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Compact sets of semi-continuous functions

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1. Definition Let X be a topological space. Write $\mathrm{LSC}(X)$ for the space of lower semi-continuous real-valued functions on X.

2. Proposition Let X be a regular countably compact topological space and $A \subseteq \mathrm{LSC}(X)$ a set which is relatively countably compact in $\mathrm{LSC}(X)$ for the pointwise topology. Then A is relatively compact in $\mathrm{LSC}(X)$ for the pointwise topology.

proof We can use the standard argument for the corresponding result for $C(X)$; see, e.g., Fremlin 03, §462.

The closure $\overline{A}$ of A in $\mathbb{R}^X$ is compact. **?** Suppose, if possible, that $\overline{A} \not\subseteq \mathrm{LSC}(X)$; take $g \in \overline{A} \setminus \mathrm{LSC}(X)$. Take $x^* \in X$ and $\epsilon > 0$ such that for every neighbourhood U of x^* in X there is a point $x \in U$ such that $g(x^*) \geq g(x) + \epsilon$.

Choose sequences $\langle f_i \rangle_{i\in\mathbb{N}}$ in A, $\langle x_i \rangle_{i\in\mathbb{N}}$ in X and a sequence $\langle U_i \rangle_{i\in\mathbb{N}}$ of neighbourhoods of x^*, as follows. Given $\langle f_i \rangle_{i<m}$ and $\langle x_i \rangle_{i<m}$, choose a neighbourhood U_m of x^* such that $f_i(x) \geq f_i(x^*) - 2^{-m}$ for every $i < m$ and $x \in \overline{U}_m$ (this is where we need to suppose that X is regular); take $x_m \in \bigcap_{i\leq m} U_i$ such that $g(x^*) \geq g(x_m) + \epsilon$. Now choose $f_m \in A$ such that $|f_m(x^*) - g(x^*)| \leq 2^{-m}$ and $|f_m(x_i) - g(x_i)| \leq 2^{-m}$ for every $i \leq m$. Continue.

At the end of the induction, take a cluster point x of $\langle x_i \rangle_{i\in\mathbb{N}}$ in X and a cluster point f of $\langle f_i \rangle_{i\in\mathbb{N}}$ in $\mathrm{LSC}(X)$. Because $x_i \in U_m$ whenever $i \geq m$, $x \in \overline{U}_m$ for every m, $f_i(x) \geq f_i(x^*)$ for every i, and $f(x) \geq f(x^*)$. Because $|f_m(x^*) - g(x^*)| \leq 2^{-m}$ for every m, $f(x^*) = g(x^*)$. Because $|f_m(x_i) - g(x_i)| \leq 2^{-m}$ whenever $i \leq m$, $f(x_i) = g(x_i)$ for every i. So

$$f(x) \geq f(x^*) = g(x^*) \geq g(x_i) + \epsilon = f(x_i) + \epsilon$$

for every i, and the open set $\{y : f(y) > f(x) - \epsilon\}$ contains x but none of the x_i. **X**

Thus the compact set $\overline{A} \subseteq \mathrm{LSC}(X)$ is the closure of A in $\mathrm{LSC}(X)$, and A is relatively compact in $\mathrm{LSC}(X)$.

Remark The requirement 'X is regular' does not appear in the result for $C(X)$; but of course there is a coarser regular topology on X with the same continuous real-valued functions, so we are not really dealing with a restricted class of spaces X.

3. Proposition For every compact Hausdorff space Z there is a compact Hausdorff space X such that Z is homeomorphic to a subspace of $\mathrm{LSC}(X)$ with the pointwise topology.

proof We may suppose that $Z \subseteq [0,1]^I$ for some set I. Let $X = I \cup \{\infty\}$ be the one-point compactification of I with its discrete topology, and define $T : Z \to \mathrm{LSC}(X)$ by setting $(Tz)(i) = z(i)$, $(Tz)(\infty) = 0$ for $z \in Z$, $i \in I$; then $T[Z] \subseteq \mathrm{LSC}(X)$ and T is a homeomorphism between Z and $T[Z]$.

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4. Theorem Let X be a countably compact regular space and $K \subseteq \mathrm{LSC}(X)$ a pointwise compact set. Then the convex hull $\Gamma(K)$ of K is relatively compact in $\mathrm{LSC}(X)$.

proof By Proposition 2, it is enough to show that every countable subset A of $\Gamma(K)$ is relatively compact in $\mathrm{LSC}(X)$. Given such an A, let $B \subseteq K$ be a countable set such that $A \subseteq \Gamma(B)$, and set $C = \overline{B} \subseteq K$; let $\overline{\Gamma(C)}$ be the closed convex hull of C taken in $\mathbb{R}^X$, so that $\overline{\Gamma(C)}$ is compact.

? If $g \in \overline{\Gamma(C)}$ is not lower semi-continuous, let $z \in X$, $\alpha \in \mathbb{R}$ be such that $g(z) > \alpha$ and $z \in \overline{\{x : g(x) \leq \alpha\}}$. We can express g as the barycentre of a Radon probability measure μ on C (Fremlin 03, 461H), so that $g(x) = \int f(x)\mu(df)$ for every $x \in X$. Let $\langle f_i \rangle_{i \in \mathbb{N}}$ be a sequence running over B. For each $n \in \mathbb{N}$ let U_n be a closed neighbourhood of z such that $f_i(x) \geq f_i(z) - 2^{-n}$ for every $i \leq n$ and $x \in U_n$; choose $x_n \in \bigcap_{j \leq n} U_j$ such that $g(x_n) \leq \alpha$. Let $x \in X$ be a cluster point of $\langle x_n \rangle_{n \in \mathbb{N}}$. Then $x \in U_n$ for every n so $f_i(x) \geq f_i(z)$ for every $i \in \mathbb{N}$; by the choice of $\langle f_i \rangle_{i \in \mathbb{N}}$, $f(x) \geq f(z)$ for every $f \in C$, and

$$\int_C f(x)\mu(df) \geq \int_C f(z)\mu(df) = g(z) > \alpha.$$

However $f(x) \leq \liminf_{i \to \infty} f(x_i)$ for every $f \in C \subseteq \mathrm{LSC}(X)$, so by Fatou's Lemma

$$\int_C f(x)\mu(df) \leq \liminf_{i \to \infty} \int_C f(x_i)\mu(df) = \liminf_{i \to \infty} g(x_i) \leq \alpha,$$

which is absurd. **X**

Thus $\overline{\Gamma(C)}$ is a compact subset of $\mathrm{LSC}(X)$ including A and A is relatively compact. As A is arbitrary, $\Gamma(K)$ is relatively compact.

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References

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