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Well-distributed sequences and Banach density

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1. Banach limits and Banach density

1A Write P for the set of all non-negative additive functionals $\nu : \mathcal{P}\mathbb{N} \to [0,1]$ such that $\nu\mathbb{N} = 1$ and ν is translation-invariant, that is, $\nu\{n+k : n \in I\} = \nu I$ for every $k \in \mathbb{N}$ and every $I \subseteq \mathbb{N}$. For $I \subseteq \mathbb{N}$, the **upper Banach density** of I is $d_s^*(I) = \sup_{\nu\in P} \nu I$. Because P is closed in $[0,1]^{\mathcal{P}\mathbb{N}}$, therefore compact, the supremum is always attained. d_s^* is a submeasure. If νI is the same for every $\nu \in P$, this common value is the **Banach density** $d_s(I)$. Note that, for any I, $\inf_{\nu\in P} \nu I = 1 - d_s^*(\mathbb{N}\setminus I)$; so $d_s(I)$ is defined iff $d_s^*(I) + d_s^*(\mathbb{N}\setminus I) = 1$.

If $\theta : \mathcal{P}\mathbb{N} \to [0,1]$ is any additive functional such that $\theta\mathbb{N} = 1$, and $\nu_0 \in P$, and we set $\nu I = \int\theta(I+j)\nu_0(dj)$ for $I \subseteq \mathbb{N}$ (following the notation of Fremlin 02, 363Lf), then $\nu \in P$. Similarly, if we write $I -' j$ for $\{i : i \in \mathbb{N},\ i+j \in I\}$, and $\nu' I = \int\theta(I -' j)\nu_0(dj)$ for $I \subseteq \mathbb{N}$, we again obtain a member ν' of P.

1B Definition For x, $y \in \mathbb{R}^{\mathbb{N}}$ set $(x*y)(n) = \sum_{i=0}^{n} x(i)y(n-i) = \sum_{i+j=n} x(i)y(j)$ for $n \in \mathbb{N}$. Then $*$ is bilinear, commutative and associative, and $\|x*y\|_1 \le \|x\|_1\|y\|_1$, $\|x*y\|_\infty \le \|x\|_1\|y\|_\infty$ for all x, $y \in \mathbb{R}^{\mathbb{N}}$.

If $x \in \mathbb{R}^{\mathbb{N}}$ its **variation** $\mathrm{Var}_{\mathbb{N}}(x)$ is $\sum_{i=0}^{\infty} |x(i+1) - x(i)|$.

1C Lemma If $I \subseteq \mathbb{N}$,

$$\begin{aligned}
d_s^*(I) &= \inf\{\|x*\chi I\|_\infty : x \in (\ell^1)^+,\ \|x\|_1 = 1\}\\
&= \inf_{m\ge 1}\sup_{k\in\mathbb{N}} \frac{1}{m}\#(I\cap[k,k+m[)\\
&= \inf_{m\ge 1}\limsup_{k\to\infty} \frac{1}{m}\#(I\cap[k,k+m[)\\
&= \lim_{m\to\infty}\sup_{k\in\mathbb{N}} \frac{1}{m}\#(I\cap[k,k+m[)\\
&= \lim_{m\to\infty}\sup_{k\in\mathbb{N}} \frac{1}{m}\#(I\cap[mk,mk+m[)\\
&= \inf_{\delta>0}\sup\{\sum_{i\in I} x(i) : x \in (\ell^1)^+,\ \|x\|_1 = 1,\ \mathrm{Var}_{\mathbb{N}}(x) \le \delta\}.
\end{aligned}$$

Remark See 3B below for a more general result.

proof Set

$$\gamma_1 = \inf\{\|x*\chi I\|_\infty : x \in (\ell^1)^+,\ \|x\|_1 = 1\},$$

$$\gamma_2 = \inf_{m\ge 1}\sup_{k\in\mathbb{N}} \frac{1}{m}\#(I\cap[k,k+m[),$$

$$\hat\gamma_2 = \inf_{m\ge 1}\limsup_{k\to\infty} \frac{1}{m}\#(I\cap[k,k+m[),$$

$$\gamma_2' = \limsup_{m\to\infty}\sup_{k\in\mathbb{N}} \frac{1}{m}\#(I\cap[k,k+m[),$$

$$\gamma_3 = \inf_{m\ge 1}\sup_{k\in\mathbb{N}} \frac{1}{m}\#(I\cap[mk,mk+m[),$$

$$\gamma_3' = \limsup_{m\to\infty}\sup_{k\in\mathbb{N}} \frac{1}{m}\#(I\cap[mk,mk+m[),$$

$$\gamma_4 = \inf_{\delta>0} \sup\{\sum_{i\in I} x(i) : x \in (\ell^1)^+,\ \|x\|_1 = 1,\ \mathrm{Var}_{\mathbb{N}}(x) \le \delta\}.$$

Note that $\gamma_2' = \limsup_{m\to\infty} \|x_m * \chi I\|_\infty$, where $x_m = \dfrac{1}{m}\chi m$ for $m \ge 1$.

(a) $d_s^*(I) \le \hat\gamma_2$. **P** Suppose that $\nu \in P$. Then

$$\nu I = \frac{1}{m}\sum_{j=0}^{m-1} \nu(I+j) = \int x_m * \chi I \, d\nu \le \limsup_{k\to\infty} (x_m * \chi I)(k)$$

$$= \limsup_{k\to\infty} \frac{1}{m}\#(\{n : n \in I,\ k \le n < k+m\}).$$

This is true for every m, so $\nu I \le \hat\gamma_2$; as ν is arbitrary, $d_s^*(I) \le \hat\gamma_2$. **Q**

(b) Of course $\hat\gamma_2 \le \gamma_2 \le \gamma_3 \le \gamma_3' \le \gamma_2'$.

(c) $\gamma_2' \le \gamma_1$. **P** Take $\epsilon > 0$. There is an $x \in (\ell^1)^+$ such that $\|x\|_1 \le 1$ and $\|x * \chi I\|_\infty \le \gamma_1 + \epsilon$. There is a $y \in (\ell^1)^+$ such that $\|y\|_1 = 1$, $\{i : y(i) \ne 0\}$ is finite and $\|x - y\|_1 \le \epsilon$. Let $r \ge 1$ be such that $y(i) = 0$ for $i \ge r$. Then for any $m \ge 1$ we have

$$|(x_m * y - x_m)(k)| = |\sum_{i+j=k} x_m(i)y(j) - x_m(k)| = 0 \text{ if } r \le k \le m \text{ or } k \ge m+r$$

$$\le \frac{1}{m} \text{ otherwise.}$$

So $\|x_m * y - x_m\|_1 \le \dfrac{2r}{m}$, and $\|x_m * y * \chi I - x_m * \chi I\|_\infty \le \dfrac{2r}{m}$. Also

$$\|x_m * x * \chi I - x_m * y * \chi I\|_\infty \le \|x_m * x - x_m * y\|_1 \le \|x - y\|_1 \le \epsilon,$$

so $\|x_m * x * \chi I - x_m * \chi I\|_\infty \le \epsilon + \dfrac{2r}{m}$. Accordingly

$$\|x_m * \chi I\|_\infty \le \epsilon + \frac{2r}{m} + \|x_m * x * \chi I\|_\infty \le \epsilon + \frac{2r}{m} + \|x * \chi I\|_\infty \le \gamma_1 + 2\epsilon + \frac{2r}{m}.$$

Letting $m \to \infty$, $\gamma_2' \le \gamma_1 + 2\epsilon$; as ϵ is arbitrary, $\gamma_2' \le \gamma_1$. **Q**

(d) $\gamma_1 \le d_s^*(I)$. **P** Let $\epsilon > 0$. Suppose that $i_0, \dots, i_r \in \mathbb{N}$. Set $x(i) = \frac{1}{r+1}\#(\{j : i = i_j\})$ for $i \in \mathbb{N}$, so that $x \in (\ell^1)^+$ and $\|x\|_1 = 1$. Then $\|x * \chi I\|_\infty \ge \gamma_1$, so there is an $n \in \mathbb{N}$ such that $(x * \chi I)(n) \ge \gamma_1 - \epsilon$, that is, $\#(\{j : n - i_j \in I\}) \ge (r+1)(\gamma_1 - \epsilon)$, that is, $\#(\{j : n \in I + i_j\}) \ge (r+1)(\gamma_1 - \epsilon)$. As $i_0, \dots, i_r$ are arbitrary, there is an additive functional $\theta : \mathcal{P}\mathbb{N} \to [0,1]$ such that $\theta\mathbb{N} = 1$ and $\theta(I + j) \ge \gamma_1 - \epsilon$ for every $j \in \mathbb{N}$ (Fremlin 02, 391F). Take any $\nu_0 \in P$ and set $\nu J = \int \theta(J + i)\nu_0(di)$ for $J \subseteq \mathbb{N}$. Then $\nu \in P$, and

$$\gamma_1 - \epsilon \le \inf_{j\in\mathbb{N}} \theta(I + j) \le \nu I \le d_s^*(I).$$

As ϵ is arbitrary, $\gamma_1 \le d_s^*(I)$. **Q**

(e) So $d_s^*(I) = \gamma_1 = \gamma_2 = \gamma_2' = \gamma_3 = \gamma_3'$. But as γ_2 and γ_2' are equal, they must both be $\lim_{m\to\infty} \sup_{k\in\mathbb{N}} \dfrac{1}{m}\#(I \cap [k, k+m[)$; and similarly both γ_3 and γ_3' are equal to $\lim_{m\to\infty} \sup_{k\in\mathbb{N}} \dfrac{1}{m}\#(I \cap [mk, mk+m[)$.

(f) $\gamma_2 \le \gamma_4$. **P** For $m \ge 1$, $k \in \mathbb{N}$ set $y_{mk}(i) = \dfrac{1}{m}$ if $k \le i < k+m$, 0 otherwise. Then $y_{mk} \in (\ell^1)^+$, $\|y_{mk}\|_1 = 1$ and $\mathrm{Var}_{\mathbb{N}}(y_{mk}) \le \dfrac{2}{m}$. Also $\gamma_2 = \inf_{m\in\mathbb{N}} \sup_{k\in\mathbb{N}} \sum_{i\in I} y_{mk}(i)$. So $\gamma_2 \le \gamma_4$. **Q**

(g) $\gamma_4 \le d_s^*(I)$. **P?** Otherwise, we can find a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in $(\ell^1)^+$ such that $\|x_n\|_1 = 1$ for every n, $\lim_{n\to\infty} \mathrm{Var}_{\mathbb{N}}(x_n) = 0$ and $\lim_{n\to\infty} \sum_{i\in I} x_n(i) > d_s^*(I)$. Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$, and define $\nu : \mathcal{P}\mathbb{N} \to [0,1]$ by setting $\nu J = \lim_{n\to\mathcal{F}} \sum_{i\in J} x_n(j)$ for every $J \subseteq \mathbb{N}$. Of course ν is a non-negative additive functional and $\nu\mathbb{N} = 1$. If $J \subseteq \mathbb{N}$, then

$$|\nu(J+1)-\nu J| = |\lim_{n\to\mathcal{F}}\sum_{i\in J+1}x_n(i)-\sum_{i\in J}x_n(i)| = |\lim_{n\to\mathcal{F}}\sum_{i\in J}x_n(i+1)-\sum_{i\in J}x_n(i)|$$
$$\le \lim_{n\to\mathcal{F}}\sum_{i\in J}|x_n(i+1)-x_n(i)| \le \lim_{n\to\mathcal{V}}\operatorname*{Var}_{\mathbb{N}}(x_n)=0.$$

So $\nu\in P$. But $\nu I=\lim_{n\to\infty}\sum_{i\in I}x_n(i)>d_s^*(I)$.

1D Corollary (a) If $I\subseteq\mathbb{N}$ and $m\ge 1$, then $d_s^*(I)\le\frac{1}{m}\limsup_{l\to\infty}\#(\{i:i\in I,\,ml\le i<m(l+1)\})$.

(b) If $I\subseteq\mathbb{N}$, $m\ge 1$ are such that $k=\lim_{l\to\infty}\#(\{i:i\in I,\,ml\le i<m(l+1)\})$ is defined, then $d_s(I)$ is defined and equal to $\frac{k}{m}$.

proof (a) Set $k=\limsup_{l\to\infty}\#(\{i:i\in I,\,ml\le i<m(l+1)\})$. Then there is an $l_0\in\mathbb{N}$ such that $\#(\{i:i\in I,\,ml\le i<m(l+1)\})\le k$ for every $l\ge l_0$. Let $r\ge l_0$. Set $x_r=\frac{1}{rm}\chi(rm)$, where in $\chi(rm)$ I am interpreting rm as the set of its predecessors in $\mathbb{N}$. Then $(x_r*\chi I)(n)=\frac{1}{rm}\#(I\cap]n-rm,n])\le\frac{m(l_0+2)+kr}{mr}$ for every n. (The interval $\mathbb{N}\cap]n-rm,n]$ must consist of at most r intervals of the form $[ml,m(l+1)[$, with $l\ge l_0$, together with at most $m(l_0+1)$ points at the left and m points at the right.) So $d_s^*(I)\le\|x_r*\chi I\|_\infty\le\frac{l_0+2}{r}+\frac{k}{m}$. Letting $r\to\infty$, we have the result.

(b) In this case, $d_s^*(I)\le\frac{k}{m}$ and $d_s^*(\mathbb{N}\setminus I)\le\frac{m-k}{m}$, so $d_s(I)$ is defined and equal to $\frac{k}{m}$.

1E Corollary (a) If $I\subseteq\mathbb{N}$ then $d_s^*(I)=\inf\{d_s(J):I\subseteq J$ and $d_s(J)$ is defined$\}$.

(b) If I, $I'\subseteq\mathbb{N}$ are disjoint and $d_s^*(I)+d_s^*(I')<1$, then there is a J such that $I\subseteq J\subseteq\mathbb{N}\setminus I'$ and $d_s(J)$ is defined.

proof (a) Take any $\gamma>d_s^*(I)$. Then there is an $m\ge 1$ such that $\#(I\cap[k,k+m[)\le\gamma m$ for every k. Let $J\subseteq\mathbb{N}$ be such that $I\subseteq J$ and $\#(\{n:n\in J,\,ml\le n<m(l+1)\})=\lfloor\gamma m\rfloor$ for every l. Then $d_s(J)$ is defined and $d_s(J)\le\gamma$. As γ is arbitrary, we have the result.

(b) This time, take γ such that $d_s^*(I)<\gamma<1-d_s^*(I')$, and $m\ge 1$ such that $\#(I\cap[k,k+m[)\le\gamma m$ and $\#(I'\cap[k,k+m[)\le(1-\gamma)m$ for every k. Let $J\subseteq\mathbb{N}$ be such that $I\subseteq J\subseteq\mathbb{N}\setminus I'$ and $\#(J\cap[ml,m(l+1)[)=\lfloor\gamma m\rfloor$ for every l; this works.

1F Corollary Suppose that $I\subseteq\mathbb{N}$ is infinite and that J is any subset of $\mathbb{N}$. Let $\langle n_i\rangle_{i\in\mathbb{N}}$ be the increasing enumeration of I, and set $M=\{n_i:i\in J\}$.

(a) $d_s^*(M)\le d_s^*(I)d_s^*(J)$.

(b) If I has Banach density, then $d_s^*(M)=d_s(I)d_s^*(J)$.

(c) If J has Banach density, then $d_s^*(M)=d_s^*(I)d_s(J)$.

(d) If both I and J have Banach density, then so has M, and $d_s(M)=d_s(I)d_s(J)$.

proof (a) Take any $\alpha'>\alpha>d_s^*(I)$ and $\beta>d_s^*(J)$. Then there is an $m_0\ge 1$ such that $\#(I\cap[k,k+m[)\le\alpha m$ and $\#(J\cap[k,k+m[)\le\beta m$ whenever $m\ge m_0$ and $k\in\mathbb{N}$. Set $f(k)=\min\{i:n_i\ge k\}$ for $k\in\mathbb{N}$. Then

$$f(k')-f(k)=\#(I\cap[k,k'[)\le\alpha(k'-k)$$

whenever $k+m_0\le k'$. Let $m_1\ge m_0$ be such that $(\alpha'-\alpha)m_0\le m_1$. Then

$$\#(M\cap[k,k+m[)=\#(\{i:i\in J,\,k\le n_i<k+m\})=\#(J\cap[f(k),f(k+m)[)$$
$$\le\beta\max(m_0,f(k+m)-f(k))\le\beta\max(m_0,\alpha m)\le\beta\alpha' m$$

whenever $m\ge m_1$ and $k\in\mathbb{N}$. Accordingly $d_s^*(M)\le\alpha'\beta$; as α, α' and β are arbitrary, $d_s^*(M)\le d_s^*(I)d_s^*(J)$.

(b) Of course $d_s^*(M)\le d_s^*(I)d_s^*(J)=d_s(I)d_s^*(J)$. Take any $\alpha<d_s(I)$ and $\beta<d_s(J)$. Then $d_s^*(\mathbb{N}\setminus I)<1-\alpha$ so there is an $m_0\in\mathbb{N}$ such that $\#(I\cap[k,k+m[)\ge\alpha m$ for every $k\in\mathbb{N}$ and $m\ge m_0$. Take any $m\ge m_0$; then

$$f(k+m)-f(k)=\#(I\cap[k,k+m[)\ge\alpha m$$

for every k. Next, $f[\mathbb{N}]=\mathbb{N}$, so there is a k such that

$$\begin{aligned}\#(M\cap[k,k+m[)&=\#(J\cap[f(k),f(k)+m[)\ge\beta m\\&\ge\beta(f(k+m)-f(k))\ge\beta\alpha m.\end{aligned}$$

Thus $\sup_{k\in\mathbb{N}}\frac{1}{m}\#(J\cap[k,k+m[)\ge\alpha\beta$. This is true for every $m\ge m_0$, so $d_s^*(M)\ge\alpha\beta$; as α and β are arbitrary, $d_s^*(M)\ge d_s(I)d_s^*(J)$ and we have equality.

(c) This time, $d_s^*(M)\le d_s^*(I)d_s^*(J)=d_s^*(I)d_s(J)$. Set $J'=\mathbb{N}\setminus J$, $M'=\{n_i:i\in J'\}$. Then $d_s^*(M')\le d_s^*(I)d_s(J')=d_s^*(I)(1-d_s(J'))$. Since $M\cup M'=I$, $d_s^*(I)\le d_s^*(M)+d_s^*(M')$; putting these together, we must have $d_s^*(M)=d_s^*(I)d_s(J)$ exactly.

(d) Applying (c) to I and $J'=\mathbb{N}\setminus J$, we see that $d_s^*(I\setminus M)=d_s(I)(1-d_s(J))$. Now M, $I\setminus M$ and $\mathbb{N}\setminus I$ cover $\mathbb{N}$ and their upper Banach densities sum to 1, so they must all have Banach densities.

1G Remarks (a) Writing $\mathcal{D}_s$ for the domain of d_s,

$$\mathbb{N}\in\mathcal{D}_s,\quad\text{if } I,\,J\in\mathcal{D}_s \text{ and } I\subseteq J\text{ then } J\setminus I\in\mathcal{D}_s,\quad\emptyset\in\mathcal{D}_s,$$

$$\text{if } I,\,J\in\mathcal{D}_s\text{ and }I\cap J=\emptyset\text{ then }I\cup J\in\mathcal{D}_s\text{ and }d_s(I\cup J)=d_s(I)+d_s(J).$$

It follows that if $\mathcal{I}\subseteq\mathcal{D}_s$ and $I\cap J\in\mathcal{I}$ for all I, $J\in\mathcal{I}$, then the subalgebra of $\mathcal{P}\mathbb{N}$ generated by $\mathcal{I}$ is included in $\mathcal{D}_s$ (Fremlin 02, 313Ga). But note that $\mathcal{D}_s$ itself is *not* a subalgebra of $\mathcal{P}\mathbb{N}$.

(b) Writing d^* for upper asymptotic density, d for density, $\mathcal{Z}\lhd\mathcal{P}\mathbb{N}$ for the asymptotic density ideal and $\mathcal{D}\subseteq\mathcal{P}\mathbb{N}$ for the domain of d (Fremlin 03, §491), we have $d^*(a)\le d_s^*(I)$ for every $I\subseteq\mathbb{N}$. **P**

$$\begin{aligned}d^*(a)&=\limsup_{m\to\infty}\frac{1}{m}\#(a\cap m)\\&\le\limsup_{m\to\infty}\sup_{k\in\mathbb{N}}\frac{1}{m}\#(\{n:n\in I,\,k\le n<k+m\}=d_s^*(a).\mathbf{Q}\end{aligned}$$

So $\mathcal{Z}_s\subseteq\mathcal{Z}$, $\mathcal{D}\supseteq\mathcal{D}_s$ and d extends d_s.

1H Proposition Let $\nu:\mathcal{P}\mathbb{N}\to[0,1]$ be an additive functional such that $\nu\mathbb{N}=1$. Then the following are equiveridical:
(i) $\nu\in P$;
(ii) $\nu I\le d_s^*(I)$ for every $I\subseteq\mathbb{N}$;
(iii) $\nu I=d_s(I)$ whenever $I\in\mathcal{D}_s$.

proof (i)⇒(ii) and **(ii)⇒(iii)** are trivial.

(iii)⇒(ii) is immediate from 1E(a).

(ii)&(iii)⇒(i) Assume that (ii) and (iii) are both true, and take any $I\subseteq\mathbb{N}$. Set $K_0=\{2i:i\in\mathbb{N}\}$, $K_1=\mathbb{N}\setminus K_0$. Then $\nu K_0=\nu K_1=\frac{1}{2}$.

? If $\nu I>\nu(I+1)$ then there is a j such that $\nu(I\cap K_j)>\nu((I\cap K_j)+1)$. Set $j'=1-j$, so that $(I\cap K_j)+1=(I+1)\cap K_{j'}$ and $\nu(I\cap K_j)>\nu((I+1)\cap K_{j'})$. Set $J=(I\cap K_j)\cup(K_{j'}\setminus(I+1))$. Then $\nu J>\frac{1}{2}$. But for any $l\in\mathbb{N}$ and $m\ge1$,

$$\begin{aligned}\#(J\cap[l,l+m[)&=\#((I\cap K_j)\cap[l,l+m[)+\#(K_{j'}\cap[l,l+m[)\\&\qquad-\#(((I\cap K_j)+1)\cap[l,l+m[)\\&\le1+\#(K_{j'}\cap[l,l+m[),\end{aligned}$$

so $d_s^*(J)\le\frac{1}{2}$. **X**

? If $\nu I<\nu(I+1)$ then there is a j such that $\nu(I\cap K_j)<\nu((I\cap K_j)+k)$. Set $j'=1-j$, $J=((I+1)\cap K_{j'})\cup(K_j\setminus I)$. Then $\nu J>\frac{1}{2}$. But for any $l\in\mathbb{N}$ and $m\ge1$,

$$\begin{aligned}\#(J\cap[l,l+m[) &= \#(((I\cap K_j)+k)\cap[l,l+m[)+\#(K_j\cap[l,l+m[)\\ &\qquad -\#((I\cap K_j)\cap[l,l+m[)\\ &\le 1+\#(K_j\cap[l,l+m[),\end{aligned}$$

so $d_s^*(J)\le\frac{1}{2}$. **X**

So $\nu I=\nu(I+1)$; as I is arbitrary, (i) is true.

1I Construction For $K\in[\mathbb{N}]^{<\omega}$, set $n_K=\sum_{i\in K}2^i$. Then $K\mapsto n_K$ is a bijection from $[\mathbb{N}]^{<\omega}$ to $\mathbb{N}$. For any set $I\subseteq\mathbb{N}$, set $A_I=\{n_K:K\in[\mathbb{N}]^{<\omega},\ \#(I\cap K)\text{ is even}\}$. Now if $I_0,\dots,I_r\subseteq\mathbb{N}$ are infinite and almost disjoint, there is an $m\in\mathbb{N}$ such that $\#(\{n:n\in\bigcap_{j\le r}A_{I_j},\ 2^ml\le n<2^m(l+1)\})=2^{m-r-1}$ for every $l\in\mathbb{N}$. **P** For $j\le r$, take $i_j\in I_j\setminus\bigcup_{k\le r,k\ne j}I_k$. Set $m=1+\max_{j\le r}i_j$. If $l\in\mathbb{N}$, then $2^ml=n_L$ where $L\cap m=\emptyset$. Set $M=\{i:i<m,\ i\ne i_j\text{ for every }j\le r\}$. Then

$$\begin{aligned}&\{n:n\in\bigcap_{j\le r}a_{I_j},\ 2^ml\le n<2^m(l+1)\}\\ &\quad=\{n_K:K\in[\mathbb{N}]^{<\omega},\ K\setminus m=L,\ \#(K\cap I_j)\text{ is even for every }j\le r\}\\ &\quad=\bigcup_{M'\subseteq M}\bigcap_{j\le r}\{n_K:K\cap M=M',\ K\setminus m=L,\ i_j\in K\text{ iff }\#((M'\cup L)\cap I_j)\text{ is odd}\}\end{aligned}$$

has $2^{\#(M)}=2^{m-r-1}$ members. **Q**

So if we take an almost disjoint family $\langle I_\xi\rangle_{\xi<\mathfrak{c}}$ of infinite subsets of $\mathbb{N}$ and set $A_\xi=A_{I_\xi}$ for every ξ, then $\langle A_\xi\rangle_{\xi<\mathfrak{c}}$ will have the property that whenever $\xi_0,\dots,\xi_r<\mathfrak{c}$ are distinct, there is an $m\in\mathbb{N}$ such that $\#(\{n:n\in\bigcap_{j\le r}A_{\xi_j},\ 2^ml\le n<2^m(l+1)\})=2^{m-r-1}$ for every $l\in\mathbb{N}$, so that $A_{\xi_0}\cap\dots\cap A_{\xi_r}$ has Banach density 2^{-r-1}.

1J Theorem Take any $\nu\in P$, and let μ be the corresponding measure on $\beta\mathbb{N}$. Then μ is Maharam homogeneous, with Maharam type $\mathfrak{c}$.

proof Writing $\widehat{A}_\xi$ for the open-and-closed subset of $\beta\mathbb{N}$ corresponding to A_ξ as defined in 1I, the family $\langle\widehat{A}_\xi\rangle_{\xi<\mathfrak{c}}$ is, with respect to μ, a stochastically independent family of cardinal $\mathfrak{c}$. So the homogeneous probability algebra $\mathfrak{B}_\mathfrak{c}$ of Maharam type $\mathfrak{c}$ is isomorphic to a subalgebra of the measure algebra $\mathfrak{A}$ of μ. At the same time, because $\beta\mathbb{N}$ has weight $\mathfrak{c}$, $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathfrak{B}_\mathfrak{c}$ (Fremlin 02, 332N). So $\mathfrak{A}$ and $\mathfrak{B}_c$ are isomorphic (Fremlin 02, 332Q).

1K Theorem Suppose that $I\subseteq\mathbb{N}$ and $d_s^*(I)>0$. Then for any finite set $J\subseteq\mathbb{N}$ there are $k\in\mathbb{N}$, $l\ge1$ such that $k+lJ\subseteq I$. In particular, I includes arbitrarily long arithmetic progressions.

proof Set $\epsilon=\frac{1}{2}d_s^*(I)$; suppose that $r\ge2$ is such that $J\subseteq r$. By Szemerédi's theorem (Szemerédi 75, or Fremlin 03, 497L[1]) there is an $m_0\ge1$ such that whenever $m\ge m_0$, $A\subseteq m$ and $\#(A)\ge\epsilon m$ there is an arithmetic progression of length r in A. Let $m\ge m_0$, $k\in\mathbb{N}$ be such that $\frac{1}{m}\#(I\cap[k,k+m[)\ge\epsilon$, and consider $A=(I\cap[k,k+m[)-m$; then there is an arithmetic progression of length r in A, so there is an arithmetic progression of length r in $A+m$, that is, there are $k\in\mathbb{N}$, $l\ge1$ such that $k+li\in A+m\subseteq I$ for every $i<r$, in which case of course $k+lJ\subseteq I$.

