Games and the Tukey classification

D.H.Fremlin

University of Essex, Colchester, England; Fields Institute, Toronto

The game $\Gamma_1(P)$ Let P be a non-empty partially ordered set. Start with $U_0 = P$. Given $U_{n-1} \subseteq P$,

Player I plays a countable cover $\mathcal{U}_n$ of U_{n-1},
Player II plays $U_n \in \mathcal{U}_n$ and a sequence $\langle x_{ni}\rangle_{i\in\mathbb{N}}$ in U_n,
Player I plays an infinite $J_n \subseteq \mathbb{N}$,
Player II plays $m_n \in \mathbb{N}$

and they continue to the next round. At the end of the game, Player I wins if $\{x_{ni} : n \ge 1, i \in J_n \cap m_n\}$ is bounded above in P; otherwise Player II wins.

Proposition Let P and Q be partially ordered sets and $f : P \to Q$ a Tukey function.
(a) If Player I has a winning strategy in $\Gamma_1(Q)$, he has a winning strategy in $\Gamma_1(P)$.
(b) If Player II has a winning strategy in $\Gamma_1(P)$, he has a winning strategy in $\Gamma_1(Q)$.

proof (a) Let σ be a winning strategy for Player I in $\Gamma_1(Q)$. Let σ' be the strategy in $\Gamma_1(P)$ defined as follows.

(i) If the initial move in $\Gamma_1(Q)$ is $\sigma() = \mathcal{V}_1$, then $\sigma'()$ is to be $\{f^{-1}[V] : V \in \mathcal{V}_1\}$.

(ii) If a game in $\Gamma_1(P)$ has proceeded with Player II's moves being U_1, $\langle x_{1i}\rangle_{i\in\mathbb{N}}$, $m_1, \dots, U_n$, $\langle x_{ni}\rangle_{i\in\mathbb{N}}$, with $U_n \in \{f^{-1}[V] : V \in \mathcal{V}_n\}$ and $U_i = f^{-1}[V_i]$ for $i \le n$, then

$$\begin{aligned}\sigma'(U_1, \langle x_{1i}\rangle_{i\in\mathbb{N}}, &m_1, \dots, U_n, \langle x_{ni}\rangle_{i\in\mathbb{N}})\\ &= \sigma(V_1, \langle f(x_{1i})\rangle_{i\in\mathbb{N}}, m_1, \dots, V_n, \langle f(x_{ni})\rangle_{i\in\mathbb{N}}) \subseteq \mathbb{N}.\end{aligned}$$

(iii) If a game in $\Gamma_1(P)$ has proceeded with Player II's moves being U_1, $\langle x_{1i}\rangle_{i\in\mathbb{N}}$, $m_1, \dots, U_n, \langle x_{ni}\rangle_{i\in\mathbb{N}}, m_n$, with $U_n \in \{f^{-1}[V] : V \in \mathcal{V}_n\}$ and $U_i = f^{-1}[V_i]$ for $i \le n$, then

$$\begin{aligned}\sigma'(U_1, \langle x_{1i}\rangle_{i\in\mathbb{N}}, &m_1, \dots, U_n, \langle x_{ni}\rangle_{i\in\mathbb{N}}, m_n)\\ &= \{f^{-1}[V] : V \in \sigma(V_1, \langle f(x_{1i})\rangle_{i\in\mathbb{N}}, m_1, \dots, V_n, \langle x_{ni}\rangle_{i\in\mathbb{N}}, m_n)\},\end{aligned}$$

a countable cover of U_n.

At the end of the game, $\{f(x_{ni} : n \ge 1, i \le m_n\}$ is bounded above in Q so $\{x_{ni} : n \ge 1, i \le m_n\}$ is bounded above in P.

(b) Let σ be a winning strategy for Player II in $\Gamma_1(P)$. Let σ' be the strategy in $\Gamma_1(Q)$ defined as follows.

