On a construction of Steprāns

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The construction Let T be an infinite tree of height ω, a single root 0, and a non-zero finite set S_τ of successors to each node τ. Suppose that λ_τ is a probability measure on S_τ for each $\tau \in T$. Let X be the set of all branches of T with its usual compact metrizable topology generated by $\{X_\tau : \tau \in T\}$ where $X_\tau = \{x : \tau \in x\}$ for $\tau \in T$. Let $\langle p_\tau\rangle_{\tau\in T}$ be a family in $[1,\infty[$. It will be helpful to have a word for the following: I will say that a function $f : T \to [0,\infty[$ is **consistent** if $f(\tau) = \bigl(\int_{S_\tau} f(\tau^\frown s)^{p_\tau}\lambda_\tau(ds)\bigr)^{1/p_\tau}$ for every $\tau \in T$. Note that the set of consistent functions is a closed set in $\mathbb{R}^T$; so, in particular, if Φ is any non-empty upwards-directed family of consistent functions and $g = \sup\Phi$ is finite at each point of T, then g is consistent, and the family of all consistent functions from T to $[0,1]$ is compact.

For each open-and-closed set $G \subseteq X$ define $\langle\nu_\tau G\rangle_{\tau\in T}$ by setting

$$\begin{aligned}\nu_\tau(G) &= 1 \text{ if } X_\tau \subseteq G\\ &= 0 \text{ if } X_\tau \cap G = \emptyset\\ &= \Bigl(\int_{S_\tau} \nu_{\tau^\frown s}(G)^{p_\tau}\,d\lambda_\tau(ds)\Bigr)^{1/p_\tau} \text{ otherwise;}\end{aligned}$$

it is easy to check (by downwards induction on $\operatorname{rank}(\tau)$) that this is well-defined for every τ. For open sets $G \subseteq X$ set $\nu_\tau(G) = \sup\{\nu_\tau(H) : H \subseteq G \text{ is open-and-closed}\}$. Observe that if $\mathcal{G}$ is an upwards-directed family of open subsets of X with union H, then $\nu_\tau(H) = \sup_{G\in\mathcal{G}}\nu_\tau(G)$ for every τ.

For every set $A \subseteq X$ and $\tau \in T$, set $\nu^*_\tau(A) = \inf\{\nu_\tau(G) : G \supseteq A \text{ is open}\}$.

Basic facts (a) If $G \subseteq X$ is open-and-closed, $\tau \mapsto \nu_\tau(G)$ is the unique consistent function such that $\nu_\tau(G)$ is 1 if $X_\tau \subseteq G$ and zero if $X_\tau \cap G = \emptyset$. If G and H are open-and-closed and $G \subseteq H$, then $\nu_\tau(G) \le \nu_\tau(H)$ for every τ. If G and H are open-and-closed, then $\nu_\tau(G \cup H) \le \nu_\tau(G) + \nu_\tau(H)$ for every τ. (Induce downwards on $\operatorname{rank}(\tau)$.)

(b) If If G is open-and-closed, the new formula $\inf\{\nu_\tau H : H \subseteq G \text{ is open-and-closed}\}$ yields $\nu_\tau(G)$ as previously defined. $\tau \mapsto \nu_\tau(G)$ is consistent for any open set $G \subseteq X$. If G and H are open and $G \subseteq H$ then $\nu_\tau(G) \le \nu_\tau(H)$ for every τ. If G and H are open, then $\nu_\tau(G \cup H) \le \nu_\tau(G) + \nu_\tau(H)$ for every τ.

(c) $\tau \mapsto \nu^*_\tau(A)$ is consistent for every $A \subseteq X$. If $A \subseteq B$ then $\nu^*_\tau(A) \le \nu^*_\tau(B)$ for every τ. If A is open then $\nu^*_\tau(A) = \nu_\tau(A)$ for every τ. If A, $B \subseteq X$, then $\nu^*_\tau(A\cup B) \le \nu^*_\tau(A)+\nu^*_\tau(B)$ for every τ.

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(d) For every $\epsilon > 0$ and $n \in \mathbb{N}$ there is a $\delta > 0$ such that whenever $f, g : T \to [0,1]$ are consistent functions and $f \leq g$ and $g(0) \leq f(0) + \delta$, then $g(\tau) \leq f(\tau) + \epsilon$ whenever $\operatorname{rank}(\tau) = n$. (Induce on n.)

Governing assumption From now on I will suppose that there is a summable sequence $\langle \gamma_n \rangle_{n \in \mathbb{N}}$ of strictly positive real numbers such that $p_\tau \geq \dfrac{1}{\gamma_n}$ whenever $\tau \in T$ and $\operatorname{rank}(\tau) = n$.

Lemma (a) Let (S, Σ, λ) be any probability space, and $p \in [1, \infty]$. If $u, v \in L^p(\lambda)$, then $\|u \vee v\|_p \leq 2^{1/p} \max(\|u\|_p, \|v\|_p)$.

