Freese-Nation numbers of finite partially ordered sets

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1. Basic theory

1A Definitions Let P be a partially ordered set. A function $f : P \to \mathcal{P}P$ is a **Freese-Nation function** if $[p,q] \cap f(p) \cap f(q) \neq 0$ whenever $p \leq q$ in P. The **Freese-Nation number** $\mathrm{FN}(P)$ of P is the smallest cardinal κ such that there is a Freese-Nation function from P to $[P]^{<\kappa}$.

Note that $p \in f(p)$ for any Freese-Nation function f on P and any $p \in P$. P has the 'Freese-Nation property' if $\mathrm{FN}(P) \leq \omega$, and the 'weak Freese-Nation property' if $\mathrm{FN}(P) \leq \omega_1$. In the language of Fuchino & Soukup 97, $\mathrm{FN}(P) \leq \kappa$ if P has the 'κ-Freese-Nation property'.

Obviously $\mathrm{FN}(P) \leq 1 + \#(P)$; and we can do better than this, see 2D.

1B Proposition Let P be a partially ordered set and $f : P \to \mathcal{P}P$ a Freese-Nation function.

(a) If $Q \subseteq P$ is order-convex