(i) If a game in $\Gamma_1(Q)$ has begun with Player I's moves being $(\mathcal{V}_1, J_1, \dots, \mathcal{V}_n)$, set $\mathcal{U}_i = \{f^{-1}[V] : V \in \mathcal{V}_i\}$ for each i; $\sigma'(\mathcal{V}_1, m_1, \dots, \mathcal{V}_n)$ is to be $(V, \langle f(x_{ni})\rangle_{i\in\mathbb{N}})$ where $V \in \mathcal{V}_n$ is such that $\sigma(\mathcal{U}_1, J_1, \dots, \mathcal{U}_n) = (f^{-1}[V], \langle x_{ni}\rangle_{i\in\mathbb{N}})$.

(ii) If a game in $\Gamma_1(Q)$ has begun with Player I's moves being $(\mathcal{V}_1, J_1, \dots, \mathcal{V}_n, J_n)$, set $\mathcal{U}_i = \{f^{-1}[V] : V \in \mathcal{V}_i\}$ for each i; $\sigma'(\mathcal{V}_1, m_1, \dots, \mathcal{V}_n, J_n)$ is to be $\sigma(\mathcal{U}_1, J_1, \dots, \mathcal{U}_n, J_n)$.

At the end of the game, $\{x_{ni} : n \geq 1,\, i \leq m_n\}$ is not bounded above in P so $\{f(x_{ni}) : n \geq 1,\, i \leq m_n\}$ is not bounded above in P.

Proposition If $\mathcal{Z} \triangleleft \mathcal{P}\mathbb{N}$ is the family of sets with zero asymptotic density, then Player I has a winning strategy in $\Gamma_1(\mathcal{Z})$.

proof Set $\nu a = \sup_{n \geq 1} \frac{1}{n} \#(a \cap n)$, $\rho(a, b) = \nu(a \triangle b)$ for a, $b \subseteq \mathcal{P}\mathbb{N}$. Then ρ is a complete metric on $\mathcal{P}\mathbb{N}$ for which $\mathcal{Z}$ is a closed separable subspace. For each n, Player I will choose $\mathcal{U}_n$ so that $\operatorname{diam} U \leq 2^{-n}$ for every $U \in \mathcal{U}_n$. When responding to a sequence $\langle a_{ni} \rangle_{i \in \mathbb{N}}$ in U_n, Player I first picks a cluster point a of $\langle a_{ni} \rangle_{i \in \mathbb{N}}$ in $\mathcal{P}\mathbb{N}$ for the usual topology of $\mathcal{P}\mathbb{N}$. Then $\nu(a \setminus a_{n0}) \leq 2^{-n}$. Now he chooses

$$J_n = \{i : \nu((a \cup a_{ni} \cup \bigcup_{j \in J \cap i} a_{nj}) \setminus a_{n0}) \leq 3 \cdot 2^{-n}\}.$$

J_n is infinite. **P** Given that $K \subseteq \mathbb{N}$ is finite and $b = \bigcup_{j \in K} a_{nj}$ is such that $\nu((a \cup b) \setminus a_{n0}) \leq 3 \cdot 2^{-n}$, let $m \in \mathbb{N}$ be such that $\#(k \cap b) \leq 2^{-n} k$ for every $k \geq m$. Take $i \in \mathbb{N}$ such that $K \subseteq i$ and $a_{ni} \cap m = a \cap m$. Set $b' = a \cup a_{ni} \cup b$. If $k \leq m$, then $\#(k \cap b' \setminus a_{n0}) = \#(k \cap ((a \cup b) \setminus a_{n0})) \leq 3 \cdot 2^{-n} k$. If $k \geq m$, then

$$\#(k \cap b' \setminus a_{n0}) \leq \#(k \cap a \setminus a_{n0}) + \#(k \cap b) + \#(k \cap a_{ni} \setminus a_{n0}) \leq 3 \cdot 2^{-n} k.$$

So $\nu(b' \setminus a_{n0}) \leq 3 \cdot 2^{-n}$ and we can add i to K. **Q**

At the end of the game, set $b_n = \bigcup_{i \in J_n \cap m_n} a_{ni}$. Then $\nu(b_n \setminus a_{n0}) \leq 3 \cdot 2^{-n}$ for each n, so $c = \bigcup_{n \geq 1} b_n \in \mathcal{Z}$. Also $\nu(a_{n+1,0} \setminus a_{n0}) \leq 2^{-n}$ for each n (because $a_{n+1,0} \in U_{n+1} \subseteq U_n$, and $\operatorname{diam}(U_n) \leq 2^{-n}$), so $d = \bigcup_{n \geq 1} a_{n0} \in \mathcal{Z}$ and $c \cup d$ is an upper bound for $\{a_{ni} : n \geq 1,\, i \in J_n \cap m_n\}$.

