A singular Borel equivalence relation

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1. Monotone functions Let S be the set of non-decreasing continuous functions from $[0,1]$ to itself, and S_1 the set $\{f : f \in S,\ f(0)=0,\ f(1)=1\}$ of surjections belonging to S.

(a) S and S_1 are closed subsets of $C([0,1])$, so are complete metric spaces under $\| \|_\infty$.

(b) On S the norm topology defined by $\| \|_\infty$ agrees with the topology of pointwise convergence defined by the functionals $f \mapsto f(t) : S \to [0,1]$ for $t \in [0,1]$. **P** Of course the topology $\mathfrak{S}$ of pointwise convergence is coarser than the norm topology $\mathfrak{T}$. If $f_0 \in S$ and $\epsilon > 0$, let $n \ge 1$ be such that $|f_0(s) - f_0(t)| \le \frac{1}{3}\epsilon$ whenever s, $t \in [0,1]$ and $|s-t| \le \frac{1}{n}$. If $f \in S$ is such that $|f(\frac{i}{n}) - f_0(\frac{i}{n})| < \frac{1}{3}\epsilon$ for every $i \le n$, then $\|f - f_0\|_\infty < \epsilon$. Thus $\{f : \|f - f_0\}_\infty < \epsilon\}$ is a neighbourhood of f for $\mathfrak{S}$. As f_0 and ϵ are arbitrary, $\mathfrak{T}$ is coarser than $\mathfrak{S}$. **Q**

(b) The semigroup operation on S is continuous. **P** If f_0, $g_0 \in S$ and $\epsilon > 0$, let $\delta > 0$ be such that $|f_0(s) - f_0(t)| \le \epsilon$ whenever $|s-t| \le \delta$; then $\|fg - f_0g_0\|_\infty \le 2\epsilon$ whenever $\|f - f_0\| \le \epsilon$ and $\|g - g_0\| \le \delta$. **Q**

(c) The group G of order-isomorphisms of $[0,1]$ is a dense G_δ subset of S_1. **P** For q, $q' \in [0,1]$, $U_{qq'} = \{f : f \in S,\ f(q) < f(q')\}$ is open; now $f \in S_1$ belongs to G iff it is in $\bigcap_{q,q' \in \mathbb{Q} \cap [0,1], q<q'} U_{qq'}$. Thus G is a G_δ set. To see that it is dense, observe that any $f \in S_1$ is $\lim_{\epsilon \downarrow 0}(1-\epsilon)f + \epsilon f_0$ where $f_0(t) = t$ for $t \in [0,1]$. **Q**

(d) $f \mapsto f^{-1} : G \to G$ is continuous. **P** By (b), it is enough to check that $f \mapsto f^{-1}(t)$ is continuous for every $t \in [0,1]$. This is trivial if $t \in \{0,1\}$. Otherwise, given $f_0 \in G$ and $\epsilon > 0$, set $s_0 = \max(0, f_0^{-1}(t) - \epsilon)$, $s_1 = \min(1, f_0^{-1}(t) + \epsilon)$ and $\delta = \min(t - f_0(s_0), f_0(s_1) - t)$; then $|f^{-1}(t) - f_0^{-1}(t)| \le \epsilon$ whenever $f \in G$ and $\|f - f_0\|_\infty \le \delta$. **Q**

(e) Accordingly G is a Polish group.

Remark G is 'extremely amenable', that is, every continuous action of G on any compact Hausdorff space has a fixed point; see Pestov p03, §6.4, Corollary 8.

2. Lemma If $f \in S$, the following are equiveridical:

(i) f is **singular**, that is, $f' =_{\text{a.e.}} 0$;

(ii) for every $\epsilon > 0$ there are $a_0, b_0, a_1, b_1, \dots, a_n, b_n$ such that $0 \le a_0 \le b_0 \le \dots \le a_n \le b_n \le 1$, $\sum_{i=0}^n f(b_i) - f(a_i) \le \epsilon$ and $\sum_{i=0}^n b_i - a_i \ge 1 - \epsilon$;

(iii) there is a conegligible set $E \subseteq [0,1]$ such that $f[E]$ is negligible.

proof (i)⇒(iii) If $f' =_{\text{a.e.}} 0$, set $E = \{t : f'(t) = 0\}$ and $\phi = f{\restriction}E$. Then ϕ is differentiable with respect to its domain, with derivative 0, at every point of E. By Fremlin 01, 263D, $f[E] = \phi[E]$ is negligible.

