectors $\mathbf{p}^{(1)}, \ldots, \mathbf{p}^{(n)}$, all within δ of each other, such that $\mathbf{p}^{(i)} \in C'_{i,\rho(i)}$ for every i. Then a limiting argument, using assumption (ii), will show that we can find a fixed ρ and a single vector $\mathbf{p}$ such that $\mathbf{p} \in C'_{i,\rho(i)}$ for every i.

The distance between two price vectors I said above that I should be looking for price vectors 'within δ of each other'. This doesn't make sense without a notion of distance between members of P. Lots of formulae can be used here, and (for interesting reasons) it won't matter which we choose; but the pictures will be more helpful, and the calculations easier, if I say that the distance $d(\mathbf{p},\mathbf{q})$ between two price vectors $\mathbf{p}$ and $\mathbf{q}$ will be $\sqrt{\sum_{i=1}^{n}(p_i-q_i)^2}$, matching ordinary Euclidean distance in two or three dimensions.

Step 1: part A The set P can be thought of as an $n-1$-dimensional **simplex** with n **vertices**. The vertices are the n points at which we have $p_j = 1$ for some j, that is, all the rent is loaded onto one of the rooms; opposite each vertex is a **facet** on which $p_j = 0$; the facet is itself an $n-2$-dimensional simplex. In one dimension, that is, when $n = 2$, we have a line segment; in two dimensions ($n = 3$), a triangle; in three dimensions ($n = 4$) a tetrahedron. Our first task is to subdivide P into smaller simplexes, all of diameter at most δ, and simultaneously to assign each vertex of the subdivision to a student in such a way that every small simplex has every vertex assigned to a different student. (For the moment we are simply ignoring the students' preferences. Those will turn up in part B.) We can do this by the following procedure. Start with the subdivision $\mathcal{K}_0$ consisting of the simplex P alone, and assign each of the vertices of P to a different student. Next, given a subdivision $\mathcal{K}_r$ in which every simplex has all its vertices assigned to different students, take (one of) the longest edge(s) in the complex and divide it in three equal parts. Suppose the edge runs from $\mathbf{v}$ to $\mathbf{w}$; let $\mathbf{x}$ and $\mathbf{y}$ be the intermediate points, so that $\mathbf{v}$, $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{w}$ are in order along the edge. Assign the new vertex $\mathbf{x}$ to the same student who has vertex $\mathbf{w}$, and the new vertex $\mathbf{y}$ to the same student who has vertex $\mathbf{v}$. Take each simplex Q of the subdivision $\mathcal{K}_r$ which has $\mathbf{v}$ and $\mathbf{w}$ for two of its vertices and cut it into three smaller simplexes Q', Q'' and Q'''; if Q has vertices $\mathbf{v},\mathbf{w},\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(n-2)}$, then

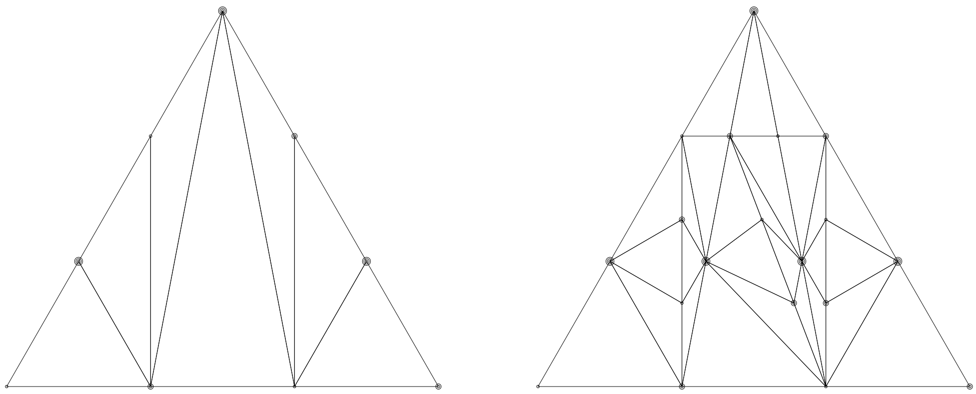
$$Q' \text{ will have vertices } \mathbf{v},\mathbf{x},\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(n-2)},$$

$$Q'' \text{ will have vertices } \mathbf{x},\mathbf{y},\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(n-2)},$$

$$Q''' \text{ will have vertices } \mathbf{y},\mathbf{w},\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(n-2)}.$$

