SSP and wQN

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1 Definitions (a) A topological space X has the **SSP** if whenever $\langle\langle f_{ni}\rangle_{i\in\mathbb{N}}\rangle_{n\in\mathbb{N}}$ is a family in $C(X)$ such that $\lim_{i\to\infty} f_{ni}(x)=0$ for every $x\in X$ and $n\in\mathbb{N}$, then there is a sequence $\langle i_n\rangle_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} f_{n,i_n}(x)=0$ for every $x\in X$.

(b) A topological space X is **wQN** ('weakly quasi-normal') if whenever $\langle f_i\rangle_{i\in\mathbb{N}}$ is a sequence in $C(X)$ such that $\lim_{i\to\infty} f_i(x)=0$ for every $x\in X$, then there is a strictly increasing sequence $\langle i_n\rangle_{n\in\mathbb{N}}$ such that $\{n:|f_{i_n}(x)|\ge 2^{-n}\}$ is finite for every $x\in X$.

(c) A topological space X is an $\mathbf{s_1}$**-space** (Fremlin 94) if whenever $A\subseteq C(X)$ and $\langle\langle f_{ni}\rangle_{i\in\mathbb{N}}\rangle_{n\in\mathbb{N}}$, $\langle f_n\rangle_{n\in\mathbb{N}}$, f are such that

$$f_{ni}\in A \text{ for all } n,\ i\in\mathbb{N},\ f_n\in C(X) \text{ for every } n\in\mathbb{N},\ f\in C(X),$$

$$\lim_{i\to\infty} f_{ni}(x)=f_n(x) \text{ for all } n\in\mathbb{N},\ x\in X,$$

$$\lim_{n\to\infty} f_n(x)=f(x) \text{ for all } x\in X,$$

then there is a sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ in A such that $\langle g_n(x)\rangle_{n\in\mathbb{N}}\to f(x)$ for all $x\in X$.

2 Proposition For a topological space X, the following are equiveridical:
(i) X is wQN;
(ii) X has the SSP;
(iii) X is an s_1-space.

proof (i)⇒(ii) Suppose that X is wQN. Let $\langle\langle f_{ni}\rangle_{i\in\mathbb{N}}\rangle_{n\in\mathbb{N}}$ be a family in $C(X)$ such that $\lim_{i\to\infty} f_{ni}(x)=0$ for every $x\in X$ and $n\in\mathbb{N}$. Set

$$g_i(x)=\sum_{n=0}^\infty \min(\frac{1}{2^n},\max(0,2^n|f_{ni}(x)|-1)).$$

Then each $g_i:X\to[0,2]$ is continuous and $\lim_{i\to\infty} g_i(x)=0$ for every $x\in X$. Because X is wQN, there is a strictly increasing sequence $\langle i_n\rangle_{n\in\mathbb{N}}$ such that $\{n:g_{i_n}(x)\ge 2^{-n}\}$ is finite for every x. But now observe that, for any n, $i\in\mathbb{N}$ and $x\in X$,

$$g_i(x)<2^{-n}\Rightarrow 2^n|f_{ni}(x)|-1<2^{-n}\Rightarrow |f_{ni}(x)|\le\frac{1+2^{-n}}{2^n}.$$

So $\lim_{n\to\infty} f_{n,i_n}(x)=0$ for every x. As $\langle f_{ni}\rangle_{n,i\in\mathbb{N}}$ is arbitrary, X has the SSP.

(ii)⇒(iii) (Bukovský, Recław & Repický 01[1]) Suppose that X has the SSP. Let A, $\langle f_{ni}\rangle_{n,i\in\mathbb{N}}$, $\langle f_n\rangle_{n\in\mathbb{N}}$, f be as in 1(c). Then $\lim_{i\to\infty} f_{ni}(x)-f_n(x)=0$ for every n and x, so there is a sequence $\langle i_n\rangle_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} f_{n,i_n}(x)-f_n(x)=0$ for every x. Now

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$\langle f_{n,i_n}\rangle_{n\in\mathbb{N}}$ is a sequence in A and $\lim_{n\to\infty} f_{n,i_n}(x) = f(x)$ for every x. As A, $\langle f_{ni}\rangle_{n,i\in\mathbb{N}}$, $\langle f_n\rangle_{n\in\mathbb{N}}$ and f are arbitrary, X is an s_1-space.

(iii)⇒(i) (Bukovský, Recław & Repický 01[2]) Suppose that X is an s_1-space. Let $\langle f_i\rangle_{i\in\mathbb{N}}$ be a sequence in $C(X)$ such that $\lim_{i\to\infty} f_i(x) = 0$ for every $x \in X$. Set $h_{ni}(x) = 2^n|f_i(x)| + 2^{-i} + 2^{-n}$, $h_n(x) = 2^{-n}$ for n, $i \in \mathbb{N}$ and $x \in X$, and $A = \{h_{ni} : n, i \in \mathbb{N}\}$. Then $\lim_{i\to\infty} h_{ni} = h_n$ for every n and $\lim_{n\to\infty} h_n = 0$, so there is a sequence $\langle g_k\rangle_{k\in\mathbb{N}}$ in A such that $\lim_{k\to\infty} g_k(x) = 0$ for every x. Express g_k as $h_{n_k j_k}$; as $h_{n_k j_k}(x) \geq 2^{-j_k} + 2^{-n_k}$ for every x and k, $\lim_{k\to\infty} j_k = \lim_{k\to\infty} n_k = \infty$; there is therefore a strictly increasing sequence $\langle k_r\rangle_{r\in\mathbb{N}}$ such that $\langle j_{k_r}\rangle_{r\in\mathbb{N}}$ is strictly increasing and $n_{k_r} \geq r$ for every r. Set $i_r = j_{k_r}$; then $g_{k_r}(x) \geq 2^r|f_{i_r}(x)|$ for every x and r, while $\lim_{r\to\infty} g_{k_r}(x) = 0$. So $\{r : |f_{i_r}(x)| \geq 2^{-r}\}$ is finite for every x. As $\langle f_i\rangle_{i\in\mathbb{N}}$ is arbitrary, X is wQN.

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References

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