The Generalized Banach Contraction Theorem

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I present a simplified proof of the Generalized Banach Contraction Theorem from Arvanitakis p02.

1. Definitions (a) For $m \geq 1$, a subset I of $\mathbb{N}$ is **m-syndetic** if $I \cap [k, k+m[$ is non-empty for every $k \in \mathbb{N}$.

(b) If (X, ρ) is a metric space, $m \geq 1$ and $\gamma \in [0,1[$, a function $f : X \to X$ is an **(m,γ)-g-contraction** if for any x, $y \in X$ there is an i with $1 \leq i \leq m$ such that $\rho(f^i(x), f^i(y)) \leq \gamma^i \rho(x,y)$.

(c) For a set $I \subseteq \mathbb{N}$, its **upper Banach density** is

$$\begin{aligned} d_s^*(I) &= \sup\{\nu P : \nu : \mathcal{P}\mathbb{N} \to [0,1] \text{ is a translation-invariant additive functional} \\ &\qquad\qquad \text{with } \nu\mathbb{N} = 1\} \\ &= \inf_{m \geq 1} \sup_{k \in \mathbb{N}} \frac{1}{m} \#(I \cap [k, k+m[) \end{aligned}$$

(Fremlin n02). Note that d_s^* is subadditive. If I is m-syndetic then $\bigcup_{i<m} I - i$ covers $\mathbb{N}$ so $d_s^*(I) \geq \frac{1}{m}$.

2. Lemma If (X, ρ) is a metric space and $f : X \to X$ is an (m,γ)-g-contraction then for any x, $y \in X$ the set $I = \{i : i \in \mathbb{N},\ \rho(f^i(x), f^i(y)) \leq \gamma^i \rho(x,y)$ is m-syndetic.

proof The point is just that $0 \in I$ and that if $i \in I$ there is a $j \in [1,m]$ such that $i + j \in I$.

3. Lemma Let (X, ρ) be a metric space and $f : X \to X$ an (m,γ)-g-contraction. If $x \in X$ is such that $f^n(x) = x$ for some $n \geq 1$, then $f(x) = x$.

proof We may suppose that n is the first integer greater than or equal to 1 such that $f^n(x) = x$. **?** If $n \geq 2$, set

$$\begin{aligned} \delta &= \min\{\rho(f^i(x), f^j(x)) : 0 \leq i < j < n\} \\ &= \min\{\rho(f^i(x), f^j(x)) : i,\ j \in \mathbb{N},\ i - j \text{ is not a multiple of } n\}. \end{aligned}$$

Take i, j such that $0 \leq i < j < n$ and $\rho(f^i(x), f^j(x)) = \delta$. Then there is a $k \in [1,m]$ such that $\rho(f^{k+i}(x), f^{k+j}(x)) \leq \gamma^k \delta < \delta$, which is impossible. **X** So $n = 1$ and $f(x) = x$.

4. Lemma (Arvanitakis p02, 4.2) Let $R \subseteq \mathbb{N} \times \mathbb{N}$ be such that

(i) $\{i : (i, 0) \in R\}$ is m-syndetic, where $m \geq 1$;
(ii) whenever $(i, j) \in R$ then $\{k : (i + k, j + k) \in R\}$ is m-syndetic.

Then there is an $I \subseteq \mathbb{N}$ such that $d_s^*(I) \geq \frac{1}{2m^2}$ and for every i, $j \in I$ there is a $k \in \mathbb{N}$ such that $(k, i) \in R$ and $(k, j) \in R$.

proof (a) Note first that R meets $[l, l + 2m[\cap [k, k + m[$ whenever $k \leq l$. **P** There is an $i \in [l - k, l - k + m[$ such that $(i, 0) \in R$, and now there is a $j \in [k, k + m[$ such that $(i + j, j) \in R$. **Q**

(b) Let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$. For each $i \in \mathbb{N}$ and $j < 2m$, set $L_{ij} = \{l : (l + j, i) \in R\}$, and for $j < 2m$ set $I_j = \{i : L_{ij} \in \mathcal{F}\}$. Observe that if $k \in \mathbb{N}$ then $\bigcup_{i \in [k,k+m[, j<2m} L_{ij} \supseteq \mathbb{N} \setminus k$ so there are some $i < m$, $j < 2m$ such that $L_{ij} \in \mathcal{F}$. This shows that $\bigcup_{j<2m} I_j$ is m-syndetic and there is some $j < 2m$ such that $d_s^*(I_j) \geq \frac{1}{2m^2}$.

(c) If i, $i' \in I_j$, then L_{ij} and $L_{i'j}$ both belong to $\mathcal{F}$, so cannot be disjoint, and there is some l such that $(l + j, i)$ and $(l + j, i')$ both belong to R. So we can take $I = I_j$.