1L Banach density on $\mathbb{Z}$ (a) We can translate 1C into a result about subsets of $\mathbb{Z}$ if we make the following changes. First, let $P_\mathbb{Z}$ be the set of translation-invariant non-negative additive functionals $\nu:\mathcal{P}\mathbb{Z}\to[0,1]$ such that $\nu\mathbb{Z}=1$, and for $I\subseteq\mathbb{Z}$ set $d_s^*(I)=\sup_{\nu\in P_\mathbb{Z}}\nu I$. (It is easy to check that this agrees with the definition in 1A if $I\subseteq\mathbb{N}$.) Now, for any $I\subseteq\mathbb{Z}$,

[1] Later editions only.

$$\begin{aligned}
d_s^*(I) &= \inf\{\|x * \chi I\|_\infty : x \in \ell^1(\mathbb{Z})^+,\ \|x\|_1 = 1\}\\
&= \inf_{m\ge 1}\sup_{k\in\mathbb{Z}}\frac{1}{m}\#(I\cap[k,k+m[)\\
&= \inf_{m\ge 1}\limsup_{|k|\to\infty}\frac{1}{m}\#(I\cap[k,k+m[)\\
&= \lim_{m\to\infty}\sup_{k\in\mathbb{Z}}\frac{1}{m}\#(I\cap[k,k+m[)\\
&= \lim_{m\to\infty}\sup_{k\in\mathbb{Z}}\frac{1}{m}\#(I\cap[mk,mk+m[)\\
&= \inf_{\delta>0}\sup\{\sum_{i\in I}x(i) : x\in\ell^1(\mathbb{Z})^+,\ \|x\|_1=1,\ \operatorname*{Var}_{\mathbb{Z}}(x)\le\delta\}
\end{aligned}$$

(use the arguments of 1C, nearly unchanged). It is worth noting that

$$d_s^*(I) = \max(d_s^*(I\cap\mathbb{N}), d_s^*((-I)\cap\mathbb{N}))$$

(using the new version

$$\inf_{m\ge1}\limsup_{|k|\to\infty}\frac{1}{m}\#(I\cap[k,k+m])$$

of $\hat\gamma_2$ in the proof of 1C, or otherwise).

(b) Following Solecki 05, we have a further characterization: for any set $I\subseteq\mathbb{Z}$,

$$d_s^*(I) = \inf_{J\in[\mathbb{Z}]^{<\omega},J\ne\emptyset}\sup_{k\in\mathbb{Z}}\frac{\#((I+k)\cap J}{\#(J)}.$$

P Set

$$\gamma_5 = \inf_{J\in[\mathbb{Z}]^{<\omega},J\ne\emptyset}\sup_{k\in\mathbb{Z}}\frac{\#((I+k)\cap J}{\#(J)}.$$

Then

$$\begin{aligned}
\inf\{\|x * \chi I\|_\infty : x\in\ell^1(\mathbb{Z})^+,\ \|x\|_1=1\} &\le \gamma_6\\
&\le \inf_{x\in\ell^1(\mathbb{Z})^+,\|x\|_1=1}\sup_{k\in\mathbb{Z}}\sum_{i\in I}x(i+k). \ \textbf{Q}
\end{aligned}$$

1M Translation-invariant functionals in $(\ell^\infty)^*$ **(a)** In the L-space $(\ell^\infty)^*$ (see Fremlin 02, 356N), we can consider the set V of functionals f such that $f(Tx)=f(x)$ for every $x\in\ell^\infty$, where $(Tx)(i)=x(i{+}1)$ for $x\in\ell^\infty$ and $i\in\mathbb{N}$. This is a weak*-closed Riesz subspace of $(\ell^\infty)^*$. **P** V is a linear subspace just because $T:\ell^\infty\to\ell^\infty$ is a linear operator, and it is weak*-closed because $f\mapsto f(x)$, $f\mapsto f(Tx)$ are weak*-continuous for every x. If $f\in V$ and $x\ge0$ in ℓ^∞, then $Tx\ge0$ and

$$|f|(Tx) = \sup_{|y|\le Tx} f(y)$$

(Fremlin 02, 356B)

$$= \sup_{-Tx\le y\le Tx} f(y) = \sup_{y\in\ell^\infty} f(\operatorname{med}(-Tx,y,Tx))$$

(where $\operatorname{med}(u,v,w)=(u\wedge v)\vee(u\wedge w)\vee(v\wedge w)$, as in Fremlin 02, 3A1Ic[2])

$$= \sup_{y\in\ell^\infty} f(\operatorname{med}(-Tx,Ty,Tx))$$

(because T is surjective)

[2]Later editions only.

$$= \sup_{y\in\ell^\infty} f(T(\mathrm{med}(-x,y,x)))$$

(because T is a Riesz homomorphism)

$$= \sup_{y\in\ell^\infty} f(\mathrm{med}(-x,y,x)) = \sup_{|y|\le x} f(y) = |f|(x).$$

As x is arbitrary, $|f|\in V$; as f is arbitrary, V is a Riesz subspace (Fremlin 02, 352Ic). **Q**

Accordingly V, with its inherited normed Riesz space structure, is an L-space (Fremlin 02, 354O).

(b) Set $V_1^+ = \{f : f\in V,\ f\ge 0,\ \|f\| = 1\}$. Then $f\mapsto f\chi$ is a bijection between V_1^+ and P. **P** In the language of Fremlin 02, §363, $\ell^\infty \cong L^\infty(\mathcal{P}\mathbb{N})$ so the L-space $(\ell^\infty)^* = (\ell^\infty)^\sim \cong L^\infty(\mathcal{P}\mathbb{N})^\sim$ is identified with the L-space M of bounded additive functionals on $\mathbb{N}$ by the map $f\mapsto f\chi : (\ell^\infty)^*\to M$. Now $f\in(\ell^\infty)^*$ is non-negative iff $f\chi\in M$ is non-negative (Fremlin 02, 363Eb), and for such f we have $\|f\| = f(\chi\mathbb{N})$, so $\{f : f\ge 0,\ \|f\| = 1\}$ corresponds to $\{\nu : \nu\ge 0,\ \nu\mathbb{N} = 1\}$. As for translation-invariance, $T(\chi(I+1)) = \chi I$ for every $I\subseteq\mathbb{N}$. So if $f\in(\ell^\infty)^*$ corresponds to $\nu = f\chi\in M$,

$$\begin{aligned} fT = f &\iff fT\chi = f\chi\\ &\iff f(T(\chi I)) = f(\chi I) \text{ for every } I\subseteq\mathbb{N}\\ &\iff f(\chi(I+1)) = f(\chi I) \text{ for every } I\subseteq\mathbb{N}\\ &\iff \nu(I+1) = \nu I \text{ for every } I\subseteq\mathbb{N}\\ &\iff \nu(I+k) = \nu I \text{ for every } I\subseteq\mathbb{N},\ k\in\mathbb{N}\\ &\iff \nu \text{ is translation-invariant.}\end{aligned}$$

Putting these together, $\nu\in P$ iff $f\in V_1^+$. **Q**

Accordingly P inherits the structure of the weak*-compact convex set V_1^+. Explicitly: if ν_0, $\nu_1\in P$ and $\alpha\in[0,1]$, we have $\nu = \alpha\nu_0 + (1-\alpha)\nu_1\in P$, with $\nu I = \alpha\nu_0 I + (1-\alpha)\nu_1 I$ for every $I\subseteq\mathbb{N}$; the corresponding topology on P is that inherited from the product topology of $\mathbb{R}^{\mathcal{P}\mathbb{N}}$ (if we give $\mathbb{N}$ its discrete topology, this is the 'narrow topology' of Fremlin 03, 437J); P is compact and the convex-combination operation

$$(\nu,\nu',\alpha)\mapsto\alpha\nu + (1-\alpha)\nu' : P\times P\times[0,1]\to P$$

is continuous; and P is the closed convex hull of its extreme points, by the Kreĭn-Mil'man theorem.

1N Extreme points of P Give P its convex structure as described in 1M. Let E be the set of extreme points of P.

(a) Just because V, as described in 1M, is an L-space, an $f\in V_1^+$ is an extreme point of V_1^+ iff it is 'atomic' in V in the sense that whenever $g\in V$ and $0\le g\le f$ then g is a multiple of f. **P** (i) If f is an extreme point and $0\le g\le f$, then either $g = 0$ or $g - f$ or $f_1 = \frac{1}{\|g\|}g$, $f_2 = \frac{1}{\|f-g\|}(f-g)$ both belong to V_1^+. In this case (because V is an L-space) $\|g\| + \|f-g\| = 1$, so f is a convex combination of f_1 and f_2, and both must be equal to f; consequently g is a multiple of f. (ii) If f is not an extreme point, express it as $\alpha f_1 + (1-\alpha)f_2$ where f_1, $f_2\in V_1^+$, $0<\alpha<1$ and neither f_1 nor f_2 is equal to f. Then $\alpha f_1\le f$ but αf_1 is not a multiple of f. **Q**

Consequently $\|f-g\| = 2$ whenever f, g are distinct extreme points of V_1^+. **P** $f\wedge g$ must be a multiple of both f and g; as neither can be a multiple of the other, $f\wedge g = 0$, $|f-g| = f+g$ and $\|f-g\| = \|f\|+\|g\| = 2$. **Q**

Translated into terms of P, this amounts to saying that if $\nu\in P$ and $\nu'\in E\setminus\{\nu\}$, then

$$\inf_{I\subseteq\mathbb{N}}(\nu I + \nu'(\mathbb{N}\setminus I)) = (\nu\wedge\nu')\mathbb{N} = 0, \quad \sup_{I\subseteq\mathbb{N}}\nu I - \nu' I = 1$$

(see Fremlin 02, 362A-362B for the structure of the L-space of bounded finitely additive functionals on $\mathcal{P}\mathbb{N}$).

(b) Let $\langle\nu_n\rangle_{n\in\mathbb{N}}$ be a sequence of distinct elements of E. Then $\langle\nu_n\rangle_{n\in\mathbb{N}}$ cannot be convergent in P for the weak*/narrow topology on P. **P?** Otherwise, let $\nu\in P$ be its limit. Choose $\langle I_k\rangle_{k\in\mathbb{N}}$, $\langle\delta_k\rangle_{k\in\mathbb{N}}$ and $\langle n_k\rangle_{k\in\mathbb{N}}$ inductively, as follows. $I_0 = \mathbb{N}$. Given that $\nu I_k > \frac{2}{3}$, set $\delta_k = \frac{1}{2}(\nu I_k - \frac{2}{3})$, and let n_k be such that $n_k\ne n_j$

for all $j < k$, $\nu_{n_k} I_k \ge \nu I_k - \delta_k$ and $\nu_{n_k} \ne \nu$, so that $(\nu - \nu_{n_k})^+ I_k = \nu I_k = \frac{2}{3} + 2\delta_k$ and there is an $I_{k+1} \subseteq I_k$ such that $\nu I_{k+1} - \nu_{n_k} I_{k+1} \ge \frac{2}{3} + \delta_k$; in particular, $\nu I_{k+1} > \frac{2}{3}$. Continue.

At the end of the induction, set $J_k = I_k \setminus I_{k+1}$ for each k. Then

$$\begin{aligned}\nu_{n_k} J_k = \nu_{n_k} I_k - \nu_{n_k} I_{k+1} &\ge \nu I_k - \delta_k - \nu_{n_k} I_{k+1}\\ &\ge \nu I_{k+1} - \nu_{n_k} I_{k+1} - \delta_k \ge \frac{2}{3}.\end{aligned}$$

So if we set $J = \bigcup_{k\in\mathbb{N}} J_{2k}$, we shall have $\nu_{n_k} J \ge \frac{2}{3}$ for even k and $\nu_{n_k} J \le \frac{1}{3}$ for odd k; in which case $\langle \nu_{n_k}\rangle_{k\in\mathbb{N}}$ cannot converge to ν. **XQ**

2. The Banach density ideal

2A Definition Set $\mathcal{Z}_s = \{I : I \subseteq \mathbb{N},\ d_s^*(I) = 0\}$. Then $\mathcal{Z}_s$ is an ideal of $\mathcal{P}\mathbb{N}$, the **Banach density ideal**. Write $\mathfrak{Z}_s$ for the quotient Boolean algebra $\mathcal{P}\mathbb{N}/\mathcal{Z}_s$, the **Banach density algebra**. The functionals d_s and d_s^* descend naturally to $\mathfrak{Z}_s$ if we set

$$\bar{d}_s^*(I^\bullet) = d_s^*(I), \quad \bar{d}_s(I^\bullet) = d_s(I) \text{ whenever } d_s(I) \text{ is defined.}$$

2B Lemma (Farah) Suppose that $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{Z}_s$. Set $\gamma = \inf_{n\in\mathbb{N}} \bar{d}_s^*(a_n)$. Then there is an $a \in \mathfrak{Z}_s$ such that $\bar{d}_s^*(a) = \gamma$ and $a \subseteq a_n$ for every n.

proof Take $I_n \subseteq \mathcal{P}\mathbb{N}$ such that $I_n^\bullet = a_n$ and $I_{n+1} \subseteq I_n$ for every n. Choose $\langle k_n\rangle_{n\in\mathbb{N}}$, $\langle K_n\rangle_{n\in\mathbb{N}}$ as follows. Given that K_j is a finite set for every $j \le n$, let k_n be such that $\bigcup_{j<n} K_j \subseteq k_n$ and $\#(K_n) \ge (\gamma - 2^{-n})n$, where $K_n = I_n \cap [k_n, k_n + n[$. Continue. Set $I = \bigcup_{n\in\mathbb{N}} K_n$, so that $I \setminus I_n$ is finite for every n, $a = I^\bullet \subseteq a_n$ for every n, and

$$\bar{d}_s^*(a) = d_s^*(I) \ge \limsup_{n\to\infty} \frac{1}{n}\#(K_n) \ge \gamma.$$

2C Proposition (a) $\bar{d}_s^*$ is a strictly positive Maharam submeasure on $\mathfrak{Z}_s$.

(b) There is a corresponding metric $\bar\rho$ on $\mathfrak{Z}_s$ defined by saying that $\bar\rho(a, b) = \bar{d}_s^*(a \mathbin{\triangle} b)$ for all a, $b \in \mathfrak{Z}_s$. Under this metric, the Boolean operations $\cup$, $\cap$, $\triangle$ and $\setminus$ and the function $\bar{d}_s^* : \mathfrak{Z}_s \to [0, 1]$ are uniformly continuous.

(c) $\mathfrak{Z}_s$ is not complete under its metric ρ.

proof (a) $\bar{d}_s^*$ is a strictly positive submeasure just because d_s^* is a submeasure and $\mathcal{Z}_s = \{I : d_s^*(I) = 0\}$. By 2B, $\bar{d}_s^*$ is a Maharam submeasure.

(b) Fremlin 02, 393B.

(c) Define $K_n \subseteq 3^n$, for $n \in \mathbb{N}$, by setting $K_0 = \emptyset$, $K_{n+1} = \{3^n i + j : j \in K_n,\ i \le 2\} \cup \{j_n\}$, where $j_n = \min(\mathbb{N} \setminus K_n)$. Then $\#(K_n) = (3^n - 1)/2$ for each n. Set $I_n = \{3^n i + j : i \in \mathbb{N},\ j \in K_n\}$, $a_n = I_n^\bullet \in \mathfrak{Z}_s$, so that $d_s(I_n) = \frac{1}{2}(1 - 3^{-n})$, and $\bar{d}_s(a_{n+1} \mathbin{\triangle} a_n) = 2 \cdot 3^{-n-1}$ for every n. Now $\langle a_n\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathfrak{Z}_s$. **?** If it has a limit $a \in \mathfrak{Z}_s$, then $\bar{d}_s^*(a) = \lim_{n\to\infty} \bar{d}_s^*(a_n) = \frac{1}{2}$. Let $I \subseteq \mathbb{N}$ be such that $I^\bullet = a$. Then $d_s^*(I) = \frac{1}{2}$, so there is an $m \in \mathbb{N}$ such that $\#(I \cap [k, k + m[) < m$ for every $k \in \mathbb{N}$. Next, $a_m \subseteq a$, so $I_m \setminus I \in \mathcal{Z}_s$, and there must be an $l \in \mathbb{N}$ such that $(I_m \setminus I) \cap [3^m l, 3^m(l + 1)[$ is empty. But now observe that $n \subseteq K_n$ for every n, so that $\{3^m l + i : i < n\} \subseteq I_m$ and $\#(I \cap [3^m l, 3^m l + m[) = \#(I_m \cap [3^m l, 3^m l + m[) = m$. **X**

Remark Part (c) can be regarded as a special case of Proposition 2 in Downarowicz & Iwanik 88.

2D Proposition (a) $\mathcal{Z}_s$ is a Borel subset of $\mathcal{P}\mathbb{N}$.

(b) $\mathcal{Z}_s$ is not a p-ideal.

proof (a)

$$\begin{aligned}\mathcal{Z}_s &= \{I : \inf_{m\ge 1}\sup_{k\in\mathbb{N}}\frac{1}{m}\#(\{n : n\in I,\, k\le n<k+m\}) = 0\}\\ &= \bigcap_{l\ge 1}\bigcup_{m\ge 1}\bigcap_{k\in\mathbb{N}}\frac{1}{m}\{x : \#(\{n : n\in I,\, k\le n<k+m\})\le\frac{1}{l}\end{aligned}$$

is $\mathrm{F}_{\sigma\delta}$ $(=\mathbf{\Pi}^0_3)$.

(b) Set $I_j=\{2^n-j : n\in\mathbb{N},\, 2^n\ge j\}$ for each j. Then $I_j\in\mathcal{Z}_s$ for every j. Let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$ and set $\theta(I)=1$ if $\{n : 2^n\in I\}\in\mathcal{F}$, 0 otherwise. Take any $\nu_0\in P$ and set $\nu I=⨍\theta(I+j)\nu_0(dj)$ for every $I\subseteq\mathbb{N}$. Then $\nu\in P$. If $I\subseteq\mathbb{N}$ is such that $I\setminus I_j$ is finite for every j, then $\theta(I+j)=1$ for every j and $d^*_s(I)=\nu I=1$ and $I\notin\mathcal{Z}_s$. Thus $\mathcal{Z}_s$ cannot be a p-ideal.

2E Proposition $\mathfrak{Z}_s$ is weakly σ-distributive.

proof ? Otherwise, we have an $a\in\mathfrak{Z}_s\setminus\{0\}$ and a double sequence $\langle a_{mn}\rangle_{m,n\in\mathbb{N}}$ such that $a_{mn}\subseteq a$ for all m and n, $\langle a_{mn}\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with infimum 0 for every m, and $\sup_{m\in\mathbb{N}}a_{m,f(m)}=a$ for every $f:\mathbb{N}\to\mathbb{N}$. Set $\gamma=\bar{d}^*_s(a)$. By 2C(a), $\inf_{n\in\mathbb{N}}\bar{d}^*_s(a_{mn})=0$ for each m; choose $f:\mathbb{N}\to\mathbb{N}$ such that $\bar{d}^*_s(e_{m,f(m)})\le 2^{-m-2}\gamma$ for every m. Setting $c_n=\sup_{m\le n}e_{m,f(m)}$, $\bar{d}^*_s(c_n)\le\frac{1}{2}\gamma$ so $\bar{d}^*_s(a\setminus c_n)\ge\frac{1}{2}\gamma$ for every n. But $\langle a\setminus c_n\rangle_{n\in\mathbb{N}}$ is non-increasing and is supposed to have infimum 0 in $\mathfrak{Z}_s$. **X**

2F Lemma Let $\mathfrak{A}$ be an atomless Dedekind σ-complete Boolean algebra and ν a strictly positive Maharam submeasure on $\mathfrak{A}$.

(a) If $a\in\mathfrak{A}$ and $0\le\gamma\le\nu a$ then there is a $d\subseteq a$ such that $\nu d=\gamma$.

(b) For every $\epsilon>0$ there is a finite partition of unity $A\subseteq\mathfrak{A}$ such that $\nu a\le\epsilon$ for every $a\in A$.

proof (a) First note that if $b\in\mathfrak{A}\setminus\{0\}$ and $\delta>0$ there is a $c\subseteq b$ such that $0<\nu c\le\delta$. **P** Choose $\langle b_n\rangle_{n\in\mathbb{N}}$ inductively by setting $b_0=b$ and for each $n\in\mathbb{N}$ taking $b_{n+1}\subseteq b_n$ such that b_{n+1} and $b_n\setminus b_{n+1}$ are both non-zero. Then $\langle b_n\setminus b_{n+1}\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}\setminus\{0\}$; as ν is exhaustive (Fremlin 02, 392Hc), there is some n such that $\nu(b_n\setminus b_{n+1})\le\delta$; as ν is strictly positive, $\nu(b_n\setminus b_{n+1})>0$. **Q**

Now let $\mathfrak{A}_a$ be the principal ideal of $\mathfrak{A}$ generated by a. Let $B\subseteq\mathfrak{A}_a$ be a maximal upwards-directed set such that $\nu b\le\gamma$ for every $b\in B$. $\mathfrak{A}$ is Dedekind complete and ν is order- continuous (see Fremlin 02, 392I, and its proof); set $d=\sup B$, so that $\nu d\le\gamma$. **?** If $\nu d<\gamma$, there is a $b\subseteq a\setminus d$ such that $0<\nu b\le\gamma-\nu d$, in which case $\nu(b\cup d)\le\gamma$ and we ought to have added $b\cup d$ to B. **X** So we have an appropriate d.

(b) Let $A_0\subseteq\mathfrak{A}$ be a maximal disjoint set such that $\nu a=\epsilon$ for every $a\in A_0$. Because ν is exhaustive, A_0 is finite. Set $c=\sup A_0$. By (a), $\nu(1\setminus c)<\epsilon$; set $A=A_0\cup\{1\setminus c\}$.

2G Proposition No atomless Dedekind σ-complete Boolean algebra can be regularly embedded in $\mathfrak{Z}_s$.

proof ? Suppose otherwise. Then there is an atomless order-closed subalgebra $\mathfrak{A}$ of $\mathfrak{Z}_s$ which is Dedekind σ-complete. Now $\bar{d}^*_s\restriction\mathfrak{A}$ is a strictly positive Maharam submeasure on $\mathfrak{A}$. By 2F(b), we can find for each $n\in\mathbb{N}$ a finite partition of unity A_n in $\mathfrak{A}\setminus\{0\}$ such that $\bar{d}^*_s(a)\le 2^{-n}$ for every $a\in A_n$; let $\mathcal{I}_n$ be a partition of $\mathbb{N}$ such that $A_n=\{I^\bullet : I\in\mathcal{I}_n\}$.

Choose $\langle I_n\rangle_{n\in\mathbb{N}}$ and $\langle L_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{P}\mathbb{N}$ as follows. Start with $L_0=\mathbb{N}$ and $I_0=\emptyset$. Suppose that we have chosen I_n and L_n such that

$$I^\bullet_n\in\mathfrak{A},\quad d^*_s(I_n)\le\frac{1}{2}-2^{-n-1},\quad d^*_s(L_n)>0,$$

$$i+j\in I_n \text{ whenever } i\in L_n \text{ and } j<n.$$

As $\mathcal{I}_{n+2}$ is a finite cover of $\mathbb{N}$, there is an $I\in\mathcal{I}_{n+2}$ such that $d^*_s(\{i : i\in L_n,\, i+n\in I\})>0$; set $I_{n+1}=I_n\cup I$ and $L_{n+1}=\{i : i\in L_n,\, i+n\in I\}$, and continue.

At the end of the induction, observe that $a=\sup_{n\in\mathbb{N}}I^\bullet_n$ ought to be defined in $\mathfrak{A}$, with $\bar{d}^*_s(a)\le\frac{1}{2}$. Let $I\subseteq\mathbb{N}$ be such that $I^\bullet=a$, so that $d^*_s(I)\le\frac{1}{2}$. If $n\in\mathbb{N}$, then $d^*_s(I_n\setminus I)=0$, so $d^*_s(\{i : i+j\in I_n\setminus I$ for some $j<n\})=0$, and there is a $k\in L_n$ such that $k+j\notin I_n\setminus I$ for every $j<n$. But this means that $k+j\in I$ for every $j<n$. Thus $\sup_{k\in\mathbb{N}}\#(I\cap[k,k+n[)=n$. As this is true for every n, $d^*_s(I)=1$. **X**

2H Proposition Write $\mathcal{Z}$ for the ideal of subsets of $\mathbb{N}$ with zero asymptotic density, and $\mathfrak{Z}$ for the quotient $\mathcal{P}\mathbb{N}/\mathcal{Z}$. Then we have a canonical sequentially order-continuous Boolean homomorphism $\pi : \mathfrak{Z}_s \to \mathfrak{Z}$, defined by saying that $\pi I^\bullet = I^\circ$ for every $I \subseteq \mathbb{N}$, where $I^\bullet$ is the equivalence class of I in $\mathfrak{Z}_s = \mathcal{P}\mathbb{N}/\mathcal{Z}_s$, and I° is the equivalence class of I in $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$.

proof Because $I \mapsto I^\circ$ is a Boolean homomorphism with kernel $\mathcal{Z} \supseteq \mathcal{Z}_s$ (1Gb), the formula gives us a Boolean homomorphism. Now suppose that $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{Z}_s$ with infimum 0. For each $n \in \mathbb{N}$ take $I_n \subseteq \mathbb{N}$ such that $I_n^\bullet = a_n$. Then, writing d^* for upper asymptotic density,

$$\inf_{n\in\mathbb{N}} d^*(I_n) \le \inf_{n\in\mathbb{N}} d_s^*(I_n)$$

(1G(b))

$$= 0$$

(2C(a)). But this means that if $I \subseteq \mathbb{N}$ and $I^\circ \subseteq \pi a_n$ for every n, $d^*(I) = 0$ and $I \in \mathcal{Z}$ and $I^\circ = 0$. Thus $\inf_{n\in\mathbb{N}} \pi a_n = 0$ in $\mathfrak{Z}$. As $\langle a_n\rangle_{n\in\mathbb{N}}$ is arbitrary, π is sequentially order- continuous.

2I Proposition (Farah 04, 1.4) There is a sequence $\langle\mathcal{K}_n\rangle_{n\in\mathbb{N}}$ of subsets of $\mathcal{P}\mathbb{N}$ such that every $\mathcal{K}_n$ is compact for the usual topology on $\mathcal{P}\mathbb{N}$ and

$$\mathcal{Z}_s = \bigcap_{n\in\mathbb{N}}\{I : I \subseteq \mathbb{N},\ I \setminus \bigcup_{i\le m} K_i \text{ is finite for some } K_0, \dots , K_m \in \mathcal{K}_n\}$$

for every $m \in \mathbb{N}$.

proof Fix $m \in \mathbb{N}$. For r, $n \in \mathbb{N}$ set

$$\mathcal{K}_{nr} = \{I : I \subseteq \mathbb{N} \setminus r,\ \#(I \cap [k, k + l[) \le \tfrac{l}{n+1} \text{ for every } l \ge r \text{ and every } k \in \mathbb{N}\}.$$

Then $\mathcal{K}_{nr}$ is compact. For $n \in \mathbb{N}$ set $\mathcal{K}_n = \bigcup_{r\in\mathbb{N}} \mathcal{K}_{nr}$; because every neighbourhood of $\emptyset$ includes all but finitely many of the $\mathcal{K}_{nr}$, $\mathcal{K}_n$ is compact. For $n \in \mathbb{N}$ write

$$\mathcal{I}_n = \{I : I \subseteq \mathbb{N},\ I \setminus \bigcup_{i\le m} K_i \text{ is finite for some } K_0, \dots , K_m \in \mathcal{K}_n\}.$$

If $I \in \mathcal{Z}_s$ and $n \in \mathbb{N}$, then there is an $r \in \mathbb{N}$ such that $\#(I \cap [k, k + l[) \le \frac{l}{n+1}$ for every $l \ge r$, and now $I \setminus r \in \mathcal{K}_n$ and $I \setminus (I \setminus r)$ is finite; accordingly $I \in \mathcal{I}_n$. Thus $\mathcal{Z}_s \subseteq \bigcap_{n\in\mathbb{N}} \mathcal{I}_n$.