(b) Set $\gamma_n' = \sum_{i=n}^{\infty} \gamma_i$ for each $n \in \mathbb{N}$. Then

$$\nu_\tau^*(A \cup B) \leq 2^{\gamma_n'} \max(\nu_\tau^*(A), \nu_\tau^*(B))$$

whenever $A, B \subseteq X$ and $\tau \in T$ and $\operatorname{rank}(\tau) = n$.

(c) For every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $A, B \subseteq X$ and $\nu_0^*(A) \leq \nu_0^*(A \cap B) + \delta$, then $\nu_0^*(A \cup B) \leq \nu_0^*(B) + \epsilon$.

proof (a) We can find u_1, v_1 such that $|u_1| \leq |u|$, $|v_1| \leq |v|$, $u_1 + v_1 = u \vee v$ and $|u_1| \wedge |v_1| = 0$. In this case, $|u \vee v|^p = |u_1|^p + |v_1|^p$ so

$$\|u \vee v\|_p^p = \|u_1\|_p^p + \|v_1\|_p^p \leq \|u\|_p^p + \|v\|_p^p \leq 2 \max(\|u\|_p^p, \|v\|_p^p)$$

and $\|u \vee v\|_p \leq 2^{1/p} \max(\|u\|_p, \|v\|_p)$.

(b)(i) Suppose first that A and B are both open-and-closed. Induce downwards on $\operatorname{rank}(\tau)$. For all sufficiently high nodes τ, $\nu_\tau(A \cup B) = \max(\nu_\tau(A), \nu_\tau(B))$. Now suppose that $\operatorname{rank}(\tau) = n$ and that we know that $\nu_{\tau^\frown s}^*(A \cup B) \leq 2^{\gamma_{n+1}'} \max(\nu_{\tau^\frown s}^*(A), \nu_{\tau^\frown s}^*(B))$ for every $s \in S_\tau$. Set $u(s) = \nu_{\tau^\frown s}(A)$, $v(s) = \nu_{\tau^\frown s}(B)$ and $w(s) = \nu_{\tau^\frown s}(A \cup B)$ for $s \in S_\tau$. Then $\nu_\tau(A) = \|u\|_p$, $\nu_\tau(B) = \|v\|_p$ and $\nu_\tau(A \cup B) = \|w\|_p$, where $p = p_\tau$ and the norms are calculated in $\mathcal{L}^p(\lambda_\tau) \equiv L^p(\lambda_\tau)$. So

$$\nu_\tau(A \cup B) = \|w\|_p \leq 2^{\gamma_{n+1}'} \|u \vee v\|_p \leq 2^{\gamma_{n+1}'} 2^{1/p} \max(\|u\|_p, \|v\|_p)$$

(by (a))

$$\leq 2^{\gamma_{n+1}'} 2^{\gamma_n} \max(\nu_\tau(A), \nu_\tau(B)) = 2^{\gamma_n'} \max(\nu_\tau(A), \nu_\tau(B)),$$

and the induction continues.

(ii) If A and B are open, let $\langle G_k \rangle_{k \in \mathbb{N}}$, $\langle H_k \rangle_{k \in \mathbb{N}}$ be non-decreasing sequences of open sets with unions A, B respectively; then

$$\begin{aligned}\nu_\tau(A \cup B) &= \sup_{k \in \mathbb{N}} \nu_\tau(G_k \cup H_k) \\ &\leq 2^{\gamma_n'} \sup_{k \in \mathbb{N}} \max(\nu_\tau(G_k), \nu_\tau(H_k)) \leq 2^{\gamma_n'} \max(\nu_\tau(A), \nu_\tau(B))\end{aligned}$$

whenever $\tau \in X$ and $\operatorname{rank}(\tau) = n$.

(iii) In general, given $\epsilon > 0$ and $\tau \in T$ and $\operatorname{rank}(\tau) = n$, there are open sets $G \supseteq A$ and $H \supseteq B$ such that $\nu_\tau(G) \le \nu^*_\tau(A) + \epsilon$ and $\nu_\tau(H) \le \nu^*_\tau(B) + \epsilon$, so that

$$\begin{aligned}\nu^*_\tau(A \cup B) &\le \nu_\tau(G \cup H)\\ &\le 2^{\gamma'_n} \max(\nu_\tau(G), \nu_\tau(H)) \le 2^{\gamma'_n}(\epsilon + \max(\nu^*_\tau(A), \nu^*_\tau(B));\end{aligned}$$

as ϵ is arbitrary, we have the result.