5. Lemma (Merryfield Rothschild & Stein 01) If (X, ρ) is a metric space, $f : X \to X$ is an (m, γ)-g-contraction and $x \in X$, then there is an $M \geq 0$ such that $I = \{i : \rho(f^i(x), x) \leq M\}$ is m-syndetic.

proof Set $M = \frac{1}{1-\gamma} \max_{i<m} \rho(f^i(x), x)$. Then $0 \in I$. If $i \in I$, there is a $j \in [1, m[$ such that $\rho(f^{j+i}(x), f^j(x)) \leq \gamma^j \rho(f^i(x), x) \leq \gamma M$, and now

$$\rho(f^{j+i}(x), x) \leq \gamma M + \rho(f^j(x), x) \leq \gamma M + (1 - \gamma)M = M,$$

so $j + i \in I$.

6. Lemma (Arvanitakis p02, 4.1) Let (X, ρ) be a metric space, $f : X \to X$ an (m, γ)-g-contraction and $x \in X$. Then there is a set $I \subseteq \mathbb{N}$ such that $d_s^*(I) > 0$ and $\langle f^i(x) \rangle_{i \in I}$ is Cauchy, in the sense that for any $\epsilon > 0$ there is an $n \in \mathbb{N}$ such that $\rho(f^i(x), f^j(x)) \leq \epsilon$ whenever i, $j \in I \setminus n$.

proof By Lemma 5, we have an $M \geq 0$ such that $\{i : \rho(f^i(x), x) \leq M\}$ is m-syndetic. Let $R \subseteq \mathbb{N} \times \mathbb{N}$ be the relation

$$\{(i, j) : \rho(f^i(x), f^j(x)) \leq M\gamma^j\}.$$

By the choice of M, $\{i : (i, 0) \in R\}$ is m-syndetic. If $(i, j) \in R$, then

$$\{k : (i + k, j + k) \in R\} \supseteq \{k : \rho(f^{i+k}(x), f^{j+k}(x)) \leq \gamma^k \rho(f^i(x), f^j(x))\}$$

is m-syndetic by Lemma 2. By Lemma 4, there is a set $I \subseteq \mathbb{N}$ with $d_s^*(I) > 0$ such that for all i, $j \in I$ there is a $k \in \mathbb{N}$ such that (k, i) and (k, j) both belong to R. In this case,

$$\rho(f^i(x), f^j(x)) \leq \rho(f^k(x), f^i(x)) + \rho(f^k(x), f^j(x)) \leq M\gamma^i + M\gamma^j \to 0,$$

as i, $j \to \infty$, so $\langle f^i(x) \rangle_{i \in I}$ is Cauchy.

7. Generalized Banach Contraction Theorem (Arvanitakis p02) Let (X, ρ) be a complete metric space and $f : X \to X$ an (m, γ)-g-contraction. Then f has a fixed point.

proof (a) Let $x \in X$. By Lemma 6, there is a set $I \subseteq \mathbb{N}$ such that $d_s^*(I) > 0$ and $\langle f^i(x)\rangle_{i\in I}$ is Cauchy. Fix a translation-invariant additive functional $\nu : \mathcal{P}\mathbb{N} \to [0,1]$ such that $\nu I > 0$. Because X is complete, $z = \lim_{i\to I} f^i(x)$ is defined in X. For $j \in \mathbb{N}$, $\delta > 0$ set $I_{\delta j} = \{i : \rho(f^i(x), f^j(z)) \le \delta\}$. Observe that if $\delta > 0$ then $I \setminus I_{\delta 0}$ is finite, so $\nu I_{\delta 0} \ge \nu I > 0$.

(b) Suppose that $k \in \mathbb{N}$ and $\delta > 0$. Then $\sum_{j=k}^{k+m-1} \nu I_{\delta j} \ge \nu I$. **P** For any $i \in I_{\delta 0}$, the set

$$\{j : i + j \in I_{\delta j}\} = \{j : \rho(f^{i+j}(x), f^j(z)) \le \delta\} \supseteq \{j : \rho(f^{i+j}(x), f^j(z)) \le \gamma^j \rho(f^i(x), z)\}$$

is m-syndetic (Lemma 1), so meets $[k, k+m[$. For $j \in [k, k+m[$ set $A_j = \{i : i \in I_{\delta 0}, i + j \in I_{\delta j}\}$. Then $\langle A_j\rangle_{j\in[k,k+m[}$ covers $I_{\delta 0}$ and $A_j + j \subseteq I_{\delta j}$ for each j, so

$$\sum_{j=k}^{k+m-1} \nu I_{\delta j} \ge \sum_{j=k}^{k+m-1} \nu(A_j + j) = \sum_{j=k}^{k+m-1} \nu A_j$$

(because ν is translation-invariant)

$$\ge \nu I_{\delta 0} \ge \nu I. \text{ } \mathbf{Q}$$

(c) Now observe that if i, j are such that $f^i(z) \ne f^j(z)$, $I_{\delta i} \cap I_{\delta j} = \emptyset$ for all sufficiently small δ. But this means that if $n > \frac{1}{\nu I}$ then $z, f(z), f^2(z), \dots, f^{mn-1}(z)$ cannot all be distinct. By Lemma 3, f has a fixed point.

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References

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