On the other hand, if $I \in \bigcap_{n\in\mathbb{N}} \mathcal{I}_n$, then $I \in \mathcal{Z}_s$. **P** Let $\epsilon > 0$. Let n be such that $2\epsilon(m + 1) \le n$. Then there are $K_0, \dots , K_m \in \mathcal{K}_n$ such that $I\setminus\bigcup_{i\le m} K_i$ is finite. Let r_i be such that $K_i \in \mathcal{K}_{nr_i}$ for each $i \le m$, and set $r = \max_{i\le m} r_i$; then $\#(K_i \cap [k, k + l[) \le \frac{l}{n+1}$ for every $l \ge r$ and every $k \in \mathbb{N}$. Set $s = \#(I \setminus \bigcup_{i\le m} K_i)$; then $\#(I \cap [k, k + l[) \le \frac{ml}{n+1} + s \le \frac{\epsilon l}{2} + s$ for every $l \ge r$ and every $k \in \mathbb{N}$. So if we take $r' \ge r$ such that $2s \le \epsilon r'$, then $\#(I \cap [k, k + l[) \le \epsilon l$ for every $l \ge r'$ and every $k \in \mathbb{N}$. As ϵ is arbitrary, $I \in \mathcal{Z}_s$. **Q**

Thus $\mathcal{Z}_s = \bigcap_{n\in\mathbb{N}} \mathcal{I}_n$, as claimed.

2J Remark In the language of Farah 04, $\mathcal{Z}_s$ is 'strongly countably determined' by $\langle\mathcal{K}_n\rangle_{n\in\mathbb{N}}$. Under the Proper Forcing Axiom this means that homomorphisms into $\mathfrak{Z}_s$ from other quotients $\mathcal{P}\mathbb{N}/\mathcal{J}$ are strikingly constrained. In fact

> **Theorem** [PFA] For every Boolean homomorphism $\pi : \mathcal{P}\mathbb{N} \to \mathfrak{Z}_s$, there are a continuous function $F : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ and a non-meager ideal $\mathcal{K} \lhd \mathcal{P}\mathbb{N}$ such that $F(I)^\bullet = \pi(I^\bullet)$ for every $I \in \mathcal{K}$.

2K Proposition $\mathfrak{Z}_s \cong \mathfrak{Z}_s^{\mathbb{N}}$.

proof Set $L_n = \{2^n(2i+1) : i \in \mathbb{N}\}$, $d_n = L_n^\bullet \in \mathfrak{Z}_s$. Because $\lim_{n\to\infty} d_s^*(\mathbb{N}\setminus\bigcup_{j\le n} L_j) = 0$, $\mathfrak{Z}_s \cong \prod_{n\in\mathbb{N}}(\mathfrak{Z}_s)_{d_n}$. But of course every $(\mathfrak{Z}_s)_{d_n}$ is isomorphic to $\mathfrak{Z}_s$, just because, for $J \subseteq \mathbb{N}$, $J \in \mathcal{Z}_s$ iff $\{2^n(2i + 1) : i \in J\} \in \mathcal{Z}_s$.

2L Lemma Suppose that $I \subseteq \mathbb{N}$ and that $0 \le \beta' < \gamma < d_s^*(I)$ and $m_0 \in \mathbb{N}$. Then there is an $m \ge m_0$ such that, setting $K = \{k : k \in \mathbb{N}, \#(I \cap [2^m k, 2^m(k+1)[\ \ge 2^m\gamma\}$ and $J = I \cap \bigcup_{k\in K}[2^m k, 2^m(k+1)[$, $d_s^*(J) \ge \beta'$.

proof Set $\beta = d_s^*(I)$ and let $\alpha > \beta$ be such that $\gamma(1 - \frac{\alpha-\beta}{\alpha-\gamma}) \ge \beta'$. Let $m \ge m_0$ be such that $\sup_{k\in\mathbb{N}} \#(I \cap [k, k+2^m[) \le 2^m\alpha$. Define K, J by the formulae given. Suppose that $l \ge m$. Then there is a k^* such that $\#(I \cap \bigl[2^l k^*, 2^l(k^*+1)\bigr[) \ge 2^l\beta$. Set $L = \{k : 2^{l-m}k^* \le k < 2^{l-m}(k^*+1)\}$. Then

$$\begin{aligned}
2^l\beta &= \sum_{k\in L\cap K} \#(I \cap [2^m k, 2^m(k+1)[) + \sum_{k\in L\setminus K} \#(I \cap [2^m k, 2^m(k+1)[) \\
&\le 2^m\alpha\#(L\cap K) + 2^m\gamma\#(L\setminus K) \\
&= 2^m\alpha\#(L) - 2^m(\alpha-\gamma)\#(L\setminus K) = 2^l\alpha - 2^m(\alpha-\gamma)\#(L\setminus K).
\end{aligned}$$

So $\#(L\setminus K) \le 2^{l-m}\frac{\alpha-\beta}{\alpha-\gamma}$. Consequently

$$\begin{aligned}
\#(J \cap \bigl[2^l k^*, 2^l(k^*+1)\bigr[) &\ge 2^m\gamma\#(L\cap K) = 2^m\gamma(2^{l-m} - \#(L\setminus K)) \\
&\ge 2^m\gamma(2^{l-m} - 2^{l-m}\tfrac{\alpha-\beta}{\alpha-\gamma}) = 2^l\gamma(1 - \tfrac{\alpha-\beta}{\alpha-\gamma}) \ge 2^l\beta'.
\end{aligned}$$

As l is arbitrary, $d_s^*(J) \ge \beta'$, as claimed.

2M Theorem Suppose that $c \in \mathfrak{Z}_s$ is non-zero. Then there is a non-zero $d \subseteq c$ such that the principal ideal $(\mathfrak{Z}_s)_d$ generated by d is isomorphic to $\mathfrak{Z}_s$.

proof (a) Choose $\langle I_n\rangle_{n\in\mathbb{N}}$, $\langle m_n\rangle_{n\in\mathbb{N}}$, $\langle \gamma_n\rangle_{n\in\mathbb{N}}$, $\langle K_n\rangle_{n\in\mathbb{N}}$, $\langle r_n\rangle_{n\in\mathbb{N}}$ as follows. Start with I_0 such that $I_0^\bullet = c$. Given that $d_s^*(I_n) > 0$, set $\beta_n = (1 - 2^{-n-1})d_s^*(I_n)$ and take $\gamma_n \in]\beta_n, d_s^*(I_n)[$. By Lemma 2L, we can find an m_n such that $m_n > m_i$ for every $i < n$ and $d_s^*(I \cap \bigcup_{k\in K_n}[2^{m_n}k, 2^{m_n}(k+1)[) \ge \beta_n$, where $K_n = \{k : \#(I_n \cap [2^{m_n}k, 2^{m_n}(k+1)[) \ge 2^{m_n}\gamma_n\}$. Take r_n such that $r_n \ge r_i$ for every $i < n$ and $r_n = 2^{m_n}k$ for some $k \ge 1 + \min K_n$, and set $I_{n+1} = I \cap (r_n \cup \bigcup_{k\in K_n}[2^{m_n}k, 2^{m_n}(k+1)[)$; then $d_s^*(I_{n+1}) \ge \beta_n > 0$. Continue.

(b) We find that if $i < n$ and $2^{m_i}k \ge r_i$ and $I_n \cap [2^{m_i}k, 2^{m_i}(k+1)[$ is not empty, then $\#(I_n \cap [2^{m_i}k, 2^{m_i}(k+1)[) \ge 2^{m_i}\gamma_i$. **P** Induce on n. For $n = i+1$, this is just the construction of I_{i+1}. For the inductive step to $n+1$ where $n > i$, if $I_{n+1} \cap [2^{m_i}k, 2^{m_i}(k+1)[$ is not empty, then either $2^{m_i}(k+1) \le r_n$ or $r_n \le 2^{m_i}k$, because r_n is a multiple of 2^{m_n} which is a multiple of 2^{m_i}. In the former case, $I_{n+1} \cap [2^{m_i}k, 2^{m_i}(k+1)[= I_n \cap [2^{m_i}k, 2^{m_i}(k+1)[$ has at least $2^{m_i}\gamma_i$ members, by the inductive hypothesis. In the latter case, $[2^{m_i}k, 2^{m_i}(k+1)[$ must meet $[2^{m_n}k', 2^{m_n}(k'+1)[$ for some $k' \in K_n$. But in this case $[2^{m_i}k, 2^{m_i}(k+1)[\subseteq [2^{m_n}k', 2^{m_n}(k'+1)[$ and $I_{n+1} \cap [2^{m_i}k, 2^{m_i}(k+1)[= I_n \cap [2^{m_i}k, 2^{m_i}(k+1)[$ again has at least $2^{m_i}\gamma_i$ members. **Q**

(c) Set $J = \bigcap_{n\in\mathbb{N}} I_n$.

(i) $d_s^*(J) > 0$. **P** Set $\gamma^* = d_s^*(I_0)\prod_{n\in\mathbb{N}}(1 - 2^{-n-1}) > 0$. Then $\gamma^* \le \gamma_n$ for every $n \in \mathbb{N}$. For $n \in \mathbb{N}$, set $k_n = \min K_n$. Because $\langle r_i\rangle_{i\in\mathbb{N}}$ is non-decreasing, $I_n \cap [2^{m_n}k_n, 2^{m_n}(k_n+1)[\subseteq I_i$ for every $i \ge n$, and

$$\#(J \cap [2^{m_n}k_n, 2^{m_n}(k_n+1)[) = \#(I_n \cap [2^{m_n}k, 2^{m_n}(k+1)[) \ge 2^{m_n}\gamma_n \ge 2^{m_n}\gamma^*.$$

As $\lim_{n\to\infty} 2^{m_n} = \infty$, $d_s^*(J) \ge \gamma^* > 0$. **Q** Set $d = J^\bullet$, so that $0 \ne d \subseteq c$.

(ii) Because $\langle I_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, (b) tells us that if $i \in \mathbb{N}$ and $2^{m_i}k \ge r_i$ and $J \cap [2^{m_i}k, 2^{m_i}(k+1)[$ is not empty, then $\#(J \cap [2^{m_i}k, 2^{m_i}(k+1)[) \ge 2^{m_i}\gamma_i$.

(d) Let $f : \mathbb{N} \to J$ be the increasing enumeration of J.

(i) $d_s^*(f[M]) \le d_s^*(M)$ for every $M \subseteq \mathbb{N}$. **P** If $\alpha > d_s^*(M)$, there is an $m \in \mathbb{N}$ such that $\#(M \cap [k, k+m[) \le \alpha m$ for every k. Now if $k \in \mathbb{N}$, $L = f^{-1}[\,[k, k+m[\,]$ is an interval with at most m members, so $\#(f[M] \cap [k, k+m[) = \#(L \cap K) \le \alpha m$. As k is arbitrary, $d_s^*(f[M]) \le \alpha$; as α is arbitrary, $d_s^*(f[M]) \le d_s^*(M)$. **Q**

(ii) $d_s^*(f[M]) \ge \frac{1}{4}\gamma^* d_s^*(M)$ for every $M \subseteq \mathbb{N}$. **P** Set $\delta = \frac{1}{4}d_s^*(M)$. If $\delta = 0$ we can stop. Otherwise, take any n such that $2^{m_n}\gamma^* \ge 1$. Then $\lfloor 2^{m_n}\gamma_n \rfloor \ge 2^{m_n-1}\gamma_n$, so there is a $k \ge r_n$ such that $\#(M\cap L) \ge 2^{m_n+1}\delta\gamma_n$ where $L = \{j : j \in \mathbb{N}, k \le j < k+\lfloor 2^{m_n}\gamma_n\rfloor\}$. Now L is an interval in $\mathbb{N}$ with $\lfloor 2^{m_n}\gamma_n\rfloor$ members. Also $f[L] \subseteq [r_n, \infty[$. So $f[L]$ cannot properly include $J \cap [2^{m_n}k, 2^{m_n}(k+1)[$ for any k such that $J \cap [2^{m_n}k, 2^{m_n}(k+1)[$ is non-empty, and $f[L]$ must be covered by two intervals $[2^{m_n}k, 2^{m_n}(k+1)[$, $[2^{m_n}k', 2^{m_n}(k'+1)[$. Since $\#(f[M\cap L]) = \#(M\cap L) \ge 2^{m_n+1}\delta\gamma_n$, one of these intervals must contain at least $2^{m_n}\delta\gamma_n$ points of $f[M]$. But this means that we have found an interval of length 2^{m_n} containing at least $2^{m_n}\delta\gamma^*$ points of $f[M]$. Since this can be done for any n large enough, $d_s^*(f[M]) \ge \delta\gamma^*$, as claimed. **Q**

(e) Now f induces an isomorphism between $\mathcal{P}\mathbb{N}$ and $\mathcal{P}J$ which takes $\mathcal{Z}_s$ to $\mathcal{Z}_s \cap \mathcal{P}J$, so induces an isomorphism between $\mathfrak{Z}_s$ and $(\mathfrak{Z}_s)_d$.

2N Corollary The automorphism group $\operatorname{Aut}\mathfrak{Z}_s$ has no outer automorphisms.

proof In the language of Fremlin 02, $\mathfrak{Z}_s$ has many involutions, so we can use Fremlin 02, 384D.

2O Corollary Writing $\widehat{\mathfrak{Z}}_s$ for the Dedekind completion of $\mathfrak{Z}_s$ (Fremlin 02, 314U), $\widehat{\mathfrak{Z}}_s$ is a homogeneous Boolean algebra; its automorphism group is simple and has no outer automorphisms.

proof (a) Let C be the set of those $c \in \mathfrak{Z}_s$ such that the principal ideal $(\mathfrak{Z}_s)_c$ is isomorphic to $\mathfrak{Z}_s$. By 2M, C is order-dense in $\mathfrak{Z}_s$ and therefore in $\widehat{\mathfrak{Z}}_s$. Take any non-zero $a \in \widehat{\mathfrak{Z}}_s$. Then there is a partition of unity C_0 in $(\widehat{\mathfrak{Z}}_s)_a$ consisting of members of C (Fremlin 02, 313K). Next, there is a partition of unity C_1 in $\mathfrak{Z}_s$ such that $C_0 \subseteq C_1 \subseteq C$. If $\#(C_1) > \#(C_0)$, note first that because $\mathfrak{Z}_s \cong \mathfrak{D}_s^{\mathbb{N}}$ we can replace one of the members of $C_1 \setminus C_0$ by a countably infinite subset of C; next, we can replace one of the members of C_0 by a copy of C_1, still lying within C. In this way, we obtain C_0', $C_1' \subseteq C$, with $\#(C_0') = \#(C_1') = \kappa$ say, which are partitions of unity in $(\widehat{\mathfrak{Z}}_s)_a$ and $\widehat{\mathfrak{Z}}_s$ respectively. Since $(\widehat{\mathfrak{Z}}_s)_c \cong \widehat{\mathfrak{Z}}_s$ for every $c \in C$, $(\widehat{\mathfrak{Z}}_s)_a \cong \widehat{\mathfrak{Z}}_s^\kappa \cong \widehat{\mathfrak{Z}}_s$. As a is arbitrary, $\widehat{\mathfrak{Z}}_s$ is homogeneous.

(b) By Fremlin 02, 381T and 383G, $\operatorname{Aut}\widehat{\mathfrak{Z}}_s$ is simple and has no outer automorphisms.

2P The shift on $\mathfrak{Z}_s$ (a) We have a Boolean automorphism $\psi : \mathfrak{Z}_s \to \mathfrak{Z}_s$ defined by saying that $\psi(I^\bullet) = (I+1)^\bullet$ for every $I \subseteq \mathbb{N}$. **P** If I, $J \subseteq \mathbb{N}$ and $I^\bullet = J^\bullet$, then $(I+1)\triangle(J+1) = (I\triangle J)+1$ belongs to $\mathcal{Z}_s$, so $(I+1)^\bullet = (J+1)^\bullet$; so the formula defines a function $\psi : \mathfrak{Z}_s \to \mathfrak{Z}_s$. If I, $J \subseteq \mathbb{N}$ then $(I\cup J)+1 = (I+1)\cup(J+1)$, so $\psi(I^\bullet \cup J^\bullet) = \psi I^\bullet \cup \psi J^\bullet$. If $I \subseteq \mathbb{N}$ then $((\mathbb{N}\setminus I)+1)\triangle(\mathbb{N}\setminus(I+1)) = \{0\}$ belongs to $\mathcal{Z}_s$, so $\psi(1\setminus I^\bullet) = 1 \setminus \psi I^\bullet$. Thus ψ is a Boolean homomorphism. If $I \notin \mathcal{Z}_s$ then $d_s^*(I+1) = d_s^*(I) \ne 0$ and $I+1 \notin \mathcal{Z}_s$; so ψ is injective. If $I \subseteq \mathbb{N}$ set $J = (I\setminus\{0\})-1$; then $\psi J^\bullet = I^\bullet$. So ψ is surjective and is a Boolean automorphism. **Q**

(b) If $\theta : \mathfrak{Z}_s \to [0,1]$ is an additive functional such that $\theta 1 = 1$, then the following are equiveridical:

(i) $\theta\psi = \theta$;

(ii) $\theta a \le \bar{d}_s^*(a)$ for every $a \in \mathfrak{Z}_s$;

(iii) $\theta a = \bar{d}_s(a)$ for every a such that $\bar{d}_s(a)$ is defined.

P Apply 1H to the functional $I \mapsto \theta I^\bullet$. **Q**

3. Well-distributed sequences

3A Definitions (a) If $z \in \ell^\infty$ is such that $⨍ z\,d\nu$ is the same for every $\nu \in P$, I will call this common value $\mathrm{WDL}_{i\to\infty} z(i)$, the **well-distributed limit** of z.

(b) A **Følner sequence** of subsets of $\mathbb{N}$ is a sequence $\langle I_m\rangle_{m\in\mathbb{N}}$ of finite non-empty subsets of $\mathbb{N}$ such that $\lim_{m\to\infty} \frac{1}{\#(I_m)}\#(I_m\triangle(k+I_m)) = 0$ for every $k \in \mathbb{N}$.

3B Theorem (a) If $z \in \ell^\infty$, then

$$\begin{aligned}\sup_{\nu\in P}\textstyle\fint z\,d\nu &= \limsup_{m\to\infty}\sup_{k\in\mathbb{N}}\frac{1}{m+1}\sum_{i=k}^{k+m} z(i)\\ &= \inf\{\sup_{k\in\mathbb{N}}\sum_{i=0}^{n}\alpha_i z(k+i) : \alpha_0,\dots,\alpha_n\ge 0,\ \sum_{i=0}^{n}\alpha_i = 1\}\\ &= \max\{\limsup_{m\to\infty}\frac{1}{\#(I_m)}\sum_{i\in I_m} z(i) :\\ &\qquad\qquad \langle I_m\rangle_{m\in\mathbb{N}} \text{ is a Følner sequence of subsets of } \mathbb{N}\}\\ &= \max\{\liminf_{m\to\infty}\frac{1}{\#(I_m)}\sum_{i\in I_m} z(i) :\\ &\qquad\qquad \langle I_m\rangle_{m\in\mathbb{N}} \text{ is a Følner sequence of subsets of } \mathbb{N}\}.\end{aligned}$$

(b) If $z\ge 0$, then $\sup_{\nu\in P}\fint z\,d\nu = \inf\{\|x*z\|_\infty : x\in\ell^1,\ x\ge 0,\ \|x\|_1 = 1\}$.

(c) If $z\in\ell^\infty$ and $\gamma\in\mathbb{R}$, then the following are equiveridical:

(α) $\mathrm{WDL}_{i\to\infty} z(i)$ is defined and equal to γ;

(β) for every $\epsilon>0$ there is an $m_0\in\mathbb{N}$ such that $|\gamma-\frac{1}{m+1}\sum_{i=k}^{k+m} z(i)|\le\epsilon$ for every $m\ge m_0$ and $k\in\mathbb{N}$;

(γ) $\lim_{m\to\infty}\frac{1}{\#(I_m)}\sum_{i\in I_m} z(i)$ is defined and equal to γ for every Følner sequence $\langle I_m\rangle_{m\in\mathbb{N}}$ of subsets of $\mathbb{N}$.

proof (a) It is enough to consider the case $0\le z\le\chi\mathbb{N}$. Set

$$\gamma_1 = \sup_{\nu\in P}\textstyle\fint z\,d\nu,$$

$$\gamma_2 = \limsup_{m\to\infty}\sup_{k\in\mathbb{N}}\frac{1}{m+1}\sum_{i=k}^{k+m} z(i),$$

$$\gamma_3 = \inf\{\sup_{k\in\mathbb{N}}\sum_{i=0}^{n}\alpha_i z(k+i) : \alpha_0,\dots,\alpha_n\ge 0,\ \sum_{i=0}^{n}\alpha_i = 1\},$$

$$\begin{aligned}\gamma_4 = \sup\{&\limsup_{m\to\infty}\frac{1}{\#(I_m)}\sum_{i\in I_m} z(i) :\\ &\langle I_m\rangle_{m\in\mathbb{N}} \text{ is a Følner sequence of subsets of } \mathbb{N}\}.\end{aligned}$$

Set $x_m = \frac{1}{m+1}\chi(m+1)$, so that $\sup_{k\in\mathbb{N}}\frac{1}{m+1}\sum_{i=k}^{k+m} z(i) = \|x_m*z\|_\infty$ for each m.

(i) $\gamma_1\le\gamma_2$. **P** If $\nu\in P$, $m\in\mathbb{N}$ then

$$\textstyle\fint z\,d\nu = \fint x_m*z\,d\nu \le \|x_m*z\|_\infty.$$

Letting $m\to\infty$, $\fint z\,d\nu\le\gamma_2$; as ν is arbitrary, $\gamma_1\le\gamma_2$. **Q**

(ii) $\gamma_2\le\gamma_3$. **P** Let $\epsilon>0$. Take $\alpha_0,\dots,\alpha_n\ge 0$ such that $\sum_{i=0}^{n}\alpha_i = 1$ and $\sup_{k\in\mathbb{N}}\sum_{i=0}^{n}\alpha_i z(k+i)\le\gamma_3+\epsilon$. Then for any k, $m\in\mathbb{N}$,

$$\begin{aligned}\frac{1}{m+1}\sum_{i=k}^{k+m} z(i) &= \frac{1}{m+1}\sum_{i=0}^{m}\sum_{j=0}^{n}\alpha_j z(k+i)\\ &\le \frac{1}{m+1}\sum_{i=0}^{m}\sum_{j=0}^{n}\alpha_j z(k+i+j) + \frac{1}{m+1}\sum_{i=0}^{n}\sum_{j=0}^{n}\alpha_j z(k+i)\\ &\le \frac{1}{m+1}\sum_{i=0}^{m}(\gamma_3+\epsilon) + \frac{1}{m+1}\sum_{i=0}^{n} z(k+i) \le \gamma_3+\epsilon+\frac{n+1}{m+1}.\end{aligned}$$

Taking the supremum over k and the (upper) limit as $m\to\infty$, $\gamma_2\le\gamma_3+\epsilon$; as ϵ is arbitary, $\gamma_2\le\gamma_3$. **Q**

(iii) For k, $i \in \mathbb{N}$ set $w_k(i) = 1 - z(k+i)$. Then there is a positive linear functional $f : \ell^\infty \to \mathbb{R}$ such that $f(\chi\mathbb{N}) = 1$ and $f(w_k) \le 1 - \gamma_3$ for every k. **P**

case 1 Suppose that there are no $\alpha_0, \dots, \alpha_n \ge 0$ such that $\sum_{i=0}^n \alpha_k w_k \ge \chi\mathbb{N}$. In this case $a_{n\delta} = \{i : w_k(i) \le \delta$ for every $k \le n\}$ is non-empty for every $n \in \mathbb{N}$, $\delta > 0$. Let $\mathcal{F}$ be an ultrafilter on $\mathbb{N}$ containing every $a_{n\delta}$, and set $f(v) = \lim_{n\to\mathcal{F}} v(n)$ for every $v \in \ell^\infty$; then $f(w_k) = 0$ for every k.

case 2 Otherwise, define a seminorm τ on ℓ^∞ by setting

$$\tau(v) = \inf\{\sum_{k=0}^n \alpha_k : \alpha_0, \dots, \alpha_n \ge 0,\ |v| \le \sum_{k=0}^n \alpha_k w_k\}$$

for $v \in \ell^\infty$. Set $\beta = \tau(\chi\mathbb{N})$. **?** If $(1-\gamma_3)\beta < 1$, take $\alpha_0, \dots, \alpha_n \ge 0$ such that $\chi\mathbb{N} \le \sum_{k=0}^n \alpha_k w_k$ and $(1-\gamma_3)\sum_{k=0}^n \alpha_k < 1$. Set $\alpha = \sum_{k=0}^n \alpha_k$; of course $\alpha > 0$; set $\alpha'_k = \alpha_k/\alpha$ for each k, so that $\sum_{k=0}^n \alpha'_k = 1$, $\frac{1}{\alpha}\chi\mathbb{N} \le \sum_{k=0}^n \alpha'_k w_k$ and $(1-\gamma_3)\alpha < 1$. Now, for any $i \in \mathbb{N}$,

$$\frac{1}{\alpha} \le \sum_{k=0}^n \alpha'_k w_k(i) = \sum_{k=0}^n \alpha'_k(1 - z(k+i)) = 1 - \sum_{k=0}^n \alpha'_k z(k+i)$$

so

$$\gamma_3 \le \sup_{i\in\mathbb{N}} \sum_{k=0}^n \alpha'_k z(i+k) \le 1 - \frac{1}{\alpha},$$

$\frac{1}{\alpha} \le 1 - \gamma_3$ and $1 \le (1-\gamma_3)\alpha$. **X**

Thus $(1-\gamma_3)\beta \ge 1$. By the Hahn-Banach theorem, there is a linear functional $g : \ell^\infty \to \mathbb{R}$ such that $g(\chi\mathbb{N}) = \beta$ and $|g(v)| \le \tau(v)$ for every $v \in \ell^\infty$. Since τ is a Riesz seminorm, $g \in (\ell^\infty)^\sim$. Take $|g|$ in $(\ell^\infty)^\sim$; then $|g|(\chi\mathbb{N}) \ge g(\chi\mathbb{N})$ and $|g|(v) \le \tau(v)$ for every $v \in \ell^\infty$. So in fact we must still have $|g|(\chi\mathbb{N}) = \beta$, while $|g|(w_k) \le \tau(w_k) \le 1$ for every k. Set $f = \frac{1}{\beta}|g|$. Then $f : \ell^\infty \to \mathbb{R}$ is a positive linear functional, $f(\chi\mathbb{N}) = 1$ and $f(w_k) \le \frac{1}{\beta} \le 1 - \gamma_3$ for every k, as required. **Q**

(iv) $\gamma_3 \le \gamma_1$. **P** Take the functional f from (iii), and set $\theta(I) = f(\chi I)$ for each $I \subseteq \mathbb{N}$. Then $\theta : \mathcal{P}\mathbb{N} \to [0,1]$ is an additive functional and $\theta(\mathbb{N}) = 1$. Take any $\nu_0 \in P$ and set $\nu I = \int \theta(I -' i)\nu_0(di)$ for $I \subseteq \mathbb{N}$, so that $\nu \in P$.