In this way we obtain a subdivision $\mathcal{K}_{r+1}$ in which every student still owns just one vertex of each simplex. So long as we always split the longest edge at each stage, we can be sure that every edge will be split, and for large enough r all the simplexes of $\mathcal{K}_r$ will be very small[1].

I give diagrams of a couple of stages in the process if $n = 3$, so that P is an equilateral triangle. I have decorated the vertices to indicate their owners.



Step 1: part B Having got our subdivision $\mathcal{K}_r$ and an assignment of each vertex $\mathbf{v}$ of $\mathcal{K}_r$ to a *student* $f(\mathbf{v})$ where $1 \le f(\mathbf{v}) \le n$, choose a *room* $g(\mathbf{v})$ for each vertex, as follows. If $\mathbf{v}$ is in the interior of P, that is, if $v_j > 0$ for every j, then take $g(\mathbf{v})$ such that $\mathbf{v} \in C_{f(\mathbf{v}),g(\mathbf{v})}$; this is possible by condition (i). Otherwise, there must be at least one $j \le n$ such that $v_j = 0$ but $v_{j^+} > 0$, where $j^+ = j+1$ if $j < n$ and $j^+ = 1$ if $j = n$; take such a j for $g(\mathbf{v})$. Observe that this ensures that $\mathbf{v} \in C'_{f(\mathbf{v}),g(\mathbf{v})}$ for every $\mathbf{v}$. But it also ensures something else:

(‡) if $\mathbf{v}^*$ is a vertex of P, and $\mathbf{v}$ is a vertex of $\mathcal{K}_r$ lying in the facet of P opposite $\mathbf{v}^*$, then $g(\mathbf{v}) \ne g(\mathbf{v}^*)$.

(For there is a j such that $v^*_j = 1$, and in this case $g(\mathbf{v}^*) = j^-$, where $j^- = j-1$ if $j > 1$, n if $j = 1$; while $v_j = 0$ so $g(\mathbf{v}) \ne j^-$.)

[1] see Appendix 1

At this point I need to call on a remarkable fact which is the key to the whole proof.

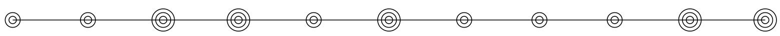
Sperner's Lemma (Sperner 28) Let P be an $n-1$-dimensional simplex, where $n \geq 2$, and $\mathcal{K}$ a subdivision of P; write V for the set of vertices of $\mathcal{K}$. Suppose we have a function $g : V \to \{1, \dots, n\}$ such that

($\ddagger$) whenever v^* is a vertex of P, and $v \in V$ lies in the facet of P opposite v^*, then $g(v) \neq g(v^*)$.

Let $\mathcal{L}$ be the set of simplexes K of $\mathcal{K}$ such that g takes a different value on each vertex of K. Then $\mathcal{L}$ has an odd number of members.

proof Induce on n.

base step If $n = 2$, so that P is just a line segment, then $\mathcal{K}$ must be just a dissection of P into intervals, and the vertices of $\mathcal{K}$ are points spread along P. Now we are told that g takes just the values 1 and 2 and that it takes different values at the two ends, as



So g must switch values an odd number of times (five, in the example here) as we move from one end to the other; and each switch corresponds to a member of $\mathcal{L}$.