Proposition Player II has a winning strategy in $\Gamma_1(\ell^1)$.

proof Player II must at each stage choose U_n so that $U_n \cap (\ell^1)^+$ is not meager in the Polish space $(\ell^1)^+$. If he does this, then it will always be somewhere dense. Since no non-empty open set in $(\ell^1)^+$ is totally bounded, he will always be able to choose $\langle x_{ni} \rangle_{i \in \mathbb{N}}$ in U_n such that $\inf_{i<j} \|x_{ni} - x_{nj}\| > 0$. This will mean that, given J_n, he will be able to choose m_n such that $\|\sup_{i \in J_n \cap m_n} x_{ni}\|_1 \geq n$. This will ensure that, at the end of the game, $\{x_{ni} : n \geq 1,\, i \in J_n \cap m_n\}$ is unbounded above in ℓ^1.

Corollary (Louveau & Veličković 99) There is no Tukey function from ℓ^1 to $\mathfrak{Z}$.

The game $\Gamma_2(P)$ Let P be a non-empty partially ordered set. Start with $U_0 = P$. Given $U_{n-1} \subseteq P$,

Player I plays a countable cover $\mathcal{U}_n$ of U_{n-1},

Player II plays $U_n \in \mathcal{U}_n$ and a finite set $I_n \subseteq U_n$

and they continue to the next round. At the end of the game, Player I wins if $\bigcup_{n \geq 1} I_n$ is bounded above in P; otherwise Player II wins.

Proposition Let P and Q be partially ordered sets and $f : P \to Q$ a Tukey function.
(a) If Player I has a winning strategy in $\Gamma_2(Q)$, he has a winning strategy in $\Gamma_2(P)$.
(b) If Player II has a winning strategy in $\Gamma_2(P)$, he has a winning strategy in $\Gamma_2(Q)$.

proof (a) Let σ be a winning strategy for Player I in $\Gamma_2(Q)$. Let σ' be the strategy in $\Gamma_2(P)$ defined as follows.

(i) If the initial move in $\Gamma_2(Q)$ is $\sigma() = \mathcal{V}_1$, then $\sigma'()$ is to be $\{f^{-1}[V] : V \in \mathcal{V}_1\}$.

(ii) If a game in $\Gamma_2(P)$ has proceeded with Player II's moves being $U_1, I_1, \ldots, U_n, I_n$, with $U_n \in \{f^{-1}[V] : V \in \mathcal{V}_n\}$ and $U_i = f^{-1}[V_i]$ for $i \le n$, then

$$\sigma'(U_1, I_1, \ldots, U_n, I_n) = \{f^{-1}[V] : V \in \sigma(V_1, f[I_1], \ldots, V_n, f[I_n])\},$$

a countable cover of U_n.

At the end of the game, $\bigcup_{n\ge1} f[I_n]$ is bounded above in Q so $\bigcup_{n\ge1} I_n$ is bounded above in P.

(b) Let σ be a winning strategy for Player II in $\Gamma_1(P)$. Let σ' be the strategy in $\Gamma_1(Q)$ defined as follows. If a game in $\Gamma_1(Q)$ has begun with Player I's moves being $(\mathcal{V}_1, \ldots, \mathcal{V}_n)$, set $\mathcal{U}_i = \{f^{-1}[V] : V \in \mathcal{V}_i\}$ for each i; $\sigma'(\mathcal{V}_1, \ldots, \mathcal{V}_n)$ is to be $f[I_n]$ where $I_n = \sigma(\mathcal{U}_1, \ldots, \mathcal{U}_n)$.

At the end of the game, $\bigcup_{n\ge1} I_n$ is not bounded above in P so $\bigcup_{n\in\mathbb{N}} f[I_n]$ is not bounded above in P.