Let $\epsilon > 0$. Then there are $a_0, \dots, I_n \subseteq \mathbb{N}$ and $\alpha_0, \dots, \alpha_n \ge 0$ such that

$$\sum_{j=0}^n \alpha_j \chi I_j \le z \le \epsilon\chi\mathbb{N} + \sum_{j=0}^n \alpha_j \chi I_j.$$

Setting $z_k = \chi\mathbb{N} - w_k$, so that $z_k(i) = z(k+i)$ for all k and i,

$$z_k(i) \le \epsilon + \sum_{j=0}^n \alpha_j \chi I_j(k+i) = \epsilon + \sum_{j=0}^n \alpha_j \chi(a_j -' k)(i)$$

for every i, and

$$\gamma_3 \le 1 - f(w_k) = f(z_k) \le \epsilon + \sum_{j=0}^n \alpha_j \theta(a_j -' k).$$

Integrating with respect to ν_0,

$$\gamma_3 \le \epsilon + \sum_{j=0}^n \alpha_j \nu(a_j) \le \epsilon + \int z\,d\nu \le \epsilon + \gamma_1.$$

As ϵ is arbitrary, $\gamma_3 \le \gamma_1$. **Q**

(v) $\gamma_2 \le \gamma_4$. **P** For each $m \in \mathbb{N}$, let k_m be such that

$$\frac{1}{m+1}\sum_{i=k_m}^{k_m+m} z(i) \ge \sup_{k\in\mathbb{N}} \frac{1}{m+1}\sum_{i=k}^{k+m} z(i) - 2^{-m},$$

and set $I_m = \{i : k_m \le i \le k_m + m\}$. Then

$$\gamma_2 = \limsup_{m\to\infty} \frac{1}{m+1}\sum_{i=k_m}^{k_m+m} z(i) = \limsup_{m\to\infty} \frac{1}{\#(I_m)}\sum_{i\in I_m} z(i) \le \gamma_4.$$

(vi) There is a Følner sequence $\langle J_m\rangle_{n\in\mathbb{N}}$ of subsets of $\mathbb{N}$ such that $\lim_{m\to\infty} \frac{1}{\#(J_m)} \sum_{i\in J_m} z(i)$ is defined and equal to γ_4. **P** For each $m \in \mathbb{N}$ there is a Følner sequence $\langle I_{mn}\rangle_{n\in\mathbb{N}}$ of subsets of $\mathbb{N}$ such that

$\limsup_{n\to\infty} \frac{1}{\#(I_{mn}} \sum_{i\in I_{mn}} z(i) > \gamma_4 - 2^{-m}$. For all n large enough,

$$\#(I_{mn}\triangle(k+I_{mn}) \le 2^{-m}\#(I_{mn}) \text{ for every } k \le m,$$

so there is a first $n(m)$ such that

$$\#(I_{m,n(m)}\triangle(k+I_{m,n(m)}) \le 2^{-m}\#(I_{m,n(m)}) \text{ for every } k \le m,$$

$$\frac{1}{\#(I_{m,n(m)}}\sum_{i\in I_{m,n(m)}} z(i) \ge \gamma_4 - 2^{-m}.$$

Set $J_m = I_{m,n(m)}$. Then $\langle J_m\rangle_{m\in\mathbb{N}}$ is a Følner sequence of subsets of $\mathbb{N}$ and

$$\gamma_4 \le \liminf_{m\to\infty} \frac{1}{\#(J_m)}\sum_{i\in J_m} z(i) \le \limsup_{m\to\infty} \frac{1}{\#(J_m)}\sum_{i\in J_m} z(i) \le \gamma_4. \textbf{ Q}$$

This shows that

$$\max\{\limsup_{m\to\infty} \frac{1}{\#(I_m)}\sum_{i\in I_m} z(i) : \langle I_m\rangle_{m\in\mathbb{N}} \text{ is a Følner sequence of subsets of } \mathbb{N}\}$$

and

$$\max\{\liminf_{m\to\infty} \frac{1}{\#(I_m)}\sum_{i\in I_m} z(i) : \langle I_m\rangle_{m\in\mathbb{N}} \text{ is a Følner sequence of subsets of } \mathbb{N}\}$$

are both defined and equal to $\gamma_4 = \lim_{m\to\infty} \frac{1}{\#(J_m)} \sum_{i\in J_m} z(i)$.

(vii) $\gamma_4 \le \gamma_1$. **P** Take $\langle J_m\rangle_{m\in\mathbb{N}}$ from (vi). Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$. Set $\nu I = \lim_{m\to\mathcal{F}} \frac{\#(I\cap J_m)}{\#(J_m)}$ for $I \subseteq \mathbb{N}$. Then $\nu : \mathcal{P}\mathbb{N} \to [0,1]$ is additive and $\nu\mathbb{N} = 1$. If $I \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, then

$$\begin{aligned}|\#((I+k)\cap J_m) - \#(I\cap J_m)| &= |\#((I+k)\cap J_m) - \#((I+k)\cap(J_m+k))| \\ &\le |\#(J_m\triangle(J_m+k))| = o(\#(J_m))\end{aligned}$$

as $m\to\infty$, so $\nu(I+k) = \nu I$; thus $\nu \in P$. As in (iv), it is easy to check that

$$⨍ z\,d\nu = \lim_{m\to\mathcal{F}} \frac{1}{\#(J_m)}\sum_{i\in J_m} z(i) = \gamma_4,$$

so $\gamma_4 \le \gamma_1$. **Q**

(viii) Putting these together, we have the result.

(b) If $x \in \ell^1$ and $x \ge 0$, then $⨍ x * z\,d\nu = \|x\|_1 ⨍ z\,d\nu$ for every $\nu \in P$; this is elementary if x is eventually zero, and the general result follows by continuity. So $d_s^*(z) = d_s^*(x*z) \le \|x*z\|_\infty$ whenever $x \in \ell^1$, $x \ge 0$ and $\|x\|_1 = 1$. On the other hand, given k, $m \in \mathbb{N}$, $\frac{1}{m+1}\sum_{i=k}^{k+m} z(i) = (x_{m+1} * z)(k+m)$ where x_{m+1} is defined as in the proof of 1C, so $\inf\{\|x*z\|_\infty : x \in (\ell^1)^+, \|x\|_1 = 1\} \le d_s^*(z)$.

(c)$(\boldsymbol{\alpha})\Rightarrow(\boldsymbol{\gamma})$ Suppose that $\mathrm{WDL}_{i\to\infty} z(i) = \gamma$, and that $\langle I_m\rangle_{m\in\mathbb{N}}$ is a Følner sequence of subsets of $\mathbb{N}$. Let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$. Set $\nu I = \lim_{m\to\mathcal{F}} \frac{\#(I\cap I_m)}{\#(I_m)}$ for every $I \subseteq \mathbb{N}$. Then $\nu : \mathcal{P}\mathbb{N} \to [0,1]$ is additive, and $\nu\mathbb{N} = 1$. If $k \in \mathbb{N}$ and $I \subseteq \mathbb{N}$, then

$$\begin{aligned}|\#((I+k)\cap I_m) - \#(I\cap I_m)| &= |\#((I+k)\cap I_m) - \#((I+k)\cap(I_m+k))| \\ &\le \#(I_m\triangle(I_m+k)) = o(\#(I_m))\end{aligned}$$

as $m\to\infty$, so

$$\nu(I+k) = \lim_{m\to\mathcal{F}} \frac{\#((I+k)\cap I_m)}{\#(I_m)} = \lim_{m\to\mathcal{F}} \frac{\#(I\cap I_m)}{\#(I_m)} = \nu I.$$

Thus $\nu \in P$, and $\gamma = ⨍ z\,d\nu$. On the other hand,

$$⨍ \chi I\,d\nu = \nu I = \lim_{m\to\mathcal{F}} \frac{1}{\#(I_m)}\sum_{i\in I_m} \chi I(i),$$

so $⨍ w\,d\nu = \lim_{m\to\mathcal{F}} \frac{1}{\#(I_m)} \sum_{i\in I_m} w(i)$ for every $w \in \ell^\infty$ and, in particular,

$$\lim_{m\to\mathcal{F}} \frac{1}{\#(I_m)}\sum_{i\in I_m} z(i) = ⨍ z\,d\nu = \gamma.$$

As $\mathcal{F}$ is arbitrary, $\lim_{m\to\infty} \frac{1}{\#(I_m)} \sum_{i\in I_m} z(i) = \gamma$; as $\langle I_m\rangle_{m\in\mathbb{N}}$ is arbitrary, (γ) is true.

not-(β)⇒not-(γ) If (β) is false, we can find an $\epsilon > 0$ and sequences $\langle k(m)\rangle_{m\in\mathbb{N}}$, $\langle l(m)\rangle_{m\in\mathbb{N}}$ such that $l(m) \ge m$ and $|\gamma - \frac{1}{l(m)+1} \sum_{i=k(m)}^{k(m)+l(m)} z(i)| > \epsilon$ for every m. Setting $I_m = \{i : k(m) \le i \le k(m) + l(m)\}$, we have $|\gamma - \frac{1}{\#(I_m)} \sum_{i\in I_m} | > \epsilon$ for every m. On the other hand, for any $k \in \mathbb{N}$,

$$\frac{\#(I_m\triangle(k+I_m))}{\#(I_m)} \le \frac{2k}{\#(I_m)} \to 0$$

as $m \to \infty$, so $\langle I_m\rangle_{m\in\mathbb{N}}$ is a Følner sequence, and (γ) is false.

(β)⇒(α) If (β) is true, then

$$\gamma = \limsup_{m\to\infty} \sup_{k\in\mathbb{N}} \frac{1}{m+1}\sum_{i=k}^{k+m} z(i) = \liminf_{m\to\infty} \inf_{k\in\mathbb{N}} \frac{1}{m+1}\sum_{i=k}^{k+m} z(i);$$

applying (a) to z and $-z$, we get $\sup_{\nu\in P} ⨍ z\,d\nu \le \gamma \le \inf_{\nu\in P} ⨍ z\,d\nu$, so $\mathrm{WDL}_{n\to\infty} z(n)$ is defined and equal to γ.

Remark Part (b) is Theorem 4.5 of Kuipers & Niederreiter 74, where it is attributed to Lorentz 48.

3C Definition Let X be a topological space and μ a probability measure on X. I say that a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in X is **well-distributed** if $\mu F \ge d_s^*(\{i : x_i \in F\})$ for every measurable closed set $F \subseteq X$.

Of course a well-distributed sequence is equidistributed in the sense of Fremlin 03, §491.

3D Proposition Let X be a topological space, μ a probability measure on X and $\langle x_i\rangle_{i\in\mathbb{N}}$ a sequence in X. Write $C_b(X)$ for the space of bounded continuous real-valued functions on X.

(a) $\langle x_i\rangle_{i\in\mathbb{N}}$ is well-distributed iff $\int f d\mu \le ⨍ f(x_i)\nu(di)$ for every measurable bounded lower semi-continuous function $f : X \to \mathbb{R}$ and every $\nu \in P$.

(b) If μ measures every zero set and $\langle x_i\rangle_{i\in\mathbb{N}}$ is well-distributed, then $\mathrm{WDL}_{i\to\infty} f(x_i)$ is defined and equal to $\int f d\mu$ for every $f \in C_b(X)$.

(c) Suppose that μ measures every zero set in X. If $\mathrm{WDL}_{i\to\infty} f(x_i)$ is defined and equal to $\int f d\mu$ for every $f \in C_b(X)$, then $d_s^*(\{n : x_n \in F\}) \le \mu F$ for every zero set $F \subseteq X$.

(d) Suppose that X is normal and μ measures every zero set and is inner regular with respect to the closed sets. If $\mathrm{WDL}_{i\to\infty} f(x_i)$ is defined and equal to $\int f d\mu$ for every $f \in C_b(X)$, then $\langle x_i\rangle_{i\in\mathbb{N}}$ is well-distributed.

(e) Suppose that μ is τ-additive and there is a base $\mathcal{G}$ for the topology of X, consisting of measurable sets and closed under finite unions, such that $\mu G \le \nu(\{i : x_i \in G\})$ for every $G \in \mathcal{G}$ and $\nu \in P$. Then $\langle x_i\rangle_{i\in\mathbb{N}}$ is well-distributed.

(f) Suppose that X is completely regular and that μ measures every zero set and is τ-additive. Then $\langle x_i\rangle_{i\in\mathbb{N}}$ is well-distributed iff $\mathrm{WDL}_{i\to\infty} f(x_i)$ is defined and equal to $\int f d\mu$ for every $f \in C_b(X)$.

(g) Suppose that X is metrizable and that μ is a topological measure. Then $\langle x_i\rangle_{i\in\mathbb{N}}$ is well-distributed iff $\mathrm{WDL}_{i\to\infty} f(x_i)$ is defined and equal to $\int f d\mu$ for every $f \in C_b(X)$.

(h) Suppose that X is compact, Hausdorff and zero-dimensional, and that μ is a Radon measure on X. Then $\langle x_i\rangle_{i\in\mathbb{N}}$ is well-distributed iff $d_s(\{n : x_n \in G\})$ is defined and equal to μG for every open- and-closed subset G of X.

proof (a)(i) Suppose that $\langle x_i\rangle_{i\in\mathbb{N}}$ is well-distributed. Let $f : X \to [0, 1]$ be a measurable lower semi-continuous function and $\nu \in P$. Take any $k \ge 1$. For each $j < k$ set $G_j = \{x : f(x) > \frac{j}{k}\}$. Then

$$\begin{aligned}\nu\{i : x_i \in G_j\} &= 1 - \nu\{i : x_i \in X \setminus G_j\}\\ &\ge 1 - d_s^*(\{i : x_i \in X \setminus G_j\}) \ge 1 - \mu(X \setminus G_j)\end{aligned}$$

(because $\langle x_i\rangle_{i\in\mathbb{N}}$ is well-distributed and $X \setminus G_j$ is a measurable closed set)

$$= \mu G_j.$$

Also $f - \frac{1}{k}\chi X \le \frac{1}{k}\sum_{j=1}^{k}\chi G_j \le f$. So

$$\int f d\mu - \frac{1}{k} \le \frac{1}{k}\sum_{j=1}^{k}\mu G_j \le \frac{1}{k}\sum_{j=1}^{k}\nu\{i : x_i \in G_j\}$$
$$= \frac{1}{k}\sum_{j=1}^{k}\fint \chi G_j(x_i)\nu(di) \le \fint f(x_i)\nu(di).$$

As k is arbitrary, $\int f d\mu \le \fint f(x_i)\nu(di)$.

The argument above depended on f taking values in $[0,1]$. But multiplying by an appropriate positive scalar we see that $\int f d\mu \le \liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n} f(x_i)$ for every bounded measurable lower semi-continuous $f : X \to [0,\infty[$, and adding a multiple of χX we see that the same formula is valid for all bounded measurable lower semi-continuous $f : X \to \mathbb{R}$.

(ii) Conversely, suppose that $\int f d\mu \le \fint f(x_i)\nu(di)$ for every bounded measurable lower semi-continuous $f : X \to \mathbb{R}$ and every $\nu \in P$. Let $F \subseteq X$ be a measurable closed set. Then $-\chi F$ is lower semi-continuous, so $-\mu F \le \fint(-\chi F)(x_i)\nu(di) = -\nu\{i : x_i \in F\}$, that is, $\nu\{i : x_i \in F\} \le \mu F$, for every $\nu \in P$. Taking the supremum over ν, $d_s^*(\{i : x_i \in F\}) \le \mu F$. As F is arbitrary, $\langle x_i\rangle_{i\in\mathbb{N}}$ is well-distributed.

(b) Apply (a) to the lower semi-continuous functions f and $-f$.

(c) Recall that if μ measures every zero set, then every bounded continuous real-valued function is integrable (Fremlin 03, 4A3L). Let $F \subseteq X$ be a zero set, and $\epsilon > 0$. Then there is a continuous $f : X \to \mathbb{R}$ such that $F = f^{-1}[\{0\}]$. Let $\delta > 0$ be such that $\mu\{x : 0 < |f(x)| \le \delta\} \le \epsilon$, and set $g = (\chi X - \frac{1}{\delta}|f|)^+$. Then $g : X \to [0,1]$ is continuous, so

$$d^*(\{i : x_i \in F\}) \le \limsup_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n} g(x_i) = \int g\, d\mu \le \mu\{x : |f(x)| \le \delta\} \le \mu F + \epsilon.$$

As ϵ and F are arbitrary, we have the result.

(d) Let $F \subseteq X$ be a measurable closed set and $\epsilon > 0$. Because μ is inner regular with respect to the closed sets, there is a measurable closed set $F' \subseteq X \setminus F$ such that $\mu F' \ge \mu(X \setminus F) - \epsilon$. Because X is normal, there is a continuous function $f : X \to [0,1]$ such that $\chi F \le f \le \chi(X \setminus F')$. Let $\nu \in P$. Then

$$\nu\{i : x_i \in F\} \le \fint f(x_i)\nu(di) = \int f d\mu \le \mu(X \setminus F') \le \mu F + \epsilon.$$

As ν and F and ϵ are arbitrary, $\langle x_i\rangle_{i\in\mathbb{N}}$ is well-distributed.

(e) Let $F \subseteq X$ be a measurable closed set, and $\epsilon > 0$. Let $\mathcal{G}_1$ be the family of members of $\mathcal{G}$ disjoint from F. Then $\mathcal{G}_1$ is upwards-directed and $\bigcup\mathcal{G}_1 = X \setminus F$; because μ is τ-additive, there is a $G \in \mathcal{G}_1$ such that $\mu G > \mu(X \setminus F) - \epsilon$. Now, for any $\nu \in P$,

$$\nu\{i : x_i \in F\} \le 1 - \nu\{i : x_i \in G\} \le 1 - \mu G \le \nu F + \epsilon.$$

As ν and F and ϵ are arbitrary, $\langle x_i\rangle_{i\in\mathbb{N}}$ is well-distributed.

(f) (i) If $\langle x_i\rangle_{i\in\mathbb{N}}$ is well-distributed then (b) tells us that $\int f d\mu = \mathrm{WDL}_{i\to\infty} f(x_i)$ for every $f \in C_b(X)$. (ii) Suppose that $\int f d\mu = \mathrm{WDL}_{i\to\infty} f(x_i)$ for every $f \in C_b(X)$. If $G \subseteq X$ is a cozero set, we can apply (c) to its complement to see that $\mu G \le \nu\{i : x_i \in G\}$ for every $\nu \in P$. So applying (e) with $\mathcal{G}$ the family of cozero sets we see that $\langle x_i\rangle_{i\in\mathbb{N}}$ is well-distributed.

(g) Because every closed set is a zero set, this follows at once from (b) and (c).

(h) If $\langle x_i\rangle_{i\in\mathbb{N}}$ is well-distributed and $G \subseteq X$ is open-and-closed, then $d_s^*(\{i : x_i \in G\}) \le \mu G$ because G is closed and $d_s^*(\{n : x_n \notin G\}) \le 1 - \mu G$ because G is open; so $d_s(\{i : x_i \in G\}) = \mu G$. If the condition is satisfied, then (e) tells us that $\langle x_i\rangle_{i\in\mathbb{N}}$ is well- distributed.

3E Proposition (a) Suppose that X and Y are topological spaces, μ a probability measure on X and $f : X \to Y$ a continuous function. If $\langle x_i \rangle_{i\in\mathbb{N}}$ is a sequence in X which is well-distributed with respect to μ, then $\langle f(x_i)\rangle_{i\in\mathbb{N}}$ is well-distributed with respect to the image measure μf^{-1}.

(b) Suppose that (X, μ) and (Y, ν) are topological probability spaces and $f : X \to Y$ is a continuous inverse-measure-preserving function. If $\langle x_i \rangle_{i\in\mathbb{N}}$ is a sequence in X which is well-distributed with respect to μ, then $\langle f(x_i)\rangle_{i\in\mathbb{N}}$ is well-distributed with respect to ν.

proof (a) Let $F \subseteq Y$ be a closed set which is measured by μf^{-1}. Then $f^{-1}[F]$ is a closed set in X measured by μ. So

$$d_s^*(\{i : f(x_i) \in F\}) = d_s^*(\{i : x_i \in f^{-1}[F]\}) \le \mu f^{-1}[F] = \mu f^{-1}[F].$$

(ii) Replace 'μf^{-1}' above by 'ν'.

3F Examples (a) For almost every $\boldsymbol{x} = \langle \xi_i \rangle_{i\in\mathbb{N}} \in [0, 1]^{\mathbb{N}}$, $\langle \xi_i \rangle_{i\in\mathbb{N}}$ is equidistributed (Fremlin 03, 491Eb) but not well-distributed (because there will be arbitrarily long gaps $[k, k + n[$ in which every $\xi_i \le \frac{1}{2}$).

(b) For $\alpha \in \mathbb{R}$ write $<\alpha> \in [0, 1[$ for its fractional part, so that $\alpha - <\alpha> \in \mathbb{N}$. Examining the proof of Weyl's Equidistribution Theorem (Fremlin 01, 281N), we see that if $\eta_1, \dots , \eta_r$ are real numbers such that $1, \eta_1, \dots , \eta_r$ are linearly independent over $\mathbb{Q}$, and we set $x_n = (<n\eta_1>, \dots , <n\eta_r>) \in [0, 1[^r$ for each n, then $\langle x_n\rangle_{n\in\mathbb{N}}$ is well-distributed for Lebesgue measure on $[0, 1[^r$.

(c) For a space with an equidistributed sequence which has no well-distributed sequence, see 3O below.

3G Proposition The usual measure μ of $\{0, 1\}^{\mathfrak{c}}$ has a well-distributed sequence.

proof Take $\langle A_\xi\rangle_{\xi<\mathfrak{c}}$ from 1I, and set $x_n(\xi) = \chi A_\xi(n)$ for each $n \in \mathbb{N}$, $\xi \in \mathfrak{c}$. If $V \subseteq \{0, 1\}^{\mathfrak{c}}$ is an open-and-closed set of the form $\{x : x{\restriction}K = u\}$ for some finite $K \subseteq \mathfrak{c}$ and $u \in \{0, 1\}^K$, then $d_s(\{i : x_i \in V\})$ is defined and equal to μV. Since any open-and-closed set in $\{0, 1\}^{\mathfrak{c}}$ is a finite disjoint union of such basic sets, $d_s(\{i : x_i \in V\})$ is defined and equal to μV for every open-and-closed set; by 3D(h), $\langle x_i\rangle_{i\in\mathbb{N}}$ is a well-distributed sequence in $\{0, 1\}^{\mathfrak{c}}$.

3H Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a topological probability space with a countable network consisting of measurable sets. Then there is a well-distributed sequence in X.

proof (a) Let $\mathcal{E}$ be a countable network consisting of measurable sets; we may suppose that $\mathcal{E}$ is a subalgebra of $\mathcal{P}X$. Then there is a Boolean homomorphism $\pi : \mathcal{E} \to \mathcal{P}\mathbb{N}$ such that $d_s(\pi E)$ is defined and equal to μE for every $E \in \mathcal{E}$. **P** Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathcal{E}$ and for $n \in \mathbb{N}$ let $\mathcal{E}_n$ be the subalgebra of $\mathcal{E}$ generated by $\{E_i : i < n\}$. I seek to define π as the union of a non-decreasing sequence $\langle \pi_n\rangle_{n\in\mathbb{N}}$ where each $\pi_n : \mathcal{E}_n \to \mathcal{P}\mathbb{N}$ is a Boolean homomorphism. The inductive hypothesis will be that $d_s(\pi_n E) = \mu E$ for every $E \in \mathcal{E}_n$. Start with $\pi_0 X = \mathbb{N}$, $\pi_0\emptyset = \emptyset$. Given $\mathcal{E}_n$ and π_n, let $\mathcal{A}_n$ be the set of atoms of $\mathcal{E}_n$. For each $A \in \mathcal{A}_n$, $\pi_n A$ has Banach density; by 1F(d), or otherwise, there is an $I_A \subseteq \pi_n A$ with Banach density equal to $\mu(E_n \cap A)$. Now set $J = \bigcup_{A\in\mathcal{A}_n} I_A$. By 312N, there is a unique Boolean homomorphism $\pi_{n+1} : \mathcal{E}_{n+1} \to \mathcal{P}\mathbb{N}$, extending π_n, such that $\pi_{n+1}E_n = J$; it is easy to check that π_{n+1} has the properties required to continue the induction. **Q**

(b) Now choose $x_n \in X$, for $n \in \mathbb{N}$, so that $x_n \in E_i$ whenever $i \le n$ and $n \in \pi E_i$. (This is always possible because if $J = \{i : i \le n,\ n \in \pi E_i\}$ then $n \in \mathbb{N} \cap \bigcap_{i\in J} \pi E_i = \pi(X \cap \bigcap_{i\in J} E_i)$ and $X \cap \bigcap_{i\in J} E_i$ cannot be empty.) If $E \in \mathcal{E}$, then there are i, $j \in \mathbb{N}$ such that $E = E_i$ and $X \setminus E = E_j$, so that

$$\{n : x_n \in E_i\}\triangle\pi E_i \subseteq \max(i, j) \text{ is finite, and } d_s(\{n : x_n \in E\}) = d_s(\pi E) = \mu E.$$

By 3D(e), $\langle x_i\rangle_{i\in\mathbb{N}}$ is well-distributed with respect to μ and the topology $\mathfrak{S}$ generated by $\mathcal{E}$. But $\mathfrak{S}$ is finer than $\mathfrak{T}$, so 3E tells us that it is also well-distributed with respect to μ and $\mathfrak{T}$.

3I Proposition (Kuipers & Niederreiter 74, p. 202, Theorem 3.9) Let X be a completely regular space and μ a Radon probability measure on X. Define $T : X^{\mathbb{N}} \to X^{\mathbb{N}}$ by setting $(Tx)(i) = x(i + 1)$ for $x \in X^{\mathbb{N}}$, $i \in \mathbb{N}$. Then if $x \in X^{\mathbb{N}}$ is a well-distributed sequence and $y \in \overline{\{T^k x : k \in \mathbb{N}\}}$ in $X^{\mathbb{N}}$, y is well-distributed.

proof Let $f \in C_b(X)$ and $\epsilon > 0$. Set $\gamma = \int f d\mu$. Then $\mathrm{WDL}_{i\to\infty} f(x_i) = \gamma$ (3D(g)). Let $m_0 \in \mathbb{N}$ be such that $\frac{1}{m+1}\sum_{i=k}^{k+m} f(x_i) \in [\gamma-\epsilon, \gamma+\epsilon]$ whenever $m \ge m_0$ and $k \in \mathbb{N}$ (3B(b)). Now take any $m \ge m_0$ and $k \in \mathbb{N}$. Then there is an $r \in \mathbb{N}$ such that $|f(y_i) - f(x_{r+i})| \le \epsilon$ whenever $k \le i \le k+m$. In this case,

$$|\frac{1}{m+1}\sum_{i=k}^{k+m} f(y_i) - \gamma| \le |\frac{1}{m+1}\sum_{i=k}^{k+m} f(y_i) - f(x_{r+i})| + |\frac{1}{m+1}\sum_{i=k}^{k+m} f(x_{r+i}) - \gamma|$$

$$\le \epsilon + |\frac{1}{m+1}\sum_{i=r+k}^{r+k+m} f(x_i) - \gamma| \le 2\epsilon.$$

This is true for every $m \ge m_0$ and every k; as ϵ is arbitrary, $\mathrm{WDL}_{i\to\infty} f(y_i) = \gamma$; as f is arbitrary, $\langle y_i\rangle_{i\in\mathbb{N}}$ is well-distributed.

3J Lemma Let $\langle(X_m, \mathfrak{T}_m, \Sigma_m, \mu_m)\rangle_{m\le n}$ be a family of τ-additive topological probability spaces, and λ the τ-additive product measure on $X = \prod_{m\le n} X_m$. For each $m \in \mathbb{N}$ let $\langle\nu_{mi}\rangle_{i\in\mathbb{N}}$ be a sequence of topological probability measures on X_m such that $\mu_m F \ge \limsup_{i\to\infty} \nu_{mi} F$ for every closed set $F \subseteq X_m$. For each $i \in \mathbb{N}$, let λ_i be the c.l.d. product measure of $\langle\nu_{mi}\rangle_{m\le n}$. Then $\lambda V \ge \limsup_{i\to\infty} \lambda_i^* V$ for every closed set $V \subseteq X$.

Remark The 'τ-additive product measure' here is supposed to be the one described in Fremlin 03, §417. The only properties we shall need of it in this note are that for any family $\langle(X_\xi, \mathfrak{T}_\xi, \Sigma_\xi, \mu_\xi)\rangle_{\xi\in I}$ of τ-additive topological probability spaces, there is a canonical τ-additive topological probability measure on $X = \prod_{\xi\in I} X_\xi$ which extends the ordinary (completed) product probability measure.

proof Induce on n. If $n = 0$ then λ_i is just the completion of ν_{0i} and the result is trivial. For the inductive step to $n \ge 1$, we can identify each λ_i with the c.l.d. product of λ'_i and ν_{ni}, where λ'_i is the product of $\langle\nu_{mi}\rangle_{m<n}$.