inductive step Suppose that we know that the result is true for n, and that we are given an n-dimensional simplex P, with a subdivision $\mathcal{K}$ and a function from the vertices of $\mathcal{K}$ to $\{1, \dots, n+1\}$ satisfying the condition ($\ddagger$). Then whenever we have two vertices of P, each lies in the facet opposite the other, so g takes a different value on each vertex, and (because P has $n+1$ vertices) g takes every value just once. Let v^* be the vertex of P such that $g(v^*) = n+1$, and let P' be the facet of P opposite v^*. Then $\mathcal{K}$ traces out a subdivision $\mathcal{K}'$ of P' (the simplexes of $\mathcal{K}'$ are those facets of the simplexes of $\mathcal{K}$ which lie in P'). The vertices of $\mathcal{K}'$ are just the vertices of $\mathcal{K}$ which lie in P', and for every such vertex v we have $g(v) \neq n+1$, that is, $g(v) \leq n$. So the restriction of g to the set V' of vertices of $\mathcal{K}'$ is a function from V' to $\{1, \dots, n\}$. Moreover, if v is a vertex of P' and $v' \in V'$ lies in the facet of P' opposite v, then v is a vertex of P and $v \in V$ lies in the facet of P opposite v, so $g(v') \neq g(v)$; thus $g \restriction V'$ satisfies ($\ddagger$). By the inductive hypothesis, the set $\mathcal{L}'$ of simplexes in $\mathcal{K}'$ such that g takes a different value on each vertex, has an odd number of members.

Now turn back to the subdivision $\mathcal{K}$. Say that a facet of a simplex in $\mathcal{K}$ is a **door** if g takes all the values $1, \dots, n$ on the n vertices of that facet. Consider the set R of pairs (K, K') where K is a simplex of $\mathcal{K}$, K' is a facet of K and K' is a door. Given a door K',

— if it lies in the interior of P there are just two simplexes $K \in \mathcal{K}$ such that $(K, K') \in R$, one on each side of K';

— if it lies in the boundary of P then it is a facet of just one simplex of $\mathcal{K}$, and belongs to $\mathcal{L}'$. **P** For every vertex v of P', $g(v) \leq n$ is equal to $g(v')$ for some vertex v' of K', so v' cannot be in the facet of P opposite v. Thus the facet of P including K' must be P', and $K' \in \mathcal{L}'$. **Q**

What this shows is that $\#(R) \equiv \#(\mathcal{L}') \bmod 2$, and $\#(R)$ is odd. On the other hand, looking at the relation R from the other side, take a simplex K in $\mathcal{K}$. Then

— if g takes all the values $1, \dots, n+1$ on the vertices of K, that is, $K \in \mathcal{L}$, then exactly one facet of K is a door;

— if g takes the values $1, \dots, n$ on the vertices of K, but not the value $n+1$, then exactly two facets of K are doors (being the facets of K opposite the two vertices of K on which g takes the same value);

— in all other cases, K has no doors.

So $\#(R) \equiv \#(\mathcal{L}) \bmod 2$ and $\#(\mathcal{L})$ is odd. Thus the induction proceeds.

Step 1: part C Returning to the main argument, we see that the function g constructed in part B satisfies the conditions of Sperner's Lemma. So there must be an odd number of simplexes in $\mathcal{K}_r$ on whose vertices g takes every value; in particular, there is at least one; call it K. But now observe that (by the construction in part A) f also takes every value on the set V_K vertices of K. So $f\restriction V_K$ and $g\restriction V_K$ and $\rho = g\circ(f\restriction V_K)^{-1}$ are bijections; and if $i \le n$ and $\mathbf{p}^{(i)} \in V_K$ is such that $f(\mathbf{p}^{(i)}) = i$, then $\mathbf{p}^{(i)} \in C'_{i,\rho(i)}$. Thus we have price vectors $\mathbf{p}^{(1)}, \dots, \mathbf{p}^{(n)}$ as called for, and Step 1 is complete.