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(iii)⇒(ii) If $E \subseteq [0,1]$ is a conegligible set such that $f[E]$ is negligible, and $\epsilon > 0$, let $F \subseteq E \setminus f[E]$ be a closed set of measure at least $1-\epsilon$ (Fremlin 00, 134Fb); then $F \cap f^{-1}[F] = \emptyset$. For $t \in [0,1]$, set $\delta(t) = [0,1] \setminus f^{-1}[F]$ if $t \in F$, $[0,1] \setminus F$ otherwise, so that $\delta(t)$ is always a neighbourhood of t. By Henstock's lemma (Fremlin 03, 481J), there is a Henstock partition of $[0,1]$ compatible with δ, that is, there are $c_0, \dots, c_m$ and $t_1, \dots, t_m$ such that $c_0 = 1$, $c_m = 1$ and

$$c_{i-1} \le t_i \le c_i, \qquad [c_{i-1}, c_i] \subseteq \delta(t_i)$$

for $1 \le i \le m$. Set $J = \{i : 1 \le i \le m,\ t_i \in F\}$ and let $\langle (a_i, b_i) \rangle_{i \le n}$ enumerate $\{(c_{i-1}, c_i) : i \in J\}$. Then $[a_i, b_i] \cap f^{-1}[F]$ is empty, so $[f(a_i), f(b_i)] \subseteq [0,1] \setminus F$, for each $i \in J$; accordingly

$$\textstyle\sum_{i=0}^n f(b_i) - f(a_i) \le \mu([0,1] \setminus F) \le \epsilon$$

(writing μ for Lebesgue measure). On the other hand, if $i \le m$ and $i \notin J$, then $[c_{i-1}, c_i]$ does not meet F. So

$$1 - \textstyle\sum_{i=0}^n b_i - a_i = \sum_{1 \le i \le m, i \notin J} c_i - c_{i-1} \le 1 - \mu F \le \epsilon.$$

As ϵ is arbitrary, (ii) is true.

(ii)⇒(i) If (ii) is true, let g be the absolutely continuous part of f (Fremlin 01, 226Cd), so that $g(0) = f(0)$ and $g' =_{\text{a.e.}} f'$. **?** If g is not constant, take $\epsilon = \frac{1}{3}(g(1) - g(0))$. Let $\delta > 0$ be such that $\int_E f' \le \epsilon$ whenever $\mu E \le \delta$ (Fremlin 01, 225A). Let $a_0, b_0, a_1, b_1, \dots, a_n, b_n$ be such that $0 \le a_0 \le b_0 \le \dots \le a_n \le b_n \le 1$, $\sum_{i=0}^n f(b_i) - f(a_i) \le \epsilon$ and $\sum_{i=0}^n b_i - a_i \ge 1 - \delta$. Then

$$g(1) - g(0) = g(a_0) - g(0) + \sum_{i=0}^n g(b_i) - g(a_i) + \sum_{i=1}^n g(a_i) - g(b_{i-1}) + g(1) - g(b_n)$$

$$= \int_E g + \sum_{i=0}^n g(b_i) - g(a_i)$$

(where $E = [0, a_0] \cup \bigcup_{1 \le i \le n} [b_{i-1}, a_i] \cup [b_n, 1]$)

$$\le \epsilon + \sum_{i=0}^n f(b_i) - f(a_i)$$

(because $\mu E = 1 - \sum_{i=1}^n b_i - a_i \le \delta$)

$$\le 2\epsilon < g(1) - g(0),$$

which is impossible. **X**

So g is constant and $f' =_{\text{a.e.}} 0$, as required.

3. Lemma (D.Preiss) For any $m \ge 1$, $\{f : f \in S_1,\ f^m \text{ is singular}\}$ is a comeager G_δ set in S_1.

proof (a) For $k \in \mathbb{N}$, let Q_k be the set of all pairs $(a, b) = ((\alpha_0, \dots, \alpha_n), (\beta_0, \dots, \beta_n))$ such that $0 \le \alpha_0 \le \beta_0 \le \alpha_1 \le \dots \le \beta_n \le 1$ and $\sum_{i=0}^n \beta_i - \alpha_i \ge 1 - 2^{-k}$. Set

$$H_k = \bigcup_{(\boldsymbol{a},\boldsymbol{b})\in Q_k} \{f : f \in S_1,\ \sum_{i=0}^n f(\beta_i) - f(\alpha_i) < 2^{-k}\},$$

so that H_k is an open set in S_1. Now $\{f : f^m \in H_k\}$ is dense in S_1. **P** If $f \in S_1$ and $\epsilon > 0$, let $\alpha_0, \dots, \alpha_n$ be such that $0 = \alpha_0 < \alpha_1 < \dots < \alpha_n = 1$ and $f(\alpha_{i+1}) - f(\alpha_i) \le \epsilon$ for every i. Set $\beta_i = (1 - 2^{-k})\alpha_{i+1} + 2^{-k}\alpha_i$ for each i. Set

$$\begin{aligned} g(t) &= f(\alpha_i) \text{ if } i < n \text{ and } \alpha_i \le t \le \beta_i, \\ &= \tfrac{t-\beta_i}{\alpha_{i+1}-\beta_i} f(\alpha_{i+1}) + \tfrac{\alpha_{i+1}-t}{\alpha_{i+1}-\beta_i} f(\alpha_i) \text{ if } i < n \text{ and } \beta_i \le t \le \alpha_{i+1}. \end{aligned}$$