Now take a closed set $V \subseteq X$ and $\epsilon > 0$. Because λ is τ-additive, there is an open set $W \subseteq X \setminus V$, expressible as a finite union of sets of the form $\prod_{m\le n} H_m$ where $H_m \subseteq X_m$ is open for every n, such that $\lambda W \ge 1 - \lambda V - \epsilon$. For each $y \in X'$, set $f(y) = \mu_m W[\{y\}]$, so that $\lambda W = \int f(y)\lambda'(dy)$, where λ' is the product of $\langle\mu_m\rangle_{m<n}$. Each $W[\{y\}]$ is open, so $f(y) \le \liminf_{i\to\infty} \nu_{mi} W[\{y\}]$. Since there are only finitely many sets which appear as $W[\{y\}]$, there is an $i_0 \in \mathbb{N}$ such that $f(y) \le \nu_{mi} W[\{y\}] + \epsilon$ whenever $y \in X'$ and $i \ge i_0$. Next, there is an $i_1 \ge i_0$ such that $\int f(y)\lambda'(dy) \le \int f(y)\lambda'_i(dy) + 2\epsilon$ for every $i \ge i_1$. **P** Take $k \ge \frac{1}{\epsilon}$ and for $l \le k$ set $U_l = \{y : f(y) > \frac{l}{k}\}$. Then U_l is a finite union of products of open sets, so is open and measured by every λ'_i. By the inductive hypothesis there is an $i_1 \ge i_0$ such that $\lambda' U_l \le \lambda'_i U_l + \epsilon$ for every $l \le k$ and $i \ge i_1$. Now, for $i \ge i_1$,

$$\int f(y)\lambda'(dy) \le \tfrac{1}{k}\textstyle\sum_{l=0}^{k-1} \lambda' U_l \le \epsilon + \tfrac{1}{k}\sum_{l=0}^{k-1} \lambda'_i U_l \le 2\epsilon + \int f(y)\lambda'_i(dy). \text{ } \mathbf{Q}$$

So for all $i \ge \max(i_0, i_1)$, we shall have

$$\lambda W = \int f(y)\lambda'(dy) \le 2\epsilon + \int f(y)\lambda'_i(dy) \le 3\epsilon + \int \nu_{mi} W[\{y\}]\lambda'_i(dy) = 3\epsilon + \lambda_i W.$$

Turning this round,

$$\lambda V \ge 1 - \lambda W + \epsilon \ge 1 - \lambda_i W + 4\epsilon = \lambda_i(X \setminus W) + 4\epsilon \ge \lambda_i^* V + 4\epsilon$$

for every $i \ge i_1$. As ϵ is arbitrary, $\lambda V \ge \limsup_{i\to\infty} \lambda_i V$, and the induction continues.

3K Lemma Let X be a topological space and μ a measure on X, and suppose that $\langle t_i\rangle_{i\in\mathbb{N}}$ is a well-distributed sequence in X. For k, $m \in \mathbb{N}$ let ν_{mk} be the point-supported measure on X defined by setting

$$\nu_{mk}(E) = 2^{-m}\#(\{K : K \subseteq m, t_{k+\#(K)} \in E\}$$

for every $E \subseteq X$. Then for every measurable open set $G \subseteq X$ and $\epsilon > 0$ there is an $m_0 \in \mathbb{N}$ such that $\nu_{mk} G \ge \mu G - \epsilon$ whenever $m \ge m_0$ and $k \in \mathbb{N}$.

proof For m, $k \in \mathbb{N}$ define $x_{mk} \in \mathbb{R}^{\mathbb{N}}$ by setting $x_{mk}(k+i) = 2^{-m}\#([m]^i)$ for $i \le m$, $x_{mk}(i) = 0$ if $i < k$ or $i > k+m$. Then $x_{mk} \in (\ell^1)^+$, $\|x_{mk}\|_1 = 1$ and $\operatorname{Var}_{\mathbb{N}} x_{mk} \le 2^{-m+1}\sup_{i\le m}\#([m]^i) = \alpha_m$ say (because $x_{mk}(i) \le x_{mk}(i+1)$ if $i < k + \frac{m}{2}$, and $x_{mk}(i) \ge x_{mk}(i+1)$ if $i \ge k + \frac{m}{2}$). Setting $I = \{i : t_i \in X \setminus G\}$, we see that

$$\liminf_{m\to\infty}\inf_{k\in\mathbb{N}} \nu_{mk}G = 1 - \limsup_{m\to\infty}\sup_{k\in\mathbb{N}}\sum_{i\in I} x_{mk}(i) \ge 1 - d^*(X\setminus G)$$

(1C, because $\limsup_{m\to\infty}\sup_{k\in\mathbb{N}}\operatorname{Var}_{\mathbb{N}}(x_{mk}) \le \lim_{m\to\infty}\alpha_m = 0$)

$$\ge 1 - \mu(X\setminus G) = \mu G,$$

as required.

3L Theorem Let $\langle (X_\xi, \mathfrak{T}_\xi, \Sigma_\xi, \mu_\xi)\rangle_{\xi\in I}$ be a family of τ-additive topological probability spaces, each of which has a well-distributed sequence. If $\#(I) \le \mathfrak{c}$, the τ-additive product measure λ on $X = \prod_{\xi\in I} X_\xi$ has a well-distributed sequence.

proof (a) For $J \subseteq I$ set $Z_J = \prod_{\xi\in J} Z_\xi$ and let λ_J be the τ-additive product measure on Z_J. Let $\langle A_\xi\rangle_{\xi\in I}$ be an almost-disjoint family of infinite subsets of $\mathbb{N}$. For each $\xi \in I$, let $\langle t_{\xi i}\rangle_{i\in\mathbb{N}}$ be a well-distributed sequence in X_ξ. For $K \in [\mathbb{N}]^{<\omega}$ set $n_K = \sum_{i\in K} 2^i$, as in 1I. Now define $\langle x_i\rangle_{i\in\mathbb{N}}$ in X by setting $x_{n_K}(\xi) = t_{\xi,\#(K\cap A_\xi)}$ for $\xi \in I$ and $K \in [\mathbb{N}]^{<\omega}$.

(b) For $J \in [I]^{<\omega}$ and disjoint L, $M \in [\mathbb{N}]^{<\omega}$ let $\nu^{(J)}_{LM}$ be the point-supported measure on Z_J defined by setting

$$\nu^{(J)}_{LM}W = 2^{-\#(L)}\#(\{K : K \subseteq L,\, x_{n_{K\cup M}}\restriction J \in W\})$$

for every $W \subseteq Z_J$. Observe that if J, L and M are such that $\langle A_\xi \cap L\rangle_{\xi\in J}$ is a disjoint cover of L, then $\nu^{(J)}_{LM}$ is just the product of the measures $\nu^{(\{\xi\})}_{A_\xi\cap L, A_\xi\cap M}$ for $\xi \in J$, interpreted as measures on $X_\xi \equiv Z_{\{\xi\}}$. **P** If $z \in Z_J$, then for any $K \subseteq L$ we have

$$x_{n_{K\cup M}}\restriction J = z \iff t_{\xi,\#(K\cap A_\xi)+\#(M\cap A_\xi)} = z(\xi) \text{ for every } \xi \in J,$$

so

$$\begin{aligned}\nu^{(J)}_{L\cup M}(\{z\}) &= 2^{-\#(L)}\#(\{K : K \subseteq L,\, x_{n_{K\cup M}}\restriction J = z\})\\ &= \prod_{\xi\in J} 2^{-\#(A_\xi\cap L)}\#(\{K : K \subseteq A_\xi \cap L,\, t_{\xi,\#(K)+\#(M\cap A_\xi)} = z(\xi)\})\\ &= \prod_{\xi\in J} 2^{-\#(A_\xi\cap L)}\#(\{K : K \subseteq A_\xi \cap L,\, x_{n_{A_\xi\cap(L\cup M)}}(\xi) = z(\xi)\})\\ &= \prod_{\xi\in J}\nu^{(\{\xi\})}_{A_\xi\cap L, A_\xi\cap M}(\{z(\xi)\}).\ \mathbf{Q}\end{aligned}$$

(c) Now suppose that $J \in [I]^{<\omega}$ and that $W \subseteq Z_J$ is open. Then for every $\epsilon > 0$ there is an $m_0 \in \mathbb{N}$ such that whenever L, $M \in [\mathbb{N}]^{<\omega}$ are disjoint, $\langle A_\xi \cap L\rangle_{\xi\in J}$ is a disjoint cover of L and $\#(A_\xi \cap L) \ge m_0$ for every $\xi \in J$, then $\nu^{(J)}_{LM}(W) \ge \lambda_J W - \epsilon$. **P?** Otherwise, we can find $\langle L_n\rangle_{n\in\mathbb{N}}$, $\langle M_n\rangle_{n\in\mathbb{N}}$ in $[\mathbb{N}]^{<\omega}$ such that, for each $n \in \mathbb{N}$, $L_n \cap M_n = \emptyset$, $\langle A_\xi \cap L_n\rangle_{\xi\in J}$ is a disjoint cover of L_n, $\#(A_\xi \cap L_n) \ge n$ for every $\xi \in J$ and $\nu^{(J)}_{L_nM_n}(W) < \lambda_J W - \epsilon$ for every n. For $\xi \in J$, $n \in \mathbb{N}$ write $\hat\nu_{\xi n}$ for the point-supported measure $\nu^{(\{\xi\})}_{A_\xi\cap L_n, A_\xi\cap M_n}$, so that $\nu^{(J)}_{L_nM_n}$ is the product of $\langle\hat\nu_{\xi n}\rangle_{\xi\in J}$ for each n, by (b). If $\xi \in J$ and $G \subseteq X_\xi$ is open, then $\mu_\xi G \le \liminf_{n\to\infty}\hat\nu_{\xi n}G$, by 3K, because $\lim_{n\to\infty}\#(A_\xi \cap L_n) = \infty$. By 3J, $\lambda_J W \le \liminf_{n\to\infty}\nu^{(J)}_{L_nM_n}(W)$; but this is impossible. **XQ**

(d) Again suppose that $J \in [I]^{<\omega}$ and that $W \subseteq Z_J$ is open. Then for every $\epsilon > 0$ there is an $m_0 \in \mathbb{N}$ such that whenever L, $M \in [\mathbb{N}]^{<\omega}$ are disjoint and there is an $L' \subseteq L$ such that $\langle A_\xi \cap L'\rangle_{\xi\in J}$ is a disjoint cover of L' and $\#(A_\xi \cap L') \ge m_0$ for every $\xi \in J$, then $\nu^{(J)}_{LM}(W) \ge \lambda_J W - \epsilon$. **P** Take the same m_0 as in (c). Suppose that L, L' and M are as stated. Then

$$\begin{aligned}\nu^{(J)}_{LM}W &= 2^{-\#(L)}\#(\{K : K \subseteq L,\, x_{n_{K\cup M}}\restriction J \in W\})\\ &= 2^{-\#(L\setminus L')}\sum_{M'\subseteq L\setminus L'} 2^{-\#(L')}\#(\{K : K \subseteq L',\, x_{n_{K\cup M'\cup M}}\restriction J \in W\})\\ &= 2^{-\#(L\setminus L')}\sum_{M'\subseteq L\setminus L'} \nu^{(J)}_{L',M\cup M'}(W)\\ &\ge 2^{-\#(L\setminus L')}\sum_{M'\subseteq L\setminus L'} \lambda_J W - \epsilon = \lambda_J W - \epsilon,\end{aligned}$$

as required. **Q**

(e) Let $W \subseteq X$ be an open set and $\epsilon > 0$. Then there is an $m \in \mathbb{N}$ such that $2^{-m}\#(\{i : x_i \in W,\, 2^m l \le i < 2^m(l+1)\}) \ge \lambda W - 2\epsilon$ for every $l \in \mathbb{N}$. **P** Because λ is τ-additive, there are a finite $J \subseteq I$ and an open set $W' \subseteq Z_J$ such that $\lambda_J W' \ge \lambda W - \epsilon$ and $W \supseteq \{x : x \in X,\, x\restriction J \in W'\}$. By (d), there is an $m_0 \in \mathbb{N}$ such that $\nu^{(J)}_{LM}(W') \ge \lambda_J W' - \epsilon$ whenever L, $M \subseteq \mathbb{N}$ are disjoint finite sets and there is an $L' \subseteq L$ such that $\langle A_\xi \cap J\rangle_{\xi\in J}$ is a disjoint cover of L' and $\#(A_\xi \cap L') \ge m_0$ for every $\xi \in J$. Because $\langle A_\xi\rangle_{\xi\in I}$ is almost disjoint, there is an $m_1 \in \mathbb{N}$ such that $A_\xi \cap A_\eta \subseteq m_1$ for all distinct ξ, $\eta \in J$. Because every A_ξ is infinite, there is an $m \ge m_1$ such that $\#(A_\xi \cap m \setminus m_1) \ge m_0$ for every $\xi \in J$. Now suppose that $l \in \mathbb{N}$. Express $2^m l$ as n_M for $M \in [\mathbb{N}]^{<\omega}$; then $M \cap m = \emptyset$. Set $L = m$, $L' = \bigcup_{\xi\in J} A_\xi \cap m \setminus m_1$. Then we see that $\nu^{(J)}_{LM}(W') \ge \lambda W - 2\epsilon$. But now

$$\begin{aligned}\lambda W - 2\epsilon \le \nu^{(J)}_{LM}(W') &= 2^{-m}\#(\{K : K \subseteq m,\, x_{n_{K\cup M}}\restriction J \in W'\})\\ &\le 2^{-m}\#(\{K : K \subseteq m,\, x_{n_{K\cup M}} \in W\})\\ &= 2^{-m}\#(\{i : 2^m l \le i < 2^m(l+1),\, x_i \in W\});\end{aligned}$$

as l is arbitrary, we have the result. **Q**

(f) This shows that $\langle x_i\rangle_{i\in\mathbb{N}}$ is well-distributed in X, which is what we have been looking for.

3M Example Set $Y = \mathbb{N} \cup \{\infty\}$ with the one-point-compactification topology, so that Y is a compact metrizable space. Give Y the point-supported Radon measure ν such that $\nu\{n\} = 3^{-n-1}$ for $n \in \mathbb{N}$, $\mu\{\infty\} = \frac{1}{2}$. For $n \in \mathbb{N}$ define K_n, j_n as in (c) of the proof of 2C, and set $I_n = \{3^{n+1}l + j_n : l \in \mathbb{N}\}$, so that $\langle I_n\rangle_{n\in\mathbb{N}}$ is disjoint and $d_s(I_n) = 3^{-n}$ for each n. Set $y_i = n$ if $i \in I_n$. (The construction ensures that $\bigcup_{n\in\mathbb{N}} I_n = \mathbb{N}$.) Then $d_s(\{i : y_i = n\}) = \nu\{n\}$ for each n, so if $\mathcal{E}$ is the subalgebra of $\mathcal{P}Y$ generated by $\{\{n\} : n \in \mathbb{N}\}$ then $d_s(\{i : y_i \in E\}) = \nu E$ for every $E \in \mathcal{E}$; as $\mathcal{E}$ is a base for the topology of Y, $\langle y_i\rangle_{i\in\mathbb{N}}$ is well-distributed in Y.

Set $X = Y \times \{0, 1\}$, and let μ be the point-supported Radon measure such that $\mu\{(n, 0)\} = 3^{-n-1}$ for each n and $\mu\{(\infty, 1)\} = \frac{1}{2}$. Then $\nu = \mu f^{-1}$, where $f(y, 0) = f(y, 1) = y$ for every $y \in Y$. **?** Suppose, if possible, that $\langle x_i\rangle_{i\in\mathbb{N}}$ is a sequence in X, well-distributed for μ, such that $f(x_i) = y_i$ for every i. Set $I = \{i : x_i \in V\}$, where $V = Y \times \{0\}$. Then $d_s(I) = \mu V = \frac{1}{2}$, while $d_s(I_n \setminus I) = d_s(\{i : x_i = (n, 1)\}) = 0$ for every n. But there is no such I, by the argument in the proof of 2C. **X**

Remark Thus the claim in Proposition 4 of Losert 78b is false.

3N Proposition Let $(X, \mathfrak{T})$ be a regular Hausdorff space, and μ a Radon probability measure on X with a well-distributed sequence. Then it has a well-distributed sequence lying within the support of μ; in particular, the support of μ is separable.

proof (a) Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be a well-distributed sequence, and Z the support of μ. For each $m \in \mathbb{N}$, let $K_m \subseteq Z$ be a compact set of measure greater than $1 - 2^{-m}$. For $F \subseteq X$, set

$$A_m(F) = \{k : k \in \mathbb{N},\, F \cap \{x_{2^m k+i} : i < 2^m\} \ne 0\}.$$

Then $d^*_s(A_m(F)) < 1$ for every closed set $F \subseteq X \setminus K$. **P**

$$\inf_{r\ge 1}\sup_{l\in\mathbb{N}}\frac{1}{r}\#(A_m(F)\cap[l,l+r[)\le 2^m\inf_{r\ge 1}\sup_{l\in\mathbb{N}}\frac{1}{2^m r}\#(\{i:2^m l\le i<2^m(l+r),\,x_i\in F\})$$
$$=2^m d_s^*(\{i:x_i\in F\})\le 2^m\mu F<1.\ \mathbf{Q}$$

Because $\{A_m(F):F\subseteq X\setminus K_m \text{ is closed}\}$ is upwards-directed, there is a non-principal ultrafilter $\mathcal{F}_m$ on $\mathbb{N}$ containing no $A_m(F)$ for closed $F\subseteq X\setminus K_m$. Now $y_{mi}=\lim_{k\to\mathcal{F}_m}x_{2^m k+i}$ is defined and belongs to K_m for every $i<2^m$. **P?** Otherwise, every point of K_m belongs to an open set G such that $\{k:x_{2^m k+i}\notin G\}\in\mathcal{F}_m$. As K_m is compact, there is an open $G\supseteq K$ such that $\{k:x_{2^m k+i}\notin G\}\in\mathcal{F}_m$, that is, $A_m(X\setminus G)\in\mathcal{F}_m$; but this is impossible. **XQ**

(b) If $F\subseteq X$ is closed and $\epsilon>0$, there is an $r\in\mathbb{N}$ such that

$$\#(\{i:2^r l\le i<2^r(l+1),\,y_{mi}\in F\})\le 2^r(\mu F+2\epsilon)$$

whenever $m\ge r$ and $0\le l<2^{m-r}$. **P** Let $K\subseteq X\setminus F$ be a compact set such that $\mu(X\setminus K)\le\mu F+\epsilon$. Because X is regular, there is an open set $G\supseteq F$ such that $\overline{G}$ does not meet K, so that $\mu\overline{G}\le\mu F+\epsilon$. Let r be such that $\#(\{i:l\le i<l+2^r,\,x_i\in\overline{G}\})\le 2^r(\mu F+2\epsilon)$ for every $l\in\mathbb{N}$. If now $m\ge r$ and $l<2^{m-r}$, set $J=\{i:2^r l\le i<2^r(l+1),\,y_{mi}\in F\}$. For every $i\in J$, the set $B_i=\{k:x_{2^m k+i}\in G\}$ belongs to $\mathcal{F}_m$; so there is a $k\in\mathbb{N}\cap\bigcap_{i\in J}B_i$, and for this k we have

$$\#(J)\le\#(\{j:2^m k+2^r l\le j<2^m k+2^r(l+1),\,x_j\in\overline{G}\})\le 2^r(\mu F+2\epsilon),$$

as required. **Q**

(c) Let $\langle z_n\rangle_{n\in\mathbb{N}}$ be a re-indexing of $\langle y_{mi}\rangle_{m\in\mathbb{N},i<2^m}$ in lexicographic order. Then for any closed set F and any $\epsilon>0$ we have $r\in\mathbb{N}$ such that $\#(\{n:2^r l-1\le n<2^r(l+1)-1,\,z_n\in F\})\le 2^r(\mu F+2\epsilon)$ for every $l\ge 1$. So $d_s^*(\{n:z_n\in F\})\le\mu F+2\epsilon$. As F and ϵ are arbitrary, $\langle z_n\rangle_{n\in\mathbb{N}}$ is well-distributed. And of course every z_n belongs to Z.

3O Corollary There is a Radon probability measure on $\{0,1\}^{\mathfrak{c}}$ which has an equidistributed sequence but no well-distributed sequence.

proof Let (X,ν) be the Stone space of Lebesgue measure on $[0,1]$. Then we can identify X, as topological space, with a subspace of $\{0,1\}^{\mathfrak{c}}$; let μ be the corresponding Radon probability measure on $\{0,1\}^{\mathfrak{c}}$ defined by setting $\mu E=\nu(X\cap E)$ whenever the latter is defined. By Fremlin 03, 491Q, μ has an equidistributed sequence. But X is the support of μ and is not separable, so μ has no well-distributed sequence, by 3N.

Remark As far as I know, the best previous example was that of Losert 79, depending on the continuum hypothesis.

3P Proposition Let $\mathfrak{A}$ be a subalgebra of $\mathfrak{Z}_s$ such that $\bar{d}_s(a)$ is defined for every $a\in\mathfrak{A}$. Then there is a subalgebra Σ of $\mathcal{P}\mathbb{N}$ such that $d_s(I)$ is defined for every $I\in\Sigma$ and $(\mathfrak{A},\bar{d}_s\restriction\mathfrak{A})\cong(\Sigma,d_s\restriction\Sigma)$; in particular, $\mathfrak{A}$ is σ-centered.

proof Set $\mathrm{T}=\{I:I\subseteq\mathbb{N},\,I^\bullet\in\mathfrak{A}\}$, so that $d_s(I)$ is defined for every $I\in\mathrm{T}$. Let X be the Stone space of T and ν the Radon probability measure on X defined by setting $\nu\widehat{I}=d_s(I)$ for every $I\in\mathrm{T}$, where $\widehat{I}\subseteq X$ is the open-and-closed set corresponding to $I\in\mathrm{T}$. For $n\in\mathbb{N}$, set $x_n(I)=\chi I(n)$ for $I\in\mathrm{T}$; then $x_n\in X$. Now $\langle x_n\rangle_{n\in\mathbb{N}}$ is well-distributed for ν. **P** If $I\in\mathrm{T}$, $d_s(\{n:x_n\in\widehat{I}\})=d_s(I)$; by 3De, $\langle x_n\rangle_{n\in\mathbb{N}}$ is well-distributed. **Q**

By 3N, there is a well-distributed sequence $\langle z_n\rangle_{n\in\mathbb{N}}$ in the support Z of ν. Let $\mathcal{E}$ be the algebra of open-and-closed subsets of Z. Then $V\mapsto V\cap Z$ is a surjective Boolean homomorphism from the algebra of open-and-closed subsets of X onto $\mathcal{E}$, so $I\mapsto Z\cap\widehat{I}$ is a surjective Boolean homomorphism from T onto $\mathcal{E}$. Its kernel is

$$\{I:Z\cap\widehat{I}=\emptyset\}=\{I:\nu\widehat{I}=0\}=\{I:d_s(I)=0\},$$

so

$$\mathcal{E}\cong\{I^\bullet:I\in\mathrm{T}\}=\mathfrak{A}.$$

On the other hand, the map $E \mapsto \{n : z_n \in E\}$ is a Boolean isomorphism between $\mathcal{E}$ and a subalgebra Σ of $\mathcal{P}\mathbb{N}$. Moreover, if $I \in \mathrm{T}$, then

$$\bar{d}_s(I^\bullet) = d_s(I) = \nu\widehat{I} = d_s(\{n : z_n \in Z \cap \widehat{I}\}),$$

so $(\Sigma, d_s{\restriction}\Sigma) \cong (\mathfrak{A}, \bar{d}_s{\restriction}\mathfrak{A})$.

Of course Σ is σ-centered, so $\mathfrak{A}$ also is.

4. The set of measures with well-distributed sequences

4A The problem For a given topological space X, to understand the set of topological probability measures on X which have well-distributed sequences.

4B Lemma Suppose that $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ is a sequence in $[0,1]$ such that $\sum_{n=0}^\infty \alpha_n = 1$. Then there is a partition $\langle I_n\rangle_{n\in\mathbb{N}}$ of $\mathbb{N}$ such that $d_s(\bigcup_{n\in L} I_n) = \sum_{n\in L}\alpha_n$ for every $L \subseteq \mathbb{N}$.

proof (a) Consider first the case in which we have $\alpha_n = 2^{-k_n}$ for each n, where $k_0 \le k_1 \le \dots$. Choose $\langle I_n\rangle_{n\in\mathbb{N}}$ inductively by setting $i_n = \min(\mathbb{N} \setminus \bigcup_{m<n} I_m$, $I_n = \{2^{k_n}l + i_n : l \in \mathbb{N}\}$ for each n. Note that we always have

$$\#(I_m \cap 2^{k_n}) \le 2^{k_n - k_m} = 2^{k_n}\alpha_m$$

for each $m < n$, so that $i_n < 2^{k_n}$. It follows that $j - 2^{k_n} \in I_m$ whenever $m < n$, $j \in I_m$ and $j \ge 2^{k_n}$; consequently $I_m \cap I_n = \emptyset$ for $m < n$. Because $i_n \in I_n$, $\bigcup_{n\in\mathbb{N}} I_n = \mathbb{N}$. Finally, for the moment, $d_s(I_n) = \alpha_n$ for every n.

(b) In general, express each non-zero α_n as $\sum_{j=0}^\infty \beta_{nj}$ where each β_{nj} is of the form 2^{-k} for some k. Re-index $\langle\beta_{nj}\rangle_{\alpha_n>0, j\in\mathbb{N}}$ as $\langle\beta_j\rangle_{j\in\mathbb{N}}$ where $\beta_0 \ge \beta_1 \ge \dots$. By (a), we have a partition $\langle J_j\rangle_{j\in\mathbb{N}}$ of $\mathbb{N}$ such that $d_s(J_j) = \beta_j$ for each j. Next, we have a partition $\langle M_n\rangle_{n\in\mathbb{N}}$ of $\mathbb{N}$ such that $\alpha_n = \sum_{j\in M_n}\beta_j$ for each n. Set $I_n = \bigcup_{j\in M_n} J_j$ for each n; then $\langle I_n\rangle_{n\in\mathbb{N}}$ is a partition of $\mathbb{N}$.

If $L \subseteq \mathbb{N}$, set $L' = \mathbb{N} \setminus L$, $M = \bigcup_{n\in L} M_n$, $M' = \bigcup_{n\in L'} M_n$. Then $\bigcup_{n\in L} I_n \cap \bigcup_{j\in M'} J_j = \emptyset$, so

$$d_s^*(\bigcup_{n\in L} I_n \le d_s(\mathbb{N} \setminus \bigcup_{j\in K} J_j) = 1 - \sum_{j\in K}\beta_j$$

for every finite $K \subseteq M'$, and

$$d_s^*(\bigcup_{n\in L} I_n \le 1 - \sum_{j\in M'}\beta_j = \sum_{j\in M}\beta_j = \sum_{n\in L}\alpha_n.$$

Similarly, $d_s^*(\bigcup_{n\in L'} I_n) \le \sum_{n\in L'}\alpha_n$. As noted in 1A, this shows that $d_s(\bigcup_{n\in L} I_n$ is defined and equal to $\sum_{n\in L}\alpha_n$, as required.

4C Lemma Let X be a topological space and $\langle\mu_n\rangle_{n\in\mathbb{N}}$ a sequence of topological probability measures on X, each with a well-distributed sequence $\langle x_{ni}\rangle_{i\in\mathbb{N}}$. If $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ is a sequence in $[0,1]$ with sum 1, and μ is any topological measure on X such that $\mu G = \sum_{n=0}^\infty \alpha_n\mu_n G$ for every open set $G \subseteq X$, then μ has a well-distributed sequence.

proof Let $\langle I_n\rangle_{n\in\mathbb{N}}$ be a partition of $\mathbb{N}$ such that $d_s(\bigcup_{n\in L} I_n) = \sum_{n\in L}\alpha_n$ for every $L \subseteq \mathbb{N}$ (4B). For $r \in \mathbb{N}$ set $y_r = x_{ni}$ where $y \in I_n$ and $i = \#(r \cap I_n)$.

If $F \subseteq X$ is closed and $\epsilon > 0$, take $m \in \mathbb{N}$ such that $\sum_{n=m+1}^\infty \alpha_n \le \epsilon$. Then

$$\begin{aligned} d_s^*(\{r : y_r \in F\}) &\le \sum_{n=0}^m d_s^*(\{r : r \in I_n,\, y_r \in F\}) + d_s^*(\bigcup_{n>m} I_n) \\ &= \sum_{n=0}^m \alpha_n d_s^*(\{i : x_{ni} \in F\}) + \sum_{n=m+1}^n \alpha_n \end{aligned}$$

(1Fb)

$$\le \sum_{n=0}^m \alpha_n\mu_n F + \epsilon \le \mu F + \epsilon.$$

As F and ϵ are arbitrary, $\langle y_r\rangle_{r\in\mathbb{N}}$ is well-distributed for μ.

4D Lemma Let $(X,\mathfrak{T},\Sigma,\mu)$ be a compact Radon probability space with a well-distributed sequence. Then there are a zero-dimensional compact Radon probability space $(Z,\mathfrak{S},\mathrm{T},\nu)$ with a well-distributed sequence and a continuous inverse-measure-preserving surjection from Z onto X.

proof Let $\mathfrak{A}$ be the set of subsets of X with negligible boundary; then $\mathfrak{A}$ is an algebra of sets (Fremlin 03, 411Yc and 491Ye). Let $(Z,\mathfrak{S})$ be its Stone space. Then Z is a zero-dimensional compact Hausdorff space. Let ν be the Radon probability measure on Z defined by setting $\nu\widehat{E}=\mu E$ for every $E\in\mathfrak{A}$, where $\widehat{E}$ is the open-and-closed subset of Z corresponding to E (416Qa).