Step 2 We can perform Step 1 for any $\delta > 0$. In particular, suppose we have done it for $\delta = 2^{-k}$ for each k, obtaining permutations ρ_k and price vectors $\mathbf{p}^{(k,i)}$ such that

$$\mathbf{p}^{(k,i)} \in C'_{i,\rho_k(i)}, \quad d(\mathbf{p}^{(k,i)}, \mathbf{p}^{(k,j)}) \le 2^{-k}$$

whenever $k \in \mathbb{N}$ and $1 \le i \le n$. By the Bolzano-Weierstrass theorem[2] we can find a strictly increasing sequence $\langle k_l \rangle_{l\in\mathbb{N}}$ such that

$$\lim_{l\to\infty} p_j^{(k_l,i)} = p_j^{(i)} \text{ for all } i,\, j \le n,$$

$$\lim_{l\to\infty} \rho_{k_l}(i) = \rho(i) \text{ for all } i \le n.$$

Because

$$|p_j^{(k_l,i)} - p_j^{(k_l,i')}| \le d(\mathbf{p}^{(k_l,i)}, \mathbf{p}^{(k_l,i')}) \le 2^{-k_l} \le 2^{-l}$$

for all i, i', j and l, $p_j^{(i)} = p_j^{(i')} = p_j$ say for all i, i' and j, and we have a single limit price vector $\mathbf{p}$. Because every $\rho_{k_l}(i)$ is an integer, so is every $\rho(i)$, and there is an l_0 such that $\rho_{k_l}(i) = \rho(i)$ whenever $l \ge l_0$ and $i \le n$; that is, $\rho = \rho_{k_l} \in S_n$ for every $l \ge l_0$. This means that

$$\mathbf{p}^{(k_l,i)} \in C'_{i,\rho_l(i)} = C'_{i,\rho(i)}$$

for every $l \ge l_0$ and $i \le n$. But now recall hypothesis (ii) of the theorem: every C_{ij}, and therefore every C'_{ij}, is closed. So $\mathbf{p} = \lim_{l\to\infty} \mathbf{p}^{(k_l,i)} \in C'_{i,\rho(i)}$ for each i, and $\mathbf{p}$ and ρ satisfy (†).

Practicalities Unfortunately, while the theorem guarantees that a solution to the original problem exists, the proof does not give us a clear idea of how to find it. Step 1 is reasonably 'constructive'; that is, given $\delta > 0$, and provided that the students can decide what they want at any particular price without dithering, we can follow doors to find appropriate simplexes by asking at most one question for each vertex of $\mathcal{K}_r$, and generally much fewer. (We start on the edge of P joining the vertices at which $p_2 = 1$ and $p_3 = 1$, so that the values of g are 1 and 2, and look for a pair of adjacent vertices of $\mathcal{K}_r$ on which g changes. This now gives us a door at dimension 1 from which we can look for a triangle in the face $\{\mathbf{p} : p_j = 0$ unless $j = 2$, 3 or 4$\}$ on which g takes all three values 1, 2 and 3; this is a door at dimension 2 from which we look for a tetrahedron which will be a door at dimension 3, and so on.) The problem is that while this (together with the calculations in Appendix 1) gives us a way to specify in advance how much data we shall need in order to get an approximate solution $\mathbf{p}^{(1)}, \dots, \mathbf{p}^{(n)}$ of neighbouring vectors with $\mathbf{p}^{(i)} \in C'_{i,\rho(i)}$ for every i, it doesn't tell us anything about how the successive approximate solutions will be spread around P, and leaves open the possibility that the approximation we get for $\delta = 10^{-6}$ is half way across P from any approximation valid for $\delta = 10^{-7}$. So if your friends want to get a computer to allocate rooms for them, they had better agree in advance that they don't care about odd pennies, and that close enough will be good enough. On the other hand, the theorem does assure you that if after an evening of wrangling you seem nowhere near a solution acceptable to everyone, it's either because someone is being obstructive (by refusing to name a preferred room at some price), or because someone is jumping at trifles (by refusing to accept a limiting solution when they accepted the approximations), or because you're looking in the wrong part of P.