Then $g \in S_1$ and $\|g - f\| \le \epsilon$. Also g, and therefore g^m, is constant on each interval $[\alpha_i, \beta_i]$, so $g^m \in H_k$. As f and ϵ are arbitrary, H_k is dense. **Q**

(b) Because $f \mapsto f^m$ is continuous, $\{f : f \in S,\ f^m \in H_k\}$ is open in S as well as dense. Now Lemma 2 tells us that $\{f : f^m \text{ is singular}\} = \bigcap_{k\in\mathbb{N}}\{f : f^m \in H_k\}$ is a comeager G_δ set.

Remark The case $m = 1$ is in ZAMFIRESCU 81.

4. Theorem (D.Preiss) $\{f : f \in G,\ f^m \text{ is singular for every } m \in \mathbb{Z}\setminus\{0\}\}$ is a comeager G_δ set in G.

proof For $m \ge 1$, Lemma 3 and 1c tell us that $\{f : f \in G,\ f^m \text{ is singular}\}$ is a comeager G_δ set in S_1 and therefore in G. So $\{f : f \in G,\ f^m \text{ is singular for every } m \ge 1\}$ is a comeager G_δ set in G. Since $f \mapsto f^{-1} : G \to G$ is a homeomorphism (1d), $\{f : f \in G,\ f^{-m} \text{ is singular for every } m \ge 1\}$ and $\{f : f \in G,\ f^m \text{ is singular for every } m \in \mathbb{Z} \setminus \{0\}\}$ are comeager G_δ sets.

5. Corollary There is an $f \in G$ such that $f(t) \ne t$ for $0 < t < 1$ and f^m is singular for every non-zero $m \in \mathbb{Z}$.

proof Because G is Polish, the comeager set V of functions with all non-trivial powers singular is non-empty; take $g \in V$. Certainly g is not the identity function, so there are $\alpha < \beta$ in $[0, 1]$ such that $g(\alpha) = \alpha$, $g(\beta) = \beta$ and $g(t) \ne t$ for $\alpha < t < \beta$. Set

$$f(t) = \tfrac{g(\beta t+\alpha(1-t))-\alpha}{\beta-\alpha}$$

for $t \in [0, 1]$; then

$$f^m(t) = \tfrac{g^m(\beta t+\alpha(1-t))-\alpha}{\beta-\alpha}$$

for every $m \in \mathbb{Z}$ and $t \in [0, 1]$, so f^m is singular for every non-zero m, while $f(t) \ne t$ for $0 < t < 1$.

6. A singular Borel equivalence relation Let f be a function as in Corollary 5, and let $W \subseteq \,]0, 1[^2$ be the corresponding orbit equivalence relation, so that $(s, t) \in W$ iff there is an $m \in \mathbb{Z}$ such that $f^m(s) = t$. For each $m \in \mathbb{Z} \setminus \{0\}$, there is a conegligible set $E_m \subseteq \,]0, 1[$ such that $f^m[E_m]$ is negligible. Setting $F = \bigcup_{m\in\mathbb{Z}\setminus\{0\}} f^m[E_m] \cup (]0, 1[\setminus E_m)$, we see that F is negligible and for every $m \in \mathbb{Z} \setminus \{0\}$, $t \in \,]0, 1[$ either $t \in F$ or $f(t) \in F$. Thus, writing Δ for the diagonal $\{(t, t) : t \in \,]0, 1[\}$,

$$W \setminus \Delta \subseteq (F \times]0,1[) \cup (]0,1[\times F).$$

On the other hand, W is non-trivial in the sense that $W[\{t\}]$ is infinite for every $t \in]0,1[$.

7. A question of H.Becker Let W be as in §6. If $h \subseteq W \setminus \Delta$ is a function such that $h(t) \neq t$ for $t \in \operatorname{dom} h$, then either $\operatorname{dom} h$ is negligible or $F_1 = \operatorname{dom} h \setminus F$ is a non-negligible set such that $h[F_1] \subseteq F$ is negligible. Thus if A, B are disjoint non-negligible subsets of $]0,1[$, W includes no function $h : A \to B$ which takes negligible sets to negligible sets.

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