We have a function $f:Z\to X$ defined by saying that $f(z)=x$ iff $x\in\overline{E}$ whenever $E\in\mathfrak{A}$ and $z\in\widehat{E}$. **P** Given $z\in Z$, then $\mathcal{E}=\{E:E\in\mathfrak{A},\,z\in\widehat{E}\}$ has the finite intersection property so $V=\bigcap_{E\in\mathcal{E}}\overline{E}$ is non-empty. If x_0, $x_1\in X$ are distinct, let $h\in C(X)$ be such that $h(x_0)<h(x_1)$; then there is an $\alpha\in\,]h(x_0),h(x_1)[$ such that $h^{-1}[\{\alpha\}]$ is negligible. Setting $F_0=\{x:h(x)\le\alpha\}$ and $F_1=\{x:h(x)\ge\alpha\}$, their boundaries ∂F_0 and ∂F_1 are both included in $h^{-1}[\{\alpha\}]$, so both F_0 and F_1 belong to $\mathfrak{A}$. Now $F_0\cup F_1=X$ so $\widehat{F}_0\cup\widehat{F}_1=Z$ and $z\in\widehat{F}_j$ for some j. But in this case $V\subseteq F_j$ and cannot contain both x_0 and x_1. Thus V must be a singleton and its single member must be $f(x)$. **Q**

Now the graph of f is

$$(Z\times X)\setminus\bigcup_{E\in\mathcal{A}}\widehat{E}\times(X\setminus\overline{E}),$$

which is closed, so f is continuous (because Z and X are compact Hausdorff spaces).

Next, f is surjective. **P** If $x\in X$, then there must be a $z\in Z$ such that $z\in\widehat{H}$ whenever $H\in\mathfrak{A}$ is open and $x\in H$; now $x=f(z)$. **Q**

f is inverse-measure-preserving. **P** If $F\subseteq X$ is closed, then $f^{-1}[F]=\bigcap\{\widehat{H}:H\in\mathfrak{A}\text{ is open, }F\subseteq H\}$, so $\nu f^{-1}[F]=\inf\{\mu H:F\subseteq H\in\mathfrak{A},\,H\text{ is open}\}=\mu F$. Now use Fremlin 03, 412K. **Q**

So if $\langle x_n\rangle_{n\in\mathbb{N}}$ is a well-distributed sequence for μ, we can choose a sequence $\langle z_n\rangle_{n\in\mathbb{N}}$ in Z such that $f(z_n)=x_n$ for every n. **P** If $W\subseteq Z$ is closed and $\epsilon>0$, then there is an $H\in\mathfrak{A}$ such that $W\subseteq\widehat{H}$ and $\nu\widehat{H}\le\nu W+\epsilon$. Now

$$d_s^*(\{n:z_n\in W\})\le d_s^*(\{n:x_n\in\overline{H}\})\le\mu\overline{H}=\mu H=\nu\widehat{H}\le\nu W+\epsilon.$$

As W and ϵ are arbitrary, $\langle z_n\rangle_{n\in\mathbb{N}}$ is well-distributed for ν.

4E Lemma If $(X,\mathfrak{T},\Sigma,\mu)$ is a zero-dimensional compact Radon probability space with a well-distributed sequence, there are a zero-dimensional atomless compact Radon probability space $(Y,\mathfrak{S},\mathrm{T},\nu)$ with a well-distributed sequence and a continuous inverse-measure-preserving function $f:X\to Y$.

proof Set $Y=X\times\{0,1\}^{\mathbb{N}}$; use 3L.

4F Lemma Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{N}$ such that $d_s(A_n)=0$ for every n. Then there is a strictly increasing function $f:\mathbb{N}\to\mathbb{N}$ such that $f^{-1}[A_n]$ is finite for each n and $d_s^*(f^{-1}[A])\le d_s^*(A)$ for every $A\subseteq\mathbb{N}$; consequently $d_s(f^{-1}[A])=d_s(A)$ whenever $d_s(A)$ is defined.

proof Choose $\langle m_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{N}$ so that $\bigcup_{j\le n}A_j\cap[m_n,m_n+2n+1[\,=\emptyset$ and $m_n+2n+1\le m_{n+1}$ for each n; this is possible because $d_s^*(\bigcup_{j\le n}A_j)<\frac{1}{2n+1}$. Set $f(i)=m_n+i-n^2$ if $n^2\le i<(n+1)^2$. Then $f:\mathbb{N}\to\mathbb{N}$ is strictly increasing and $f^{-1}[A_n]\subseteq n^2$ is finite for each n.

Let $A\subseteq\mathbb{N}$ and $\epsilon>0$. Set $\gamma=d_s^*(A)$. Then there is an $M\in\mathbb{N}$ such that $\#(A\cap J)\le M+(\epsilon+\gamma)\#(J)$ for every interval $J\subseteq\mathbb{N}$. Let n be such that $\epsilon n\ge 2M$. If $k\ge n^2$, then $f[\,[k,k+n[\,]$ is of the form $J_1\cup J_2$ where J_1 and J_2 are intervals and $\#(J_1+\#(J_2)=n$. So

$$\#(f^{-1}[A]\cap[k,k+n[\,]=\#(A\cap J_1)+\#(A\cap J_2)\le 2M+n(\epsilon+\gamma)\le n(2\epsilon+\gamma).$$

By 1D, $d_s^*(f^{-1}[A])\le 2\epsilon+\gamma$; as ϵ is arbitrary, $d_s^*(f^{-1}[A])\le d_s^*(A)$.

Applying this to A and $\mathbb{N}\setminus A$ we see that $d_s(f^{-1}[A])=d_s(A)$ whenever the right-hand-side is defined.

Remark In the language of §5, f is inverse-Banach-density-preserving; the proof here can be short-circuited by applying Proposition 5C.

4G Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a compact Radon probability space with a well-distributed sequence, and $f:X\to[0,\infty]$ a lower semi-continuous function such that $\int f d\mu=1$. Then the indefinite-integral measure defined by f has a well-distributed sequence.

proof (a) Consider first the case in which $f=\frac{1}{\mu V}\chi V$ where V is open-and-closed. Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be well-distributed for μ, and set $I=\{n:x_n\in V\}$; then $d_s(I)=\mu V$. Let $\langle n_i\rangle_{i\in\mathbb{N}}$ be the increasing enumeration of I, and set $y_i=x_{n_i}$ for each i. If $F\subseteq X$ is closed, then

$$\mu(F\cap V)\ge d_s^*(\{n:n\in I,\,x_n\in F\})=d_s(I)d_s^*(\{i:y_i\in F\})$$

by 1Fb; dividing both sides by $d_s(I)=\mu V$, $\nu F\ge d_s^*(\{i:y_i\in F\})$; as F is arbitrary, $\langle y_i\rangle_{i\in\mathbb{N}}$ is well-distributed for ν.

(b) Now suppose that X is zero-dimensional. In this case there are a sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ of open-and-closed sets of non-zero measure and a sequence $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ in $]0,\infty[$ such that $f=_{\text{a.e.}}\sum_{n=0}^\infty\alpha_n\chi V_n$. **P** Choose $\langle f_n\rangle_{n\in\mathbb{N}}$, $\langle g_n\rangle_{n\in\mathbb{N}}$ as follows. $f_0=f$. Given that $f_n:X\to[0,\infty]$ is integrable and lower semi-continuous, set $G_{ni}=\{x:f_n(x)>2^{-n}i\}$ for $i\in\mathbb{N}$; then $\int f_n d\mu\le 2^{-n}+\sum_{i=1}^\infty\mu G_{ni}$; let $m_n\ge 1$ be such that $\int f_n d\mu\le 2^{-n+1}+\sum_{i=1}^{m_n}\mu G_{ni}$; choose open-and-closed sets $V_{ni}\subseteq G_{ni}$ such that $\mu V_{ni}\ge\mu G_{ni}+\frac{1}{2^n m_n}$ for each i; set $g_n=\sum_{i=1}^{m_n}2^{-n}\chi V_{ni}$ and $f_{n+1}=f_n-g_n$. Continue.

Observe that $\int f_{n+1}d\mu\le 2^{-n+2}$ for each n, so that $f=_{\text{a.e.}}\sum_{n=0}^\infty g_n$. We can therefore take $\langle(\alpha_n,V_n)\rangle_{n\in\mathbb{N}}$ to be an enumeration of $\{(2^{-n},V_{ni}):n\in\mathbb{N},\,1\le i\le m_n\}$, deleting any terms in which V_{ni} is empty. (If this leaves us with only finitely many terms, break one of them up by replacing a $(2^{-n},V)$ by $\langle(2^{-k-n-1},V)\rangle_{k\in\mathbb{N}}$.) **Q**

Now (a) and 4C tell us that ν has a well-distributed sequence.

(c) Finally, for the general case, let $(Z,\mathfrak{S},\mathrm{T},\lambda)$ be a zero-dimensional compact Radon probability space with a well-distributed sequence $\langle z_n\rangle_{n\in\mathbb{N}}$ and a continuous inverse-measure-preserving function $\phi:Z\to X$ (4D). Then $f\phi$ is lower semi-continuous and $\int f\phi\,d\lambda=1$, so (b) tells us that there is a sequence $\langle z'_n\rangle_{n\in\mathbb{N}}$ in Z such that $d_s^*(\{n:z'_n\in W\})\le\int_W f\phi\,d\lambda$ for every closed set $W\subseteq Z$. But in this case

$$\nu F=\int_F f d\mu=\int_{\phi^{-1}[F]}f\phi\,d\lambda\ge d_s^*(\{n:z'_n\in\phi^{-1}[F]\}=d_s^*(\{n:\phi(z'_n)\in F\})$$

for every closed set $F\subseteq X$, and $\langle\phi(z'_n)\rangle_{n\in\mathbb{N}}$ is well-distributed for ν.

4H Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon probability space with a well-distributed sequence, and E a non-negligible measurable set. Then the normalized subspace measure $(\mu E)^{-1}\mu_E$ has an equidistributed sequence.

proof (a) To begin with (down to the end of (d)) let us suppose that E is compact. Set $\alpha=1-\mu E$ and let $\langle x_i\rangle_{i\in\mathbb{N}}$ be a sequence in X which is well-distributed for μ. For $n\ge 1$, $\epsilon>0$ and closed $F\subseteq X\setminus E$ set

$$A(n,F,\epsilon)=\{k:k\in\mathbb{N},\,\#(\{i:i<n,\,x_{kn+i}\in F\})\ge n(\alpha+\epsilon)\}.$$

Then $d_s^*(A(n,F,\epsilon))<1$. **P?** Otherwise, let $\nu\in P$ be such that $\nu A(n,F,\epsilon)=1$. Set

$$\nu' I=\int\frac{1}{n}\#(I\cap[kn,kn+n[)\nu(dk)$$

for each $I\subseteq\mathbb{N}$; then $\nu'\in P$. But now

$$\alpha+\epsilon\le\nu'(\{i:x_i\in F\})\le d_s^*(\{i:x_i\in F\})\le\mu F\le\alpha,$$

which is impossible. **XQ**

Because $\{A(n,F,\epsilon):\epsilon>0,\,F\subseteq X\setminus E\text{ is closed}\}$ is upwards-directed, there is a non-principal ultrafilter $\mathcal{F}_n$ on $\mathbb{N}$ not containing $A(n,F,\epsilon)$. Let K_n be the set of those $i<n$ such that $y_{ni}=\lim_{k\to\mathcal{F}_n}x_{kn+i}$ is defined and belongs to E.

(b) $\limsup_{n\to\infty}\frac{1}{n}\#(K_n)\le\mu E$. **P** For any $\epsilon>0$, there is an open set $G\subseteq X$ such that $E\cap\overline{G}=\emptyset$ and $\mu G>\alpha-\epsilon$. Let $n_0\ge 1$ be such that $\#(\{i:i<n,\,x_{kn+i}\in G\})\ge n(\alpha-2\epsilon)$ for every $k\in\mathbb{N}$, $n\ge n_0$. Setting $K'_n=\lim_{k\to\mathcal{F}_n}\{i:i<n,\,x_{kn+i}\in G\}$, we have $\#(K'_n)\ge n(\alpha-2\epsilon)$ and $K'_n\cap K_n=\emptyset$; so

$$\frac{1}{n}\#(K_n) \le 1 - \alpha + 2\epsilon = \mu E + 2\epsilon$$

for every $n \ge n_0$. **Q**

(c) Suppose that $n \ge 1$, $F \subseteq X$ and $\beta > 0$ are such that F is closed and $\#(\{i : i < n,\, x_{kn+i} \in F\}) \ge (\alpha+\beta)n$ for every $k \in \mathbb{N}$. Then $\#(\{i : i \in K_n,\, y_{ni} \in F\}) \ge \beta n$. **P?** Otherwise, for every $i \in n \setminus K_n$ there is a closed set F_i disjoint from E such that $\{k : x_{kn+i} \in F_i\} \in \mathcal{F}_n$. Set $\epsilon = \beta - \frac{1}{n}\#(\{i : i \in K_n,\, y_{ni} \in F\}) > 0$, $F' = \bigcup_{i \in n\setminus K_n} F_i$. Then

$$I = \mathbb{N} \cap \bigcap\nolimits_{i \in n \setminus K_n} \{k : x_{kn+i} \in F_i\} \setminus A(n, F', \epsilon)$$

belongs to $\mathcal{F}_n$. But if $k \in I$, then $x_{kn+i} \in F'$ for every $i \in n \setminus K_n$, and

$$\begin{aligned}\#(\{i : i \in K_n,\, x_{kn+i} \in F\}) &\ge \#(\{i : i < n,\, x_{kn+i} \in F \cup F'\}) \\ &\qquad\qquad - \#(\{i : i < n,\, x_{kn+i} \in F'\}) \\ &> (\alpha+\beta)n - (\alpha+\epsilon)n.\end{aligned}$$

So

$$\begin{aligned}\#\{i : i \in K_n,\, y_{ni} \in F\}) &\ge \lim_{k \to \mathcal{F}_n} \#(\{i : i \in K_n,\, x_{kn+i} \in F\}) \\ &> (\beta - \epsilon)n = \#(\{i : i \in K_n,\, y_{ni} \in F\},\end{aligned}$$

which is impossible. **XQ**

(d) In particular, taking $F = X$ and $\beta = \mu E$, we have $\#(K_n) \ge n\mu E > 0$ for every $n \ge 1$. So we have point-supported measures μ_n on E defined by setting $\mu_n D = \frac{1}{\#(K_n)}\#(\{i : i \in K_n,\, y_{ni} \in D\})$ for every $D \subseteq E$. Now $\limsup_{n\to\infty} \mu_n F \le \frac{\mu F}{\mu E}$ for every closed set $F \subseteq E$. **P** Let $\epsilon > 0$. Then there is a closed set $F_0 \subseteq X \setminus F$ such that $\mu(\operatorname{int} F_0) \ge 1 - \mu F - \epsilon$. Now there is an n_0 such that $\frac{1}{n}\#(\{i : i < n,\, x_{kn+i} \in \operatorname{int} F_0\}) \ge \mu(\operatorname{int} F_0) - \epsilon$ whenever $n \ge n_0$ and $k \in \mathbb{N}$; by (b), we can suppose also that $\frac{1}{n}\#(K_n) \le \mu E + \epsilon$ for $n \ge n_0$. Take $n \ge n_0$. Then

$$\frac{1}{n}\#(\{i : i < n,\, x_{kn+i} \in F_0\}) \ge 1 - \mu F - 2\epsilon$$

for every k. By (c),

$$\#(\{i : i \in K_n,\, y_{ni} \in F_0\}) \ge n(1 - \mu F - \alpha - 2\epsilon) = \mu(E \setminus F) - 2\epsilon,$$

$$\begin{aligned}\#(\{i : i \in K_n,\, y_{ni} \in F\}) &\le \#(K_n) - n(\mu(E\setminus F) - 2\epsilon) \\ &\le n(\mu E + \epsilon - \mu(E \setminus F) + 2\epsilon) = n(\mu F + 3\epsilon),\end{aligned}$$

$$\mu_n F = \tfrac{\#(\{i : y_{ni} \in F\})}{\#(K_n)} \le \tfrac{\mu F + 3\epsilon}{\mu E}.$$

As ϵ is arbitrary, we have the result. **Q**

By Fremlin 03, 491D, there is an equidistributed sequence in E.

(e) For the general case, if there is a compact set $F \subseteq E$ with the same measure as E we can apply the result of (a)-(d) to F. Otherwise, let $\langle F_m \rangle_{m\in\mathbb{N}}$ be a disjoint sequence of compact non-negligible subsets of E such that $E \setminus \bigcup_{m\in\mathbb{N}} F_m$ is negligible. For each $m \in \mathbb{N}$ let $\langle z_{mi} \rangle_{i\in\mathbb{N}}$ be a well-distributed sequence for the normalized subspace measure on F_m. For $n \in \mathbb{N}$ let μ_n be the point-supported probability measure on E defined by setting

$$\mu_n D = \sum\nolimits_{m=0}^{\infty} \tfrac{\mu F_m}{(n+1)\mu E}\#(\{i : i \le n,\, z_{mi} \in D\})$$

for $D \subseteq E$. If $F \subseteq E$ is relatively closed, then

$$\begin{aligned}\limsup_{n\to\infty} \mu_n F &\le \sum_{m=0}^{\infty} \frac{\mu F_m}{\mu E} \limsup_{n\to\infty} \frac{1}{n+1} \#(\{i : i \le n,\ z_{mi} \in F\}) \\ &\le \sum_{m=0}^{\infty} \frac{\mu F_m}{\mu E} \frac{\mu(F \cap F_m)}{\mu F_m} = \frac{\mu F}{\mu E}.\end{aligned}$$

By Fremlin 03, 491D again, $\frac{1}{\mu E}\mu_E$ has an equidistributed sequence.

4I Theorem (Rindler 76) Let X be a separable compact Hausdorff topological group, and μ the Haar probability measure on X. Then μ has a well-distributed sequence.

proof Following Rindler 76, I use what is in effect a refinement of the proof of existence of equidistributed sequences given in Fremlin 03, 491H. We can cut a step out by actually quoting that result. Let $\langle y_m \rangle_{m\in\mathbb{N}}$ be an equidistributed sequence for μ; we can suppose that $y_0 = e$, the identity of X. Define $\langle x_i \rangle_{i\in\mathbb{N}}$ inductively by saying that

$$x_0 = e,$$

$$\text{if } i = j + kn!, \text{ where } n \in \mathbb{N},\ 1 \le k \le n \text{ and } j < n!, \text{ then } x_i = y_k x_j.$$

Then we find that whenever $n, k \in \mathbb{N}$ and $j < n!$, $x_{j+kn!} = x_{kn!}x_j$ (induce on k simultaneously for all n, j).

Take $f \in C(X)$ and $\epsilon > 0$. For $a, b \in X$, set $f_{ab}(x) = f(axb)$. Then the function $(a, b) \mapsto f_{ab} : X \times X \to C(X)$ is continuous, so $\{f_{ab} : a, b \in X\}$ is compact for the norm topology on $C(X)$. We have

$$\int f = \int f_{ab} = \lim_{m\to\infty} \frac{1}{m}\sum_{i=0}^{m-1} f_{ab}(y_i)$$

for every $a, b \in X$, so there is an $n \ge 1$ such that

$$|\frac{1}{n}\sum_{i=0}^{n-1} f_{ab}(y_i) - \int f| \le \epsilon$$

for all $a, b \in X$. In this case, for any $k \in \mathbb{N}$,

$$|\frac{1}{n!}\sum_{i=0}^{n!-1} f(x_{i+kn!}) - \int f| = |\frac{1}{n!}\sum_{i=0}^{n!-1} f(ax_i) - \int f|$$

(where $a = x_{kn!}$)

$$\begin{aligned}&= |\frac{1}{n!}\sum_{j=0}^{n-1}\sum_{i=0}^{(n-1)!-1} f(ax_{i+j(n-1)!}) - \int f| \\ &\le \frac{1}{(n-1)!}\sum_{i=0}^{(n-1)!-1} |\frac{1}{n}\sum_{j=0}^{n-1} f(ax_{i+j(n-1)!}) - \int f| \\ &= \frac{1}{(n-1)!}\sum_{i=0}^{(n-1)!-1} |\frac{1}{n}\sum_{j=0}^{n-1} f(ay_jx_i) - \int f| \\ &\le \frac{1}{(n-1)!}\sum_{i=0}^{(n-1)!-1} \epsilon = \epsilon.\end{aligned}$$

Now suppose that $m \ge \frac{2}{\epsilon}\|f\|_\infty n!$ and $l \in \mathbb{N}$. Then

$$|\sum_{i=l}^{l+m}(f(x_i)-\int f)|\le|\sum_{i=l}^{k_0n!-1}(f(x_i)-\int f)|$$
$$+\sum_{k=k_0}^{k_1-1}|\sum_{i=kn!}^{(k+1)n!-1}(f(x_i)-\int f)|+|\sum_{i=k_1n!}^{l+m}(f(x_i)-\int f)|$$

(where $(k_0-1)n!<l\le k_0n!$ and $k_1n!\le l+m+1<(k_1+1)n!$)

$$\le 2n!\|f\|_\infty+(k_1-k_0)n!\epsilon+2n!\|f\|_\infty$$
$$\le m\epsilon+\epsilon(m+1)+m\epsilon\le 3(m+1)\epsilon,$$

and

$$\frac{1}{m+1}|\sum_{i=l}^{l+m}f(x_i)-\int f|\le 3\epsilon.$$

As ϵ is arbitrary, $\int f=\mathrm{WDL}_{i\to\infty}f(x_i)$, by 3A(c-$\beta$). By 3Df, $\langle x_i\rangle_{i\in\mathbb{N}}$ is well-distributed.

5 Inverse-Banach-density-preserving functions

5A Definition I will say that $f:\mathbb{N}\to\mathbb{N}$ is **inverse-Banach-density-preserving** if $d_s(f^{-1}[I])$ is defined and equal to $d_s(I)$ whenever $d_s(I)$ is defined.

5B Theorem Let $f:\mathbb{N}\to\mathbb{N}$ be a function. For an additive functional $\nu:\mathcal{P}\mathbb{N}\to\mathbb{R}$ and $I\subseteq\mathbb{N}$, write $(\nu f^{-1})(I)=\nu(f^{-1}[I])$. Then the following are equiveridical:

(i) f is inverse-Banach-density-preserving;
(ii) $d_s^*(f^{-1}[I])\le d_s^*(I)$ for every $I\subseteq\mathbb{N}$;
(iii) $d_s^*(f^{-1}[I])\le d_s(I)$ whenever $d_s(I)$ is defined;
(iv) $\nu f^{-1}\in P$ for every $\nu\in P$.

If f is injective, we can add

(v) $\lim_{n\to\infty}d_s^*(A_n)=0$,

where $A_n=\{i:i\in\mathbb{N},\ f(i)+1\ne f(j)$ whenever $|j-i|\le n\}$ for $n\in\mathbb{N}$.

proof (a)(i)⇒(iv) Assume (i). Let $\nu\in P$, and take $I\subseteq\mathbb{N}$ such that $d_s(I)$ is defined. Then

$$(\nu f^{-1})(I)=\nu(f^{-1}[I])\le d_s^*(f^{-1}[I])=d_s(I).$$

As we also have $(\nu f^{-1})(\mathbb{N})=1$ and $(\nu f^{-1})(\mathbb{N}\setminus I)\le d_s(\mathbb{N}\setminus I)$, $(\nu f^{-1})(I)=d_s(I)$. As I is arbitrary, $\nu f^{-1}\in P$, by 1H.

(b)(iv)⇒(ii) If (iv) is true and $I\subseteq\mathbb{N}$, then

$$d_s^*(f^{-1}[I])=\sup_{\nu\in P}\nu(f^{-1}[I])=\sup_{\nu\in P}(\nu f^{-1})(I)\le\sup_{\nu\in P}\nu I=d_s^*(I).$$

(c)(ii)⇒(iii) is trivial.

(d)(iii)⇒(i) Assume (iii). If $d_s(I)$ is defined, then $d_s^*(f^{-1}[I])\le d_s(I)$ and

$$d_s^*(\mathbb{N}\setminus f^{-1}[I])=d_s^*(\mathbb{N}\setminus I)\le d_s(\mathbb{N}\setminus I)=1-d_s(I),$$

so $d_s(f^{-1}[I])$ is defined and equal to $d_s(I)$.

(e)(v)⇒(ii) Suppose that f is injective and that (v) is true. Set $B=\{i:f(i)+1\in f[\mathbb{N}]\}$, and for $i\in B$ set $g(i)=f^{-1}(f(i)+1)$. Take $I\subseteq\mathbb{N}$ and $\epsilon>0$. Then there is an $m\ge 1$ such that $\frac{1}{m}\#(I\cap[l,l+m[)\le d_s^*(I)+\epsilon$ for every $l\in\mathbb{N}$. Let $n\ge 1$ be such that $d_s^*(A_n)<\frac{\epsilon}{m}$. Set

$$C_k=\{i:i\in\mathbb{N},\ g^j(i)\text{ is defined and }|g^j(i)-i|\le jn\text{ for }1\le j\le k\}$$

for $k\in\mathbb{N}$. Then $d_s^*(\mathbb{N}\setminus C_k)\le\frac{(k+1)\epsilon}{m}$ for each k. **P** Let $r\in\mathbb{N}$ be such that $2mn\le\epsilon r$ and $\#(A_n\cap[l,l+r[)\le\frac{rn\epsilon}{m}$ for every l. Take any $l\in\mathbb{N}$. Then

$$\#([l, l+r[\setminus C_1) \le 2n + \#([l, l+r[\cap A_n) \le 2n + \frac{r\epsilon}{m}.$$

Next, for any $k \ge 1$,

$$\begin{aligned}\#([l, l+r[\cap C_k \setminus C_{k+1}) &\le \#([l, l+r[\cap C_1 \setminus g^{-1}[C_k]) \\ &\le 2n + \#(g^{-1}[\,[l, l+r[\setminus C_k])\end{aligned}$$

(because if $l+n \le i < l+r-n$ and $i \in C_1$, then $g(i) \in [l, l+r[)$

$$\le 2n + \#(\,[l, l+r[\setminus C_k)$$

because g is injective. Inducing on k, we see that

$$\#([l, l+r[\setminus C_k) \le 2kn + \frac{r\epsilon}{m} \le \frac{(k+1)r\epsilon}{m}$$

for every k. And this is true for every l, so $d_s^*(\mathbb{N} \setminus C_k) \le \frac{(k+1)\epsilon}{m}$ for every k. **Q**

Set $C = C_{m-1}$, so that $|g^j(i) - i| \le jn$ whenever $i \in C$ and $j < m$; that is, whenever $i \in C$ and $j < m$ there is an i' such that $|i - i'| \le jn$ and $f(i') = f(i) + j$. Then $d_s^*(\mathbb{N} \setminus C) \le \epsilon$. Let s be so large that $2mn \le \epsilon s$ and $\#(C \cap [l, l+s[) \ge (1 - 2\epsilon)s$ for every l.

Take any $l \in \mathbb{N}$. Define $J_0, J_1, \dots$ as follows. Given J_j for $j < k$, set

$$l_k = \min(f[\,[l+mn, l+s-mn[\cap C] \setminus \bigcup_{j<k} J_j)$$

if this set is not empty, and $J_k = [l_k, l_k + m[$. Stop when $f[\,[l+mn, l+s-mn[\cap C] \subseteq \bigcup_{j\le k} J_j$. Note that the J_j are disjoint and that $J_j \subseteq f[\,[l, l+s[\,]$ for every j; while $(k+1)m \ge \#([l+mn, l+s-mn[\cap C) \ge s - 2mn - 2\epsilon s$. Now consider

$$\begin{aligned}\#([l, l+s[\cap f^{-1}[I]) = \#(f[\,[l, l+s[\,] \cap I) &\le \sum_{j=0}^{k} \#(J_j \cap I) + s - (k+1)m \\ &\le (k+1)m(d_s^*(I) + \epsilon) + 2mn + 2\epsilon s \\ &\le s(d_s^*(I) + \epsilon) + 2mn + 2\epsilon s \le s(d_s^*(I) + 4\epsilon).\end{aligned}$$

As this is true for every l, $d_s^*(f^{-1}[I]) \le d_s^*(I) + 4\epsilon$. As this is true for every $I \subseteq \mathbb{N}$ and $\epsilon > 0$, (ii) is proved.