Proposition If $\mathcal{F}$ is the family of nowhere dense sets in $\mathbb{N}^{\mathbb{N}}$, Player I has a winning strategy in $\Gamma_2(\mathcal{F})$.

proof Enumerate a countable base $\mathcal{G}$ for the topology of $\mathbb{N}^{\mathbb{N}}$, not containing $\emptyset$, as $\langle H_n\rangle_{n\in\mathbb{N}}$. For $G \in \mathcal{G}$ set $W_G = \{F : F \in \mathcal{F},\ F \cap G = \emptyset\}$. Player I will take

$$\mathcal{V}_n = \{U_{n-1} \cap W_G : G \in \mathcal{G},\ G \subseteq H_{n-1}\}$$

for each $n \ge 1$. Now if Player II's moves are $u_n = U_{n-1} \cap W_{G_n}$, $\mathcal{I}_n$ for each n, where $G_n \subseteq H_{n-1}$, and $E = \bigcup_{n\ge1} \bigcup \mathcal{I}_n$, then $E \cap G_n \subseteq \bigcup_{m\le n} \bigcup \mathcal{I}_m$ is nowhere dense for every n, so E is nowhere dense.

Proposition Player II has a winning strategy in $\Gamma_2(\mathcal{Z})$.

proof Give $\mathcal{Z}$ the Polish topology derived from the metric ρ. Player II must choose U_n so that U_n is never meager. Then he will always be able to find a finite set $I_n \subseteq U_n$ such that $\nu(\bigcup I_n \setminus n) \ge \frac{1}{2}$. **P** There must be $c \in \mathcal{Z}$, $\delta > 0$ such that U_n is dense in $\{a : a \in \mathcal{Z},\ \rho(a, c) < \delta\}$. Take $m \in \mathbb{N}$ so large that $2^m \ge n$ and $2^{-m} < \delta$. Then for $2^m \le i < 2^{m+1}$ there is an $a_i \in U_n$ such that $i \in a_i$; set $I_n = \{a_i : 2^m \le i < 2^{m+1}\}$. **Q** If he does this, $\bigcup_{n\ge1} \bigcup I_n \notin \mathcal{Z}$.

Corollary (Fremlin 91) There is no Tukey function from $\mathcal{Z}$ to $\mathcal{F}$.

The game $\Gamma_3(P)$ Let P be a non-empty partially ordered set. Start with $U_0 = P$. Given $U_{n-1} \subseteq P$,

Player I plays an upwards-directed countable cover $\mathcal{U}_n$ of U_{n-1},

Player II plays $U_n \in \mathcal{U}_n$ and a finite set $I_n \subseteq U_n$

and they continue to the next round. At the end of the game, Player I wins if $\bigcup_{n\ge1} I_n$ is bounded above in P; otherwise Player II wins.

Proposition Let P and Q be partially ordered sets and $f : P \to Q$ a Tukey function.

(a) If Player I has a winning strategy in $\Gamma_3(Q)$, he has a winning strategy in $\Gamma_3(P)$.

(b) If Player II has a winning strategy in $\Gamma_3(P)$, he has a winning strategy in $\Gamma_3(Q)$.

Proposition Player I has a winning strategy in $\Gamma_3(\mathbb{N}^{\mathbb{N}})$.

proof Given U_{n-1}, set $V_{ni} = \{\alpha : \alpha \in U_{n-1},\ \alpha(n) \le i\}$, $\mathcal{U}_n = \{V_{ni} : i \in \mathbb{N}\}$.

Proposition If $\mathcal{E}$ is the family of subsets of $[0,1]$ with Lebesgue negligible closures, Player II has a winning strategy in $\Gamma_3(\mathcal{E})$.

proof Play so that $\mu(\overline{\bigcup U_n}) > \frac{1}{2}$ for every n and $\bigcup I_n$ meets every interval of length 2^{-n} which $\bigcup U_n$ meets.

Corollary (Fremlin 91) There is no Tukey function from $\mathcal{E}$ to $\mathbb{N}^{\mathbb{N}}$.

Reference

Fremlin D.H. [91] 'The partially ordered sets of measure theory and Tukey's ordering,' Note di Matematica 11 (1991) 177-214.

Louveau A. & Veličković B. [99] 'Analytic ideals and cofinal types', Ann. Pure and Applied Logic 99 (1999) 171-195.