Something you might have noticed The argument here does *not* suppose that people are consistent in their choices. It allows someone to say, for instance, that at price vector $\mathbf{p}$ only room j will do, but that at price vector $\mathbf{q}$, with $q_j < p_j$ and $q_k > p_k$, only room k will do. Ordinarily this would seem very odd. But of course this is because we are talking of *monetary* rents, and (by definition) money is what you can't have too much of. If payments were in cabbages, you might feel that under certain circumstances you would be glad to have the landlord take some more.

For other 'envy-free allocation' problems see Su 99, from which most of the ideas above are cribbed.

[2] see Appendix 2.

Appendix 1: the subdivision process in step 1, part A I said that 'for large enough r all the simplexes of $\mathcal{K}_r$ will be very small'. Perhaps a little more explanation is called for. The problem is that when we move from $\mathcal{K}_r$ to $\mathcal{K}_{r+1}$, the original edge $(\mathbf{v},\mathbf{w})$ is divided into definitely shorter pieces, but we simultaneously introduce many new edges $(\mathbf{u},\mathbf{x})$ and $(\mathbf{u},\mathbf{y})$ without looking at their lengths, and we had better make sure that they too are usefully shorter than $(\mathbf{v},\mathbf{w})$. In fact, if the edge $(\mathbf{v},\mathbf{w})$ has length c, that is, $d(\mathbf{v},\mathbf{w}) = c$, then the new edges all have length at most $c\sqrt{\frac{7}{9}}$. **P** Of course

$$d(\mathbf{v},\mathbf{x}) = d(\mathbf{x},\mathbf{y}) = d(\mathbf{y},\mathbf{w}) = \frac{c}{3} \le c\sqrt{\tfrac{7}{9}}.$$

As for $d(\mathbf{u},\mathbf{x})$, consider the triangle $(\mathbf{u},\mathbf{v},\mathbf{w})$; set $a = d(\mathbf{u},\mathbf{v})$, $b = d(\mathbf{u},\mathbf{w})$, $B = \widehat{\mathbf{uvw}}$ and $s = d(\mathbf{u},\mathbf{x})$. Then the cos rule tells us that

$$s^2 = a^2 + \frac{c^2}{9} - \frac{2ac}{3}\cos B, \qquad b^2 = a^2 + c^2 - 2ac\cos B.$$

Eliminating $\cos B$, we get

$$s^2 = \frac{2}{3}a^2 + \frac{1}{3}b^2 - \frac{2}{9}c^2.$$

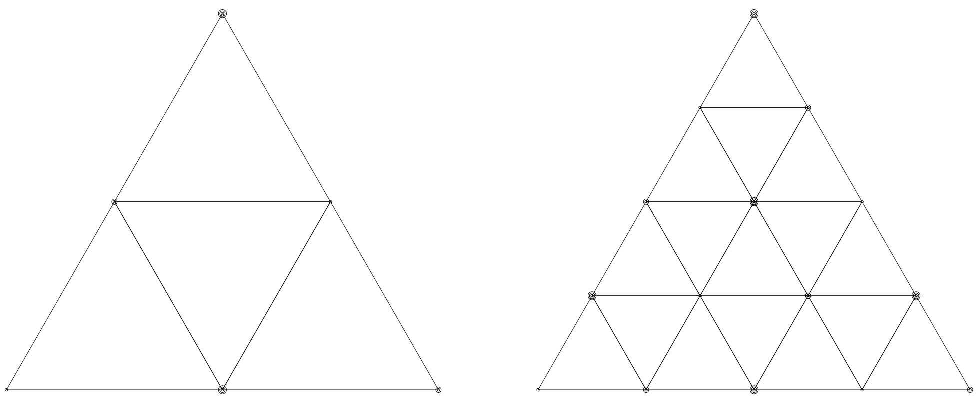
But a, $b \le c$, because c is (one of) the longest edge(s) of $\mathcal{K}_r$, so $s^2 \le \frac{7}{9}c^2$ and $s \le c\sqrt{\frac{7}{9}}$. Similarly, $d(\mathbf{u},\mathbf{y}) \le c\sqrt{\frac{7}{9}}$. **Q**