(c) Let $n \in \mathbb{N}$ be such that $2^{\gamma'_n} \le 1 + \frac{1}{2}\epsilon$. Let $\delta > 0$ be such that whenever f, $g : T \to [0,1]$ are consistent functions and $f \le g$ and $g(0) \le f(0) + \delta$ then $g(\tau) \le f(\tau) + \frac{1}{2}\epsilon$ whenever $\operatorname{rank}(\tau) = n$. Now suppose that $\nu^*_0(A) \le \nu^*_0(A \cap B) + \delta$. Then $\nu^*_\tau(A) \le \nu^*_\tau(A \cap B) + \frac{1}{2}\epsilon \le \nu^*_\tau(B) + \frac{1}{2}\epsilon$ whenever $\operatorname{rank}(\tau) = n$. So

$$\nu^*_\tau(A \cup B) \le 2^{\gamma'_n} \max(\nu^*_\tau(A), \nu^*_\tau(B)) \le \max(\nu^*_\tau(A), \nu^*_\tau(B)) + \frac{1}{2}\epsilon \le \nu^*_\tau(B)) + \epsilon$$

whenever $\operatorname{rank}(\tau) = n$. Now a downwards induction on $\operatorname{rank}(\tau)$ shows that $\nu^*_\tau(A \cup B) \le \nu^*_\tau(B) + \epsilon$ whenever $\operatorname{rank}(\tau) \le n$, and in particular when $n = 0$.

Theorem $\nu^*_0 : \mathcal{P}X \to [0,1]$ is a Choquet capacity.

proof (For the definition of 'Choquet capacity', see Fremlin 03, 432I.)

(a) We know already that $\nu^*_0(A) \le \nu^*(B)$ whenever $A \subseteq B$.

(b) Let $\langle A_n \rangle_{n \in \mathbb{N}}$ be a non-decreasing sequence of subsets of X with union A. Set $\alpha = \sup_{n \in \mathbb{N}} \nu^*_0(A_n) = \lim_{n \to \infty} \nu^*_0(A_n)$. Then $\alpha \le \nu^*_0(A)$. Let $\epsilon > 0$. For each $n \in \mathbb{N}$, let $\delta_n \in]0, 2^{-n}]$ be such that $\nu^*_0(B \cup C) \le \nu^*_0(B) + 2^{-n-1}\epsilon$ whenever B, $C \subseteq X$ and $\nu^*_0(C) \le \nu^*_0(B \cap C) + \delta_n$. Let $G_n \supseteq A_n$ be an open set such that $\nu_0(G_n) \le \nu^*_0(A_n) + \delta_n$.

Set $H_{mn} = \bigcup_{m \le i \le n} G_i$ for $m \le n$. Then $\nu_0 H_{mn} \le \nu_0 G_n + 2^{-m}\epsilon$ whenever $m \le n$. **P** Induce downwards on m. If $m = n$ the result is trivial. For the inductive step to $m < n$, we have $H_{mn} = G_m \cup H_{m+1,n}$ and $A_m \subseteq G_m \cap H_{m+1,n}$. Now

$$\nu_0(G_m) \le \nu^*_0(A_m) + \delta_m \le \nu_0(G_m \cap H_{m+1,n}) + \delta_m,$$

so

$$\nu_0(H_{mn}) \le \nu_0(G_m \cup H_{m+1,n}) \le \nu_0(H_{m+1,n}) + 2^{-m-1}\epsilon$$

(by the choice of δ_m)

$$\le \nu_0(G_n) + 2^{-m-1}\epsilon + 2^{-m-1}\epsilon$$

(by the inductive hypothesis)

$$= \nu_0(G_n) + 2^{-m}\epsilon$$

and the induction proceeds. **Q**

In particular,

$$\nu_0(\bigcup_{i \le n} G_i) = \nu_0(H_{0n}) \le \nu_0(G_n) + \epsilon \le \nu^*_0(A_n) + 2^{-n} + \epsilon \le \alpha + 2^{-n} + \epsilon$$

for every $n \in \mathbb{N}$. Setting $G = \bigcup_{n\in\mathbb{N}} G_n$,

$$\nu_0^*(A) \le \nu_0(G) = \lim_{n\to\infty} \nu_0(\bigcup_{i\le n} G_i) \le \alpha + \epsilon.$$

As ϵ is arbitrary, $\nu_0^*(A) = \alpha = \sup_{n\in\mathbb{N}} \nu_0^*(A_n)$.

(c) If $K \subseteq X$ is compact, then

$$\nu_0^* K = \inf\{\nu_0(G) : G \supseteq K \text{ is open}\} = \inf\{\nu_0^*(G) : G \supseteq K \text{ is open}\}.$$

Definition Set $\nu E = \nu_0^*(E)$ for every Borel set $E \subseteq X$.

Theorem (i) ν is a submeasure.
(ii) $\nu E = \sup\{\nu K : K \subseteq E \text{ is compact}\} = \inf\{\nu G : G \supseteq E \text{ is open}\}$ for every Borel set E.
(iii) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of Borel sets with union E, $\nu E = \lim_{n\to\infty} \nu E_n$. ∎

proof The only non-trivial bit now is the first equality of (ii), and this is Choquet's theorem, 432J in Fremlin 03.

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References

Fremlin D.H. [03] *Measure Theory*, vol. 4. To be published by Torres Fremlin. *Drafts available from* `http://www.essex.ac.uk/maths/staff/fremlin/mtcont.htm`.