(f)$\neg$(v)$\Rightarrow$ $\neg$(ii) Again taking f to be injective, suppose that (v) is not true; set

$$\delta = \lim_{m\to\infty} d_s^*(A_m) = \inf_{m\in\mathbb{N}} d_s^*(A_m) > 0.$$

Then we can find a sequence $\langle l_m \rangle_{m\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$\#(A_m \cap [l_m, l_m + m[) \ge \frac{\delta m}{2},$$

$$l_{m+1} > l_m + m, \quad f(j) > f(i) + 1 \text{ whenever } i < l_m + m,\ j \ge l_{m+1} - 1$$

for every $m \in \mathbb{N}$. Set $K_m = [l_m, l_m + m[$ and $L_m = \{i : i \in K_m,\ f(i) + 1 \notin f[K_m]\}$ for each m; then $\#(L_m) \ge \#(A_m \cap K_m) \ge \frac{\delta m}{2}$; also $f(i) + 1 \notin f[K_{m'}]$ whenever m, m' are different and $i \in K_m$.

Define $I \subseteq \mathbb{N}$ as follows. For each $m \in \mathbb{N}$, let J_m consist of the first, third, fifth… members of $f[K_m \setminus L_m]$; now take $I = \bigcup_{m\in\mathbb{N}} J_m \cup f[L_m]$. Then $d_s^*(I) \le \frac{1}{2}$. **P** For each $i \in I$, set $i' = i + 1$ if $i \in f[L_m]$ for some m, and otherwise let i' be the next number above i such that neither i' nor $i' - 1$ belongs to $f[L_m]$. Then $i' \notin I$ for every $i \in I$, and $i \mapsto i'$ is injective. Also, for any interval $J \subseteq \mathbb{N}$, there can be at most two points $i \in I \cap J$ such that $i' \notin J$. (Note that if $K_m \setminus L_m$ has an odd number of members, then the top member i of $f[K_m \setminus L_m]$ is put into I; but now $i' \le l_{m+1} - 1$ is still safely away from $f[K_{m+1}]$.) **Q** But if we look at $f^{-1}[I] \cap K_m$, we see that this includes L_m and also at least $\lceil \frac{1}{2}\#(K_m \setminus L_m) \rceil$ members of $K_m \setminus L_m$; so

$$\begin{aligned}d_s^*(f^{-1}[I]) &\ge \limsup_{m\to\infty} \frac{1}{m}\#(f^{-1}[I] \cap K_m) \ge \limsup_{m\to\infty} \frac{1}{m}(\#(L_m) + \frac{1}{2}\#(K_m \setminus L_m) - 1) \\ &= \limsup_{m\to\infty} \frac{1}{2} + \frac{1}{2m}\#(L_m) \ge \frac{1}{2} + \frac{\delta}{4} > \frac{1}{2}.\end{aligned}$$

So (ii) is false.

5C Proposition Suppose that $f:\mathbb{N}\to\mathbb{N}$ is such that

(*) there is an $r\ge 1$ such that for every $m\ge 1$ there is an $l_0\in\mathbb{N}$ such that for every $l\ge l_0$ there is a partition of $[l,l+m[$ into at most r sets J such that $f\restriction J$ is injective and $f[J]$ is an interval.

Then f is inverse-Banach-density-preserving.

proof Take $I\subseteq\mathbb{N}$ and $\gamma>d_s^*(I)$. Then there is an $n_0\in\mathbb{N}$ such that $\#(I\cap[l,l+n[)\le\gamma n$ whenever $n\ge n_0$ and $l\in\mathbb{N}$; consequently $\#(I\cap K)\le n_0+\gamma\#(K)$ for every interval $K\subseteq\mathbb{N}$. Now take any $m\ge 1$. Then $\limsup_{l\to\infty}\#(f^{-1}[I]\cap[l,l+m[)\le rn_0+\gamma m$. **P** Let l_0 be as in (*). If $l\ge l_0$, let $\mathcal{J}$ be a partition of $[l,l+m[$ into at most r sets such that $f\restriction J$ is injective and $f[J]$ is an interval for each $J\in\mathcal{J}$. Now

$$\#(f^{-1}[I]\cap[l,l+m[)=\sum_{J\in\mathcal{J}}\#(f^{-1}[I]\cap J)$$

(because $\mathcal{J}$ is a partition of $[l,l+m[$)

$$=\sum_{J\in\mathcal{J}}\#(I\cap f[J])$$

(because $f\restriction J$ is injective for each J)

$$\le\sum_{J\in\mathcal{J}}n_0+\gamma\#(f[J])$$

(because $f[J]$ is an interval for each J)

$$\le rn_0+\sum_{J\in\mathcal{J}}\gamma\#(f[J])$$

(because $\#(\mathcal{J})\le r$)

$$=rn_0+\gamma\sum_{J\in\mathcal{J}}\#(J)=rn_0+\gamma m.$$

And this is true for every $l\ge l_0$. **Q**

Letting $m\to\infty$, we get $d_s^*(f^{-1}[I])\le\gamma$. As I and γ are arbitrary, f is inverse-Banach-density-preserving, by 5B(ii).

5D Examples (a) See 4F.

(b) Define $f:\mathbb{N}\to\mathbb{N}$ by setting

$$f(i)=i-n^2 \text{ whenever } n\in\mathbb{N} \text{ and } n^2\le i<(n+1)^2.$$

Then f is inverse-Banach-density-preserving. **P** In 5C, we can take $r=2$. **Q**

(c) Define $f:\mathbb{N}\to\mathbb{N}$ by setting

$$f(i)=(n+1)!-i-1+n! \text{ whenever } n\in\mathbb{N} \text{ and } n!\le i<(n+1)!.$$

Then f is an involution, and is inverse-Banach-density-preserving. **P** Once again, we can take $r=2$ in 5C. **Q**

But note that there is a set I such that $d(I)$ is defined but $d(f^{-1}(I))$ is not. **P** Set

$$I=\{2i:i\in\mathbb{N}\}\cup\bigcup_{n\in\mathbb{N}}[(n+1)!-n!,(n+1)![.$$

Then $d(I)=\frac{1}{2}=d_*(f^{-1}[I])$ but $d^*(f^{-1}[I])=\frac{3}{4}$. **Q**

(d) Set $K=\bigcup_{n\in\mathbb{N}}\bigl[(2n)^2,(2n+1)^2\bigr[$, $L=\bigcup_{n\in\mathbb{N}}\bigl[(2n+1)^2,(2n+1)^2\bigr[$. Let $\langle k_n\rangle_{n\in\mathbb{N}}$, $\langle l_n\rangle_{n\in\mathbb{N}}$ be the increasing enumerations of K, L respectively. Define $f:\mathbb{N}\to\mathbb{N}$ by setting $f(2i)=k_i$, $f(2i+1)=l_i$ for $i\in\mathbb{N}$, so that f is a bijection. If $m\ge 1$ and $l\ge m^2$ then $f[\,[l,l+m[\,]$ is made up of at most four intervals, so 5C tells us that ν is inverse-Banach-density-preserving. But if E is the set of even numbers then $d_s^*f[E]=1>\frac{1}{2}=d_s^*E$, so f^{-1} is not inverse-Banach-density-preserving.

6 Amenable groups Some of the ideas of §3, in particular, can be expressed in terms of general groups.

6A Definition Let G be any group. Let $Q_G \subseteq \ell^1(G)$ be the set of functions $z : G \to [0,1]$ such that $\{a : z(a) \neq 0\}$ is finite and $\sum_{a\in G} z(a) = 1$, that is, the convex hull of $\{\chi\{a\} : a \in G\}$. For any linear space W, $z \in Q_G$ and $f \in W^G$, set $(z * f)(a) = \sum_{b\in G, z(b)\neq 0} z(b)f(b^{-1}a)$ for $a \in G$. Observe that if $b \in G$ then $(\chi\{b\} * f)(a) = f(b^{-1}a)$ for every a, that is, $\chi\{b\} * f = b\bullet_l f$ in the sense of Fremlin 03, 441A. We can use the same formula to define $z_1 * z_2$ for z_1, $z_2 \in Q_G$, and now $\chi\{a\} * \chi\{b\} = \chi\{ab\}$ for a, $b \in G$; hence, or otherwise, $(z_1 * z_2) * f = z_1 * (z_2 * f)$ for all z_1, $z_2 \in Q_G$ and $f \in W^G$. $(Q_G, *)$ is a semigroup with identity $\chi\{e\}$, where e is the identity of G, and we can think of $*$ as a semigroup action of Q_G on W^G (Fremlin 03, 449Ya).

6B Lemma Let G be a group and W a locally convex Hausdorff linear topological space.

(a) For every $f \in W^G$, there is at most one $w \in W$ such that

(*) for every neighbourhood V of w in W and every $z \in Q_G$ there is a $z' \in Q_G$ such that $(z' * z * f)[G] \subseteq V$.

We may therefore define a function WDL by saying that, for $f \in W^G$, WDL(f) is defined and equal to w iff w satisfies the condition (*).

(b)(i) The domain D of WDL is a linear subspace of W^G, and WDL $: D \to W$ is a linear operator.

(ii) If $f \in D$ and $z \in Q_G$, then $z * f \in D$ and WDL$(z * f) =$ WDL(f).

(iii) In particular, WDL$(a\bullet_l f)$ is defined and equal to WDL(f) whenever $f \in D$ and $a \in G$.

(iv) WDL(f) belongs to the closed convex hull $\overline{\Gamma(f[G])}$ of $f[G]$ for every $f \in D$.

(v) If $f : G \to W$ is a constant function, then $f \in D$ and WDL(f) is the constant value of f.

(c) Suppose that W' is another locally convex Hausdorff linear topological space and $T : W \to W'$ is a continuous linear operator. If $f \in D$, then WDL(Tf) is defined and equal to $T($WDL$(f))$.

(d) Suppose that G is abelian.

(i) If $w \in W$ and $f \in W^G$ are such that for every neighbourhood V of w there is a $z' \in Q_G$ such that $(z' * f)[G] \subseteq V$, WDL(f) is defined and equal to w.

(ii) For $f \in W^G$, define $\tilde{f} \in W^G$ by setting $\tilde{f}(a) = f(a^{-1})$ for every $a \in G$. Then WDL$(f) =$ WDL$(\tilde{f})$ if either is defined.

proof (a) Suppose that w, w' satisfy the condition and that V, V' are convex neighbourhoods of w, w' respectively. Then there is a $z \in Q_G$ such that $(z * f)[G] \subseteq V$. Next, there is a $z' \in Q_G$ such that $(z' * z * f)[G] \subseteq V'$. But $(z' * z * f)(a) \in \Gamma((z * f)[G]) \subseteq V$ for every $a \in G$, so V meets V'. As V and V' are arbitrary, $w = w'$.

(b)(i) Suppose that f, $g \in D$, with WDL$(f) = w_1$ and WDL$(g) = w_2$, and $z \in Q_G$. Let V be a convex neighbourhood of 0 in W. Then there are a $z_1 \in Q_G$ such that $(z_1 * z * f)[G] \subseteq w_1 + V$ and a $z_2 \in Q_G$ such that $(z_2 * z_1 * z * g)[G] \subseteq w_2 + V$ for every $a \in G$. In this case, $(z_2 * z_1 * z * f)(a) \in \Gamma((z_1 * z * f)[G]) \subseteq w_1 + V$ for every $a \in G$, so $(z_2 * z_1 * z * (f + g))[G] \subseteq w_1 + V + w_2 + V$. As V and z are arbitrary, $f + g \in D$ and WDL$(f + g) = w_1 + w_2$.

Thus D is closed under addition and WDL is additive. The check that $\alpha f \in D$ and WDL$(\alpha f) = \alpha\,$WDL(f) is easy.

(ii) Immediate from the definition, just because $*$ is associative in the right way.

(iii) Special case of (ii).

(iv)-(v) Also immediate from the definition.

(c) The point is that $T(z * f) = z * Tf : G \to W'$ for every $z \in Q_G$, because

$$(T(z * f))(a) = T((z * f)(a)) = T(\sum_{z(b)\neq 0} z(b)f(b^{-1}a)) = \sum_{z(b)\neq 0} z(b)T(f(b^{-1}a))$$
$$= \sum_{z(b)\neq 0} z(b)(Tf)(b^{-1}a) = (z * Tf)(a)$$

for every $a \in G$. Now take a neighbourhood V of $T($WDL$(f))$, and $z \in Q_G$. Then $T^{-1}[V]$ is a neighbourhood of WDL(f), so there is a $z' \in Q_G$ such that $(z' * z * f)[G] \subseteq T^{-1}[V]$, that is,

$$V \supseteq T[(z' * z * f)[G]] = (T(z' * z * f))[G] = ((z' * z * (Tf))[G].$$

As V and z are arbitrary, $T(\mathrm{WDL}(f)) = \mathrm{WDL}(Tf)$.

(d)(i) If $z \in Q_G$ and V is a convex neighbourhood of w, take $z' \in Q_G$ such that $(z' * f)[G] \subseteq V$. Because G is abelian, $z' * z = z * z'$, so

$$(z' * z * f)[G] = (z * (z' * f))[G] \in \Gamma((z' * f)[G]) \subseteq \Gamma(V) = V.$$

As V and z are arbitrary, $w = \mathrm{WDL}(f)$.

(ii) Because G is abelian, $(z * f)^{\sim} = \tilde{z} * \tilde{f}$ for all $z \in Q_G$ and $f \in W^G$. So (i) gives the result.

6C Definition In the context of 6B, I will call $\mathrm{WDL}(f)$ the **well-distributed limit** of f. I will write wWDL for the well-distributed limit associated with the weak topology of W; of course wWDL extends WDL.

6D Homomorphic images: Proposition Let G_1, G_2 be groups and $\phi : G_1 \to G_2$ a surjective homomorphism. If W is a locally convex Hausdorff linear topological space and $f : G_2 \to W$ is a function, then $\mathrm{WDL}(f\phi) = \mathrm{WDL}(f)$ if either is defined.

In particular, if G is a group and $\phi : G \to G$ is an automorphism, then $\mathrm{WDL}(f\phi) = \mathrm{WDL}(f)$ whenever $f \in W^G$ and $\mathrm{WDL}(f)$ is defined.

proof (a) For $z \in Q_{G_1}$, $c \in G_2$ set $\hat{z}(c) = \sum_{a\in\phi^{-1}[\{c\}]} z(a)$. Then $\hat{z} \in Q_{G_2}$. If z_1, $z_2 \in Q_{G_1}$, then $\widehat{z_1 * z_2} = \hat{z}_1 * \hat{z}_2$. **P** If $c \in G_2$,

$$\begin{aligned}(\hat{z}_1 * \hat{z}_2)(c) &= \sum_{d_1 d_2 = c} \hat{z}_1(d_1)\hat{z}_2(d_2) = \sum_{d_1 d_2 = c}\sum_{\phi(a_1)=c_1}\sum_{\phi a_2 = c_2} z_1(a_1)z_2(a_2) \\ &= \sum_{\phi b = c}\sum_{a_1 a_2 = b} z_1(a_1)z_2(a_2) = \sum_{\phi b = c}(z_1 * z_2)(b) = \widehat{z_1 * z_2}(c). \textbf{ Q}\end{aligned}$$

Similarly, for any $f : G_2 \to W$ and $z \in Q_{G_1}$, $z * f\phi = (\hat{z} * f)\phi$. **P** If $a \in G_1$,

$$\begin{aligned}(\hat{z} * f)\phi(a) &= \sum_{d\in G_2} \hat{z}(d)f(d^{-1}\phi a) = \sum_{d\in G_2}\sum_{\phi b = d} z(b)f(d^{-1}\phi a) \\ &= \sum_{d\in G_2}\sum_{\phi b = d} z(b)f((\phi b)^{-1}\phi a) = \sum_{d\in G_2}\sum_{\phi b = d} z(b)f(\phi(b^{-1}a)) \\ &= \sum_{b\in G_1} z(b)f(\phi(b^{-1}a)) = (z * f\phi)(a). \textbf{ Q}\end{aligned}$$

(b)(i) If $w = \mathrm{WDL}(f\phi)$ is defined, $y \in Q_{G_2}$ and V is a neighbourhood of w, let $z \in Q_{G_1}$ be such that $\hat{z} = y$; such exists because ϕ is surjective. Then there is a $z' \in Q_{G_1}$ such that $(z' * z * f\phi)[G_1] \subseteq V$. Now (a) tells us that $(\hat{z}' * \hat{z} * f)\phi = z' * z * f\phi$, so

$$(\hat{z}' * y * f)[G_2] = (\hat{z}' * \hat{z} * f)[\phi[G_1]] = ((\hat{z}' * \hat{z} * f)\phi)[G_1] = (z' * z * f\phi)[G_1] \subseteq V.$$

As y and V are arbitrary, $\mathrm{WDL}(f) = w$.

(ii) If $w = \mathrm{WDL}(f)$ is defined, $z \in Q_{G_1}$ and V is a neighbourhood of w, let $y \in Q_{G_2}$ be such that $(y * \hat{z} * f)[G_2] \subseteq V$. Let $z' \in Q_{G_1}$ be such that $\hat{z}' = y$; then

$$(z' * z * f\phi)[G_1] = ((y * \hat{z} * f)\phi)[G_1] = (y * \hat{z} * f)[G_2] \subseteq V.$$

As z and V are arbitrary, $\mathrm{WDL}(f\phi) = w$.

6E Examples (a) If we set $f(i) = (-1)^i i$ for $i \in \mathbb{Z}$, then $\mathrm{WDL}(f)$ is defined and equal to 0. **P** Setting $z(0) = \frac{1}{2}$, $z(-1) = z(1) = \frac{1}{4}$ and $z(i) = 0$ for other $i \in \mathbb{Z}$, $z * f = \mathbf{0}$; by 6B(d-i), $\mathrm{WDL}(f) = 0$. **Q** Thus an unbounded sequence can have a well-distributed limit in the sense of 6A.

(b) Let G be the free group on two generators a, b. For $x \in G$, let $n_x \in \mathbb{Z}$ be such that $x = a^{n_x}y$ where y is either the identity e or has reduced expression beginning with a power of b. Observe that $n_{ax} = n_x + 1$

for every $x \in G$. Set $f(x) = (-1)^{n_x}$. If we set $z_0(e) = z_0(a) = \frac{1}{2}$ and $z_0(x) = 0$ for other $x \in G$, then $z_0 * f = \mathbf{0}$, and $\mathrm{WDL}(z_0 * f) = 0$. On the other hand, if we set $z_1(e) = z_1(b) = z_1(b^{-1}) = \frac{1}{3}$ and $z_1(x) = 0$ for other $x \in G$, then $(z_1 * f)(x) \ge \frac{1}{3}$ for every $x \in G$. **P** If $x = a^n y$ where y is either the identity e or has reduced expression beginning with a power of b, and $n \ne 0$, then $bx = ba^n y$ and $b^{-1}x = b^{-1}a^n y$ have reduced expressions beginning with powers of b, so

$$(z_1 * f)(x) = \frac{2}{3} + \frac{1}{3}(-1)^n \ge \frac{1}{3}.$$

If $n = 0$, express y as $b^m y'$ where y' is either the identity or has reduced expression beginning with a power of a. If $m = 0$ then $x = y = y' = e$ and $(z_1 * f)(x) = 1$. If $m = \pm 1$, then $(z_1 * f)(x) = \frac{2}{3} + \frac{1}{3}f(y') \ge \frac{1}{3}$; if $|m| \ge 2$, then again $(z_1 * f)(x) = 1$. **Q** But this means that $\Gamma((z_1 * f)[G])$ and $\Gamma((z_0 * f)[G])$ have disjoint closures; by 6B(b-iv), $\mathrm{WDL}(f)$ is undefined.

6F Lemma Let G be a group and W a Banach space. Write $\ell^\infty(G; W)$ for the set of bounded functions $f : G \to W$ with the norm $\|f\|_\infty = \sup_{a \in G} \|f(a)\|$. Set

$$D = \{f : f \in \ell^\infty(G; W),\ \mathrm{WDL}(f) \text{ is defined in } W\},$$

$$D_{\mathrm{w}} = \{f : f \in \ell^\infty(G; W),\ \mathrm{wWDL}(f) \text{ is defined in } W\}.$$

(i) $\mathrm{wWDL} : D_{\mathrm{w}} \to W$ has norm at most 1.
(ii) D and D_{w} are $\| \|_\infty$-closed linear subspaces of $\ell^\infty(G; W)$.

proof Elementary.

6G More definitions (a) I think I had better recall a definition from Fremlin 03. A topological group G is **amenable** if whenever X is a compact Hausdorff space and $\bullet$ is a continuous action of G on X then there is a G-invariant Radon probability measure on X. A locally compact Hausdorff group G, with a left Haar measure μ, is amenable in this sense iff for every finite $I \subseteq G$ and $\epsilon > 0$ there is a non-negligible compact set $K \subseteq G$ such that $\mu(K \triangle aK) \le \epsilon \mu K$ for every $a \in I$ (Fremlin 03, 449I).

(b) Now suppose that G is a topological group and that W is a locally convex linear topological space. Let U_W be the set of functions $f : G \to W$ which are uniformly continuous for the right uniformity of G (Fremlin 03, 4A5H) and such that $f[G]$ is relatively weakly compact in W. Let P_W be the family of linear operators $p : U_W \to W$ such that

$$p(f) \text{ belongs to the closed convex hull } \overline{\Gamma(f[G])} \text{ of } f[G],$$

$$p(a \bullet_l f) = p(f)$$

whenever $a \in G$ and $f \in U_W$.

6H Proposition Let G be an amenable topological group and W a Banach space.
(a) U_W is a closed linear subspace of $\ell^\infty(G; W)$.
(b) $\bullet_l$ is a continuous action of G on U_W.
(c) P_W is non-empty.
(d) $p(z * f) = p(f)$ whenever $f \in U_W$, $p \in P_W$ and $z \in Q_G$.

proof (a) is elementary if you have seen weak compactness in Banach spaces before.

(b) For any $a \in G$, the function $b \mapsto a^{-1}b : G \to G$ is uniformly continuous for the right uniformity. **P** $(a^{-1}b)(a^{-1}c)^{-1} = a^{-1}(bc^{-1})a \doteq e$ whenever $bc^{-1} \doteq e$. **Q** So $a \bullet_l f \in U_W$ whenever $f \in U_W$. As usual, it follows that $\bullet_l$ is an action of G on U_W.

Suppose that $a_0 \in G$, $f_0 \in U_W$ and $\epsilon > 0$. Then there is a neighbourhood H of the identity in G such that $\|f_0(a) - f_0(b)\| \le \epsilon$ whenever $a, b \in G$ and $ab^{-1} \in H$. Suppose that $a \in a_0 H^{-1}$ and that $\|f - f_0\| \le \epsilon$. Then, for any $b \in G$,

$$\begin{aligned}\|(a \bullet_l f)(b) - (a_0 \bullet_l f_0)(b)\| &= \|f(a^{-1}b) - f_0(a_0^{-1}b)\| \\ &\le \epsilon + \|f_0(a^{-1}b) - f_0(a_0^{-1}b)\| \le 2\epsilon\end{aligned}$$

because $(a^{-1}b)(a_0^{-1}b)^{-1} = a^{-1}a_0 \in H$. As ϵ, a_0 and f_0 are arbitrary, $\bullet_l$ is continuous.

(c) Because G is amenable, there is a positive linear functional $p_0 : U_{\mathbb{R}} \to \mathbb{R}$ such that $p_0(\chi G) = 1$ and $p_0(a\bullet_l f) = p(f)$ whenever $a \in G$ and $f \in U_{\mathbb{R}}$ (Fremlin 03, 449D/449E). Because p_0 is positive and $p_0(\chi G) = 1$, p_0 has norm at most 1. For $f \in U_W$ and $h \in W^*$, $hf \in U_{\mathbb{R}}$; set $\phi_f(h) = p_0(hf)$. Then $\phi_f : W^* \to \mathbb{R}$ is linear, and $\phi_f(h) \le \sup_{w\in f[G]} h(w)$. Because $f[G]$ is relatively weakly compact and W is a Banach space, $\overline{\Gamma(f[G])}$ is weakly compact, so there is a (unique) $p(f) \in \overline{\Gamma(f[G])}$ such that $\phi_f(h) = h(p(f))$ for every $h \in W^*$.

Clearly $p : U_W \to W$ is linear. It has norm at most 1 because

$$|h(p(f))| = |\phi_f(h)| \le \sup_{w\in f[G]} |h(w)| \le \sup_{a\in G} \|h\|\|f(a)\| = \|h\|\|f\|_\infty$$

whenever $h \in W^*$ and $f \in U_W$. If $f \in U_W$, $a \in G$ and $h \in W^*$, then

$$h(p(a\bullet_l f)) = p_0(h(a\bullet_l f)) = p_0(a\bullet_l(hf)) = p_0(hf) = h(p(f)).$$

Thus we have an appropriate operator p.

(d) If $I = \{a : z(a) \ne 0\}$, $z * f = \sum_{a\in I} z(a)a\bullet_l f$.

6I Følner filters Let G be an amenable locally compact Hausdorff group. Let $\mathcal{K}$ be the family of compact subsets of G which are not Haar negligible, and for $a \in G$ and $\epsilon > 0$ set

$$\begin{aligned}\mathcal{K}_{a\epsilon} &= \{K : K \in \mathcal{K},\ \mu(K\triangle aK) \le \epsilon\mu K \text{ for every left Haar measure } \mu \text{ on } G\}\\ &= \{K : K \in \mathcal{K},\ \mu(K\triangle aK) \le \epsilon\mu K \text{ for some left Haar measure } \mu \text{ on } G\}.\end{aligned}$$

Because G is amenable, $\{\mathcal{K}_{a\epsilon} : a \in G, \epsilon > 0\}$ generates a filter on $\mathcal{K}$ (Fremlin 03, 449I(ix)); I will call this the **left Følner filter** of G.

Similarly, the **right Følner filter** of G is generated by sets of the form

$$\tilde{\mathcal{K}}_{a\epsilon} = \{K : K \in \mathcal{K},\ \mu(K\triangle Ka) \le \epsilon\mu K \text{ for every right Haar measure } \mu \text{ on } G\}.$$

The map $a \mapsto a^{-1} : G \to G$ exchanges the two filters. If G is abelian, the left and right Følner filters are the same, and I will call them just the **Følner filter**.

6J Lemma Let G be an amenable locally compact Hausdorff group, $\mathcal{K}$ the family of its non-Haar-negligible compact subsets, and $\mathcal{F}ø$ its left Følner filter. Let μ be a Haar measure on G, and $\mathcal{F}$ an ultrafilter on $\mathcal{K}$ including $\mathcal{F}ø$. Let W be a Banach space.

(a) For $f \in U_W$ the weak limit

$$p(f) = \text{w-lim}_{K\to\mathcal{F}} \frac{1}{\mu K} \oint_K f d\mu$$

is defined, where $\oint_K f d\mu$ is the Bochner integral of $f\restriction K$ with respect to the subspace measure on K.

(b) $p \in P_W$.

proof (a) Because W is a Banach space and $f[G]$ is relatively weakly compact, the closed convex hull $\overline{\Gamma(f[G])}$ is weakly compact. As $\frac{1}{\mu K}\oint_K f d\mu \in \overline{\Gamma(f[G])}$ for every $K \in \mathcal{K}$, and $\mathcal{F}$ is an ultrafilter, the limit $p(f)$ is defined and belongs to $\overline{\Gamma(f[G])}$.

(b) Of course p is a linear operator, so we have only to check its translation-invariance. If $f \in U_W$, $c \in G$, $\epsilon > 0$ and $K \in \mathcal{K}_{c^{-1},\epsilon}$, then

$$\frac{1}{\mu K}\oint_K c\bullet_l f\, d\mu = \frac{1}{\mu K}\oint_K f(c^{-1}a)\mu(da) = \frac{1}{\mu K}\oint_{c^{-1}K} f(a)\mu(da),$$

so

$$\Big\|\frac{1}{\mu K}\oint_K c\bullet_l f\, d\mu - \frac{1}{\mu K}\oint_K f\, d\mu\Big\| \le \frac{1}{\mu K}\|f\|_\infty \mu(K\triangle c^{-1}K) \le \epsilon\|f\|_\infty.$$

As $\mathcal{K}_{c^{-1},\epsilon} \in \mathcal{F}$, $\|p(c\bullet_l f) - p(f)\| \le \epsilon\|f\|_\infty$; as c, f and ϵ are arbitrary, $p \in P_W$.

6K Theorem Let G be a unimodular amenable locally compact Hausdorff group, $\mathcal{K}$ the family of its non-Haar-negligible compact subsets, and $\mathcal{F}\text{ø}$ its left Følner filter. For Banach spaces W, define WDL, U_W and P_W as in 6B and 6G. Let μ be a Haar measure on G.

(a) If $f \in U_{\mathbb{R}}$,

$$\sup_{p\in P_{\mathbb{R}}} p(f) = \inf_{z\in Q_G} \sup_{a\in G}(z * f)(a) = \limsup_{K\to\mathcal{F}\text{ø}} \frac{1}{\mu K}\int_K f d\mu.$$

(b) For $f \in U_W$ and $K \in \mathcal{K}$, write $\oint\!\!\!\!\!\!\!\!\!\!\!/\;\;_K f d\mu$ for the Bochner integral of $f\restriction K$ with respect to the subspace measure μ_K on K. If $f \in U_W$ and $w^* \in W$, the following are equiveridical:

(i) wWDL(f) is defined and equal to w^*;

(ii) w-$\lim_{K\to\mathcal{F}\text{ø}} \frac{1}{\mu K} \oint\!\!\!\!\!\!\!\!\!\!\!/\;\;_K f d\mu$ is defined and equal to w^*;

(iii) $p(f) = w^*$ for every $p \in P_W$.

(c) If $f : G \to W$ is bounded and uniformly continuous for the right uniformity of G, then

$$\mathrm{WDL}(f) = \lim_{K\to\mathcal{F}\text{ø}} \frac{1}{\mu K} \oint\!\!\!\!\!\!\!\!\!\!\!/\;\;_K f d\mu$$

in the sense that if one is defined so is the other, and they are then equal.

proof (a) It is enough to consider the case $0 \le f \le \chi G$. Set

$$\gamma_1 = \sup_{p\in P_{\mathbb{R}}} p(f),$$

$$\gamma_2 = \inf_{z\in Q_G} \sup_{a\in G}(z * f)(a),$$

$$\gamma_3 = \limsup_{K\to\mathcal{F}\text{ø}} \frac{1}{\mu K}\int_K f d\mu.$$

(i) $\gamma_1 \le \gamma_2$. **P** If $p \in P_{\mathbb{R}}$ and $z \in Q_G$, then $p \in U_{\mathbb{R}}^*$ has norm at most 1 and $p(\chi G) = 1$, so p is a positive linear functional on the M-space $U_{\mathbb{R}}$. By 6Hd, $p(z * f) = p(f)$. Since $p(z * f) \in \overline{\Gamma((z * f)[G])}$, $p(f) = p(z * f) \le \sup_{a\in G}(z * f)(a)$. As p and z are arbitrary, $\gamma_1 \le \gamma_2$. **Q**

(ii) $\gamma_2 \le \gamma_3$. **P** Take any $\epsilon > 0$ and $\mathcal{L} \in \mathcal{F}\text{ø}$. Then there are a finite $I \subseteq G$ and a $\delta > 0$ such that $\mathcal{L} \supseteq \mathcal{L}' = \bigcap_{a\in I} \mathcal{K}_{a\delta}$, where

$$\mathcal{K}_{a\delta} = \{K : K \in \mathcal{K},\ \mu(K\triangle aK) \le \delta\mu K\}.$$

Take $K \in \mathcal{L}'$. Observe that for any $c \in G$,

$$\mu(Kc\triangle aKc) = \mu(K\triangle aK) \le \delta\mu K = \delta\mu(Kc)$$

for every $a \in I$, so $Kc \in \mathcal{L}'$. (This is where I need to suppose that G is unimodular.)

There is a neighbourhood H of the identity e of G such that $|f(a) - f(b)| \le \epsilon$ whenever $ab^{-1} \in H$. Let $b_0, \dots , b_n \in G$ be such that $K \subseteq \bigcup_{i\le n} Hb_i$, and let $E_0, \dots , E_n$ be a partition of K into Borel sets such that $E_i \subseteq Hb_i$ for each $i \le n$; then $|f(a) - f(b_i)| \le \epsilon$ for every $a \in E_i$. Set $\alpha_i = \frac{\mu E_i}{\mu K}$ for each $i \le n$. For any $c \in G$, $\langle E_ic\rangle_{i\le n}$ is a partition of Kc, and if $a \in E_ic$ then $a(b_ic)^{-1} = (ac^{-1})b_i^{-1} \in H$, so $|f(a) - f(b_ic)| \le \epsilon$. Accordingly

$$\begin{aligned}\Bigl|\frac{1}{\mu(Kc)}\int_{Kc} f d\mu - \sum_{i=0}^{n}\alpha_i f(b_ic)\Bigr| &\le \frac{1}{\mu(Kc)}\sum_{i=0}^{n}\Bigl|\int_{E_ic} f d\mu - f(b_ic)\mu E_i\Bigr| \\ &= \frac{1}{\mu(Kc)}\sum_{i=0}^{n}\Bigl|\int_{E_ic} f d\mu - f(b_ic)\mu(E_ic)\Bigr| \\ &\le \frac{1}{\mu(Kc)}\sum_{i=0}^{n}\epsilon\mu(E_ic) = \epsilon.\end{aligned}$$

Set $z = \sum_{i=0}^{n} \alpha_i\chi\{b_i^{-1}\}$. By the definition of γ_2, there is a $c \in G$ such that

$$\gamma_2 - \epsilon \le (z * f)(c) = \textstyle\sum_{i=0}^{n} \alpha_i f(b_ic),$$

so that $\frac{1}{\mu(Kc)}\int_{Kc} f d\mu \ge \gamma_2 - 2\epsilon$. Thus $\sup_{L\in\mathcal{L}} \frac{1}{\mu L}\int_L f d\mu \ge \gamma_2 - 2\epsilon$. As $\mathcal{L}$ is arbitrary, $\gamma_3 \ge \gamma_2 - 2\epsilon$; as ϵ is arbitrary, $\gamma_3 \ge \gamma_2$. **Q**

(iii) $\gamma_3 \le \gamma_1$. **P** Let $\mathcal{F}$ be an ultrafilter on $\mathcal{K}$ including $\mathcal{F}\emptyset$ and such that $\lim_{K\to\mathcal{F}} \frac{1}{\mu K}\int_K f d\mu = \gamma_3$. Let $p \in P_{\mathbb{R}}$ be defined by the formula in Lemma 6J. Then $\gamma_3 = p(f) \le \gamma_1$. **Q**

(b)(i)⇒(iii) Suppose that $w^* = \mathrm{wWDL}(f)$, and that $p \in P_W$. Let $\epsilon > 0$ and $h \in W^*$. Then there is a $z \in Q_G$ such that $|h(w^*) - h((z*f)(a))| \le \epsilon$ for every $a \in G$. Now $p(f) = p(z*f) \in \overline{\Gamma((z*f)[G])}$, so $|h(w^*) - h(p(f))| \le \epsilon$. As h and ϵ are arbitrary, $p(f) = w^*$.

(iii)⇒(ii) Suppose that $p(f) = w^*$ for every $p \in P_W$. Let $\mathcal{F}$ be any ultrafilter on $\mathcal{K}$ extending $\mathcal{F}\emptyset$. Set $p(g) = \text{w-}\lim_{K\to\mathcal{F}} \frac{1}{\mu K}\mathrm{B}\!\!\int_K g\,d\mu$ for $g \in U_W$. Then $p \in P_W$ (6J). So $p(f) = w^*$. As $\mathcal{F}$ is arbitrary, $w^* = \text{w-}\lim_{K\to\mathcal{F}\emptyset} \frac{1}{\mu K}\mathrm{B}\!\!\int_K f\,d\mu$.

(ii)⇒(i) Suppose that $w^* = \text{w-}\lim_{K\to\mathcal{F}\emptyset} \frac{1}{\mu K}\mathrm{B}\!\!\int_K f\,d\mu$. Applying 6J with any ultrafilter $\mathcal{F}$ extending $\mathcal{F}\emptyset$, we see that $w^* = \text{w-}\lim_{K\to\mathcal{F}\emptyset} \frac{1}{\mu K}\mathrm{B}\!\!\int_K a\bullet_l f\,d\mu$ for every $a \in G$, and therefore $w^* = \text{w-}\lim_{K\to\mathcal{F}\emptyset} \frac{1}{\mu K}\mathrm{B}\!\!\int_K z*f\,d\mu$ for every $z \in Q_G$.

Let V be any weak neighbourhood of W. Then there are $h_0, \dots, h_n \in W^*$ and $\epsilon > 0$ such that $V \supseteq \{w : h_i(w) \le \epsilon + h_i(w^*)$ for every $i \le n\}$. Take any $z_0 \in Q_G$, and choose $z_1, \dots, z_{n+1} \in Q_G$ as follows. Given z_k where $k \le n$, then

$$\begin{aligned} h_k(w^*) &= h_k(\operatorname*{w-lim}_{K\to\mathcal{F}\emptyset}\frac{1}{\mu K}\mathrm{B}\!\!\int_K z_k * f\,d\mu) \\ &= \lim_{K\to\mathcal{F}\emptyset}\frac{1}{\mu K}\mathrm{B}\!\!\int_K z_k * h_k f\,d\mu \le \inf_{z\in Q_G}\sup_{a\in G}(z*z_k*h_k f)(a) \end{aligned}$$

by (a). Let $z \in Q_G$ be such that $\sup_{a\in G}(z*z_k*h_k f)(a) \le h_k(w^*) + \epsilon$, and set $z_{k+1} = z*z_k$; continue.

At the end, we see that for every $k \le n$ we can express z_{n+1} as $z*z_{k+1}$ for some $z \in Q_G$, so that

$$h_k((z_{n+1}*f)(b)) = (z_{n+1}*h_k f)(b)) \le \sup\nolimits_{a\in G}(z_{k+1}*h_k f)(a) \le h_k(w^*) + \epsilon$$

for every $b \in G$ and $k \le n$; consequently $(z_{n+1}*f)[G] \subseteq V$, while z_{n+1} is also expressible as $z*z_0$ for some $z \in Q_G$. As z_0 and V are arbitrary, $\mathrm{wWDL}(f)$ is defined and equal to w^*.

(c)(i) Suppose that $w^* = \mathrm{WDL}(f)$ is defined. Then for any $\epsilon > 0$ there is a $z \in Q_G$ such that $\|w^* - (z*f)(a)\| \le \epsilon$ for every $a \in G$. In this case, $\|w^* - \frac{1}{\mu K}\mathrm{B}\!\!\int_K z*f\,d\mu\| \le \epsilon$ for every $K \in \mathcal{K}$.

Set $I = \{c : z(c) \ne 0\}$. Then $\mathcal{L} = \bigcap_{c\in I}\mathcal{K}_{c^{-1},\epsilon}$ belongs to $\mathcal{F}\emptyset$. If $K \in \mathcal{L}$, then

$$\begin{aligned} \|\frac{1}{\mu K}\mathrm{B}\!\!\int_K z*f\,d\mu - \frac{1}{\mu K}\mathrm{B}\!\!\int_K f d\mu\| &\le \sum_{c\in I} z(c)\|\frac{1}{\mu K}\mathrm{B}\!\!\int_K f(c^{-1}a)\mu(da) - \frac{1}{\mu K}\mathrm{B}\!\!\int_K f(a)\mu(da)\| \\ &= \sum_{c\in I} z(c)\|\frac{1}{\mu K}\mathrm{B}\!\!\int_{c^{-1}K} f(a)\mu(da) - \frac{1}{\mu K}\mathrm{B}\!\!\int_K f(a)\mu(da)\| \\ &\le \sum_{c\in I} z(c)\|f\|_\infty\frac{\mu(K\triangle c^{-1}K)}{\mu K} \le \epsilon\|f\|_\infty, \end{aligned}$$

and $\|w^* - \frac{1}{\mu K}\mathrm{B}\!\!\int_K f d\mu\| \le \epsilon(1 + \|f\|_\infty)$. As ϵ is arbitrary, $w^* = \lim_{K\to\mathcal{F}\emptyset}\frac{1}{\mu K}\mathrm{B}\!\!\int_K f d\mu$.

(ii) Suppose that $w^* = \lim_{K\to\mathcal{F}\emptyset}\frac{1}{\mu K}\mathrm{B}\!\!\int_K f d\mu$ is defined.

(α) We need to know that for any $z \in Q_G$ we also have

$$w^* = \lim\nolimits_{K\to\mathcal{F}\emptyset}\frac{1}{\mu K}\mathrm{B}\!\!\int_K z*f\,d\mu.$$

P For any $c \in G$ and $\epsilon > 0$, $\mathcal{K}_{c^{-1},\epsilon} \in \mathcal{F}\emptyset$. For $K \in \mathcal{K}_{c^{-1},\epsilon}$,

$$\begin{aligned}\|\frac{1}{\mu K}\mathrm{P}\!\!\int_K c\bullet_l f d\mu - \frac{1}{\mu K}\mathrm{P}\!\!\int_K f d\mu\| &= \|\frac{1}{\mu K}\mathrm{P}\!\!\int_K f(c^{-1}a)\mu(da) - \frac{1}{\mu K}\mathrm{P}\!\!\int_K f(a)\mu(da)\| \\ &= \|\frac{1}{\mu K}\mathrm{P}\!\!\int_{c^{-1}K} f(a)\mu(da) - \frac{1}{\mu K}\mathrm{P}\!\!\int_K f(a)\mu(da)\| \\ &\le \frac{\mu(K\triangle c^{-1}K)}{\mu K}\|f\|_\infty \le \epsilon\|f\|_\infty.\end{aligned}$$

As ϵ is arbitrary,

$$w^* = \lim_{K\to\mathcal{F}\emptyset}\frac{1}{\mu K}\mathrm{P}\!\!\int_K c\bullet_l f\,d\mu.$$

It follows at once that

$$w^* = \lim_{K\to\mathcal{F}\emptyset}\frac{1}{\mu K}\mathrm{P}\!\!\int_K z * f\,d\mu$$

for every $z \in Q_G$. **Q**

(β) Suppose that $z_0 \in Q_G$ and $\epsilon > 0$. Set $g = z_0 * f$. Then there are a finite $I \subseteq G$ and a $\delta > 0$ such that $\|w^* - \frac{1}{\mu K}\mathrm{P}\!\!\int_K g\,d\mu\| \le \epsilon$ whenever K belongs to $\mathcal{L} = \bigcap_{c\in I}\mathcal{K}_{c\delta}$. As in (a-ii) above, $Ka \in \mathcal{L}$ whenever $K \in \mathcal{L}$ and $a \in G$.

There is a neighbourhood H of the identity e of G such that $\|g(a) - g(b)\| \le \epsilon$ whenever $ab^{-1} \in H$. Let $b_0, \dots, b_n \in G$ be such that $K \subseteq \bigcup_{i\le n} Hb_i$, and let $E_0, \dots, E_n$ be a partition of K into Borel sets such that $E_i \subseteq Hb_i$ for each $i \le n$; then $\|g(a) - g(b_i)\| \le \epsilon$ for every $a \in E_i$. Set $\alpha_i = \frac{\mu E_i}{\mu K}$ for each $i \le n$. For any $c \in G$, $\langle E_i c\rangle_{i\le n}$ is a partition of Kc, and if $a \in E_i c$ then $a(b_i c)^{-1} = (ac^{-1})b_i^{-1} \in H$, so $\|g(a) - g(b_i c)\| \le \epsilon$. Accordingly

$$\begin{aligned}\Big\|\frac{1}{\mu(Kc)}\mathrm{P}\!\!\int_{Kc} g\,d\mu - \sum_{i=0}^n \alpha_i g(b_i c)\Big\| &\le \frac{1}{\mu(Kc)}\sum_{i=0}^n\Big\|\mathrm{P}\!\!\int_{E_i c} g\,d\mu - g(b_i c)\mu E_i\Big\| \\ &= \frac{1}{\mu(Kc)}\sum_{i=0}^n\Big\|\mathrm{P}\!\!\int_{E_i c} g\,d\mu - g(b_i c)\mu(E_i c)\Big\| \\ &\le \frac{1}{\mu(Kc)}\sum_{i=0}^n \epsilon\mu(E_i c) = \epsilon.\end{aligned}$$

Set $z = \sum_{i=0}^n \alpha_i\chi\{b_i^{-1}\}$. If $c \in G$, then

$$\|(z*g)(c) - \frac{1}{\mu(Kc)}\mathrm{P}\!\!\int_{Kc} g\,d\mu\| = \|\frac{1}{\mu(Kc)}\mathrm{P}\!\!\int_{Kc} g\,d\mu - \textstyle\sum_{i=0}^n \alpha_i g(b_i c)\| \le \epsilon.$$

Thus $\|(z * z_0 * f)(c) - w^*\| \le 2\epsilon$ for every $c \in G$. As ϵ and z_0 are arbitrary, WDL(f) is defined and equal to w^*.

6L Theorem (compare Jung Park & Park 97[3]) Let G be a unimodular amenable locally compact Hausdorff group, and $\bullet$ an action of G on a Banach space W such that

$w \mapsto a\bullet w$ is a linear operator of norm at most 1 for every $a \in G$,

$a \mapsto a\bullet w : G \to W$ is norm-continuous for every $w \in W$,

$\{a\bullet w : a \in G\}$ is relatively weakly compact for every $w \in W$.

For $w \in W$ and $a \in G$ set $g_w(a) = a^{-1}\bullet w$. Then WDL(g_w) is defined, and $a\bullet$ WDL(g_w) = WDL(g_w), for every $a \in G$ and $w \in W$.

proof (a) Define U_W as in 6G. Then $g_w \in U_W$ for every $w \in W$. **P** Our hypotheses guarantee that $g_w[G]$ is relatively weakly compact. For any $\epsilon > 0$, there is a neighbourhood H of the identity in G such that $\|w - a\bullet w\| \le \epsilon$ whenever $a \in H$. Now

[3] I am indebted to A.S.Vernitski for the reference.

$$\|g_w(a) - g_w(b)\| = \|a^{-1}\bullet w - b^{-1}\bullet w\| = \|w - (ab^{-1})\bullet w\| \le \epsilon$$

whenever $ab^{-1} \in H$. As ϵ is arbitrary, g_w is uniformly continuous for the right uniformity. **Q**

(b) Set $W_0 = \{w : w \in W,\, a\bullet w = w \text{ for every } a \in G\}$, and let V be the linear subspace of W generated by

$$W_0 \cup \{w - a\bullet w : w \in W,\, a \in G\}.$$

Then V is norm-dense in W. **P?** Otherwise, there are an $h \in W^*$ and a $w \in W$ such that $h(v) = 0$ for every $v \in V$ and $h(w) \ne 0$. Let μ be a left Haar measure on G, $\mathcal{K}$ the family of non-negligible compact subsets of G, and $\mathcal{F}$ an ultrafilter on $\mathcal{K}$ including the Følner ultrafilter of G. Let w^* be the weak limit $\lim_{K\to\mathcal{F}} \frac{1}{\mu K} \mathrm{P}\!\!\!\int_K a\bullet w\, \mu(da)$. We have $h(a\bullet w) = h(w)$ for every $a \in G$, so

$$h\Big(\frac{1}{\mu K} \mathrm{P}\!\!\!\int_K a\bullet w\, \mu(da)\Big) = \frac{1}{\mu K} \int_K h(a\bullet w)\, \mu(da) = h(w)$$

for every $K \in \mathcal{K}$, and $h(w^*) = h(w)$. On the other hand, for any $c \in G$,

$$\begin{aligned} c\bullet w^* &= c\bullet\Big(\lim_{K\to\mathcal{F}} \frac{1}{\mu K} \mathrm{P}\!\!\!\int_K a\bullet w\, \mu(da)\Big) \\ &= \lim_{K\to\mathcal{F}} \frac{1}{\mu K} \mathrm{P}\!\!\!\int_K (ca)\bullet w\, \mu(da) = \lim_{K\to\mathcal{F}} \frac{1}{\mu K} \mathrm{P}\!\!\!\int_{cK} a\bullet w\, \mu(da), \end{aligned}$$

so

$$\|w^* - c\bullet w^*\| \le \lim_{K\to\mathcal{F}} \frac{1}{\mu K} \|w\| \mu(K \triangle cK) = 0$$

and $w^* = c\bullet w^*$. As c is arbitrary, $w^* \in W_0$ and $0 = h(w^*) = h(w)$. **XQ**

(c) The set

$$V_1 = \{w : w \in W,\, \mathrm{WDL}(g_w) \text{ is defined and belongs to } W_0\}$$

is a norm-closed linear subspace of W. If $w \in W_0$ then g_w is the constant function with value w, so $\mathrm{WDL}(g_w) = w$ and $w \in V_1$. If $w \in W$ and $c \in G$ then $w - c\bullet w \in V_1$. **P** For $K \in \mathcal{K}$,

$$\begin{aligned} \frac{1}{\mu K} \mathrm{P}\!\!\!\int_K g_{w-c\bullet w}\, d\mu &= \frac{1}{\mu K} \mathrm{P}\!\!\!\int_K a^{-1}\bullet w - (a^{-1}c)\bullet w\, \mu(da) \\ &= \frac{1}{\mu K} \mathrm{P}\!\!\!\int_K a^{-1}\bullet w - (c^{-1}a)^{-1}\bullet w\, \mu(da) \\ &= \frac{1}{\mu K} \Big(\mathrm{P}\!\!\!\int_K a^{-1}\bullet w\, \mu(da) - \mathrm{P}\!\!\!\int_{c^{-1}K} a^{-1}\bullet w\, \mu(da) \Big) \end{aligned}$$

has norm at most $\frac{\|w\|}{\mu K} \mu(K \triangle c^{-1}K)$. By Theorem 6Kc,

$$\mathrm{WDL}(g_{w-c\bullet w}) = \lim_{K\to\mathcal{F}_\emptyset} \frac{1}{\mu K} \mathrm{P}\!\!\!\int_K g_{w-c\bullet w}\, d\mu = 0$$

and $w - c\bullet w \in V_1$. **Q**

But this means that $V_1 \supseteq V$ and $V_1 = W$, as required.

6M Remarks (a) Presumably the ideas above can be adapted to an appropriate class of semigroups to give a true generalization of Theorem 3B.

(b) An alternative expression of the idea of (*) in 6Ba would be

(†) for every neighbourhood V of w and every $z \in Q_G$ there is a $z' \in Q_G$ such that $(z'' * z' * z * f)[G] \subseteq V$ for every $z'' \in Q_G$.

Put like this, it becomes something we can consider whenever X is a set, W is a Hausdorff space, S is a semigroup and $\bullet$ is a semigroup action of S on W^X. 6B(b-i) would then become a result about product actions on $W_1^X \times W_2^X \cong (W_1 \times W_2)^X$.

6N For abelian groups, we can approach from a different direction.

Lemma Let G be an abelian group, W a Hausdorff linear topological space and $\bullet$ an action of G on W such that $w \mapsto a\bullet w = T_a(w)$ is a linear operator for every $a \in G$ and the family $\{T_a : a \in G\}$ is equicontinuous. For $w \in W$ and $a \in G$ set $g_w(a) = a^{-1}\bullet w$. Then for w, $w^* \in W$ the following are equiveridical:

(i) $\mathrm{WDL}(g_w)$ is defined and equal to w^*;

(ii) $w^* \in \overline{\Gamma(g_w[G])}$ and $a\bullet w^* = w^*$ for every $a \in G$.

proof (i)⇒(ii) Suppose that $w^* = \mathrm{WDL}(g_w)$. By 6B(b-iv), $w^* \in \overline{\Gamma(g_w[G])}$. If $c \in G$, then

$$c\bullet g_w(a) = c\bullet a^{-1}\bullet w = a^{-1}\bullet c\bullet w = g_w(c^{-1}a) = (c\bullet_l g_w)(a)$$

for every $a \in G$. Applying 6Bc to the map $u \mapsto c\bullet u$ we see that

$$c\bullet w^* = \mathrm{WDL}(c\bullet_l g_w) = \mathrm{WDL}(g_w) = w^*$$

by 6B(b-iii). So w^* is G-invariant and (ii) is true.

(ii)⇒(i) Let V be a convex neighbourhood of w^*. Let V_1 be a closed convex neighbourhood of 0 in W such that $T_a(w) \in V - w^*$ whenever $w \in V_1$ and $a \in G$. Let $a_0, \ldots , a_n \in G$ and $\alpha_0, \ldots , \alpha_n \geq 0$ be such that $\sum_{i=0}^n \alpha_i = 1$ and $\sum_{i=0}^n \alpha_i g_w(a_i) \in w^* + V_1$. Set $z = \sum_{i=0}^n \alpha_i\chi\{a_i^{-1}\} \in Q_G$. Then, for any $a \in G$,

$$(z' * g_w)(a) = \sum_{i=0}^n \alpha_i g_w(a_i a) = \sum_{i=0}^n \alpha_i(a^{-1}\bullet a_i^{-1}\bullet w) = a^{-1}\bullet\sum_{i=0}^n \alpha_i(a_i^{-1}\bullet w),$$

so

$$\begin{aligned}(z * g_w)(a) - w^* &= a^{-1}\bullet\sum_{i=0}^n \alpha_i(a_i^{-1}\bullet w) - w^* = a^{-1}\bullet((\sum_{i=0}^n \alpha_i(a_i^{-1}\bullet w)) - w^*)\\ &= h_{a^{-1}}\bullet((\sum_{i=0}^n \alpha_i(a_i^{-1}\bullet w)) - w^*) \in V - w^*\end{aligned}$$

because $\sum_{i=0}^n \alpha_i(a_i^{-1}\bullet w) \in w^* + V_1$. Thus $(z * g_w)[G] \subseteq V$. As V is arbitrary, $\mathrm{WDL}(g_w) = w^*$, by 6B(d-i).

6O Theorem Let G be an abelian group, W a Banach space and $\bullet$ an action of G on W such that $w \mapsto a\bullet w$ is a linear operator of norm at most 1 for every $a \in G$. For $w \in W$ and $a \in G$, set $g_w(a) = a^{-1}\bullet w$. If $w \in W$ is such that $g_w[G]$ is relatively weakly compact, then $w^* = \mathrm{WDL}(g_w)$ is defined and $a\bullet w^* = w^*$ for every $a \in G$.

proof Set $K = \overline{\Gamma(g_w[G])}$; because W is complete, K is weakly compact. Since $c\bullet g_w(a) = g_w(ac^{-1})$ for every $a \in G$, $c\bullet g_w[G] = g_w[G]$ for every $c \in G$, and $c\bullet u \in K$ for every $u \in K$. If $a \in G$, set $T_a w = a\bullet w$ for $w \in W$; then T_a is weakly continuous, so the action of G on K is continuous if we give K its weak topology and G its discrete topology. As G is abelian, it is amenable in any topology (Fremlin 03, 449Cf), and there is a G-invariant probability measure μ on K which is Radon for the weak topology. Let w^* be the barycenter of μ. Then $T_a w^*$ is the barycenter of the image measure $\mu T_a^{-1} = \mu$ (Fremlin 03, 461B), so $a\bullet w^* = w^*$ for every $a \in G$. By 6N, $w^* = \mathrm{WDL}(g_w)$.

Remark I have kept the formula $g_w(a) = a^{-1}\bullet w$ from 6L. But in the present context it would be more natural to look at $\tilde{g}_w(a) = a\bullet w$; of course $\tilde{g}_w[G] = g_w[G]$ and $\mathrm{WDL}(g_w) = \mathrm{WDL}(\tilde{g}_w)$ if either is defined, by 6B(d-ii), so we get the same results.

7 Problems (a) Is it consistent to suppose that every Radon probability measure on $\{0,1\}^{\omega_1}$ has a well-distributed sequence? What if $\mathfrak{m} > \omega_1$?

(b) If $(X, \mathfrak{T}, \Sigma, \mu)$ is a compact Radon probability space with a well-distributed sequence and ν is a probability measure on X which is an indefinite-integral measure over μ, must ν have a well-distributed sequence? (Compare 4G and Fremlin 03, 491R.)

(c) In 6K and 6L, do we really need G to be unimodular?

(d)(i) Find a useful characterization of the set E of extreme points of P. (See 1M.) (ii) Is E compact?

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[4]For 'later editions' see `http://www.essex.ac.uk/maths/people/fremlin/mtcont.htm`.