For each r, let c_r be the length of the longest edge in $\mathcal{K}_r$, so that $\langle c_r\rangle_{r\in\mathbb{N}}$ is a non-increasing sequence and has a limit c^* say. Then $c^* \le c_r\sqrt{\frac{7}{9}}$ for every r. **P** Let m be the number of edges of $\mathcal{K}_r$ of length greater than $c_r\sqrt{\frac{7}{9}}$. Then each of the moves $\mathcal{K}_r \to \mathcal{K}_{r+1} \to \mathcal{K}_{r+2} \to \dots$ will subdivide one of these edges until they have all gone (because all the new edges introduced will have length at most $c_r\sqrt{\frac{7}{9}}$). So after m moves, when we have reached $\mathcal{K}_{r+m}$, they will all have disappeared. Now $c^* \le c_{r+m} \le c_r\sqrt{\frac{7}{9}}$. **Q** As r is arbitrary, $c^* \le c^*\sqrt{\frac{7}{9}}$ and $c^* = 0$.

Remarks The following questions arise in this context.

1. The set P of price vectors, as described above, is a regular $n-1$-dimensional simplex; all its edges are the same length $\sqrt{2}$. The simplexes in $\mathcal{K}_r$ come in many different shapes; I believe that, as r increases, we get new shapes at each level (except, of course, in the trivial cases $n = 1$ and $n = 2$). What restrictions are there on the shapes of the simplexes which appear? It is easy to see that for any simplex K in any $\mathcal{K}_r$, the ratio of the longest side to the shortest side is at most 3. But is there (for given n) a non-zero lower bound to the angles of the triangles which appear?

2. Even if the process described here gives indefinitely thin triangles as the subdivision proceeds, is there some alternative way of subdividing a simplex into small simplexes for which thin triangles do not appear, but there is still a way to assign ownership of the vertices so that every small simplex is shared between all n owners? Note that if $n = 3$ this is easy, using equilateral triangles throughout, starting with



But this method doesn't work for a tetrahedron.

3. More advanced still, we have the following. Suppose we start with an equilateral triangle and use the trisection method to generate subdivisions $\mathcal{K}_r$ as above. For large r, we have an abundant collection of triangles, and we can ask for the *distribution* of their shapes. Here the 'shape' of a triangle is most naturally described by a triple (θ,ϕ,ψ) where $\theta\ge\phi\ge\psi\ge 0$ and $\theta+\phi+\psi=\pi$; the set of shapes is itself a triangle T in three-dimensional space. [Alternatively, we can describe it by a pair (a,b) such that $0\le b\le a\le 1\le a+b$, taking the side lengths of the triangle to be $1\ge a\ge b$. This is easier to generalize to higher dimensions.] I conjecture that there is a well-defined probability measure ν on T which is the limit, as $r\to\infty$, for the narrow topology, of the 'empirical' measures ν_r, where ν_r gives a mass to each point in T equal to the proportion of triangles in $\mathcal{K}_r$ with that shape.

Appendix 2: the Bolzano-Weierstrass theorem This is the following:

If $\langle\mathbf{x}^{(k)}\rangle_{k\in\mathbb{N}}$ is a sequence in m-dimensional Euclidean space which is **bounded** in the sense that there is some M such that $|x_i^{(k)}|\le M$ for every $k\in\mathbb{N}$ and every $i\le m$, then it has a subsequence $\langle\mathbf{x}^{(k_l)}\rangle_{l\in\mathbb{N}}$ which is **convergent** in the sense that $x_i=\lim_{l\to\infty}x_i^{(k_l)}$ is defined for every $i\le m$.

The standard proof is by induction on m; see almost any first course in analysis except my own. In the application here (Step 2 above), we need all the $p_j^{(k_l,i)}$ and all the $\rho_{k_l}(i)$ to converge, so we are working with $m=n^2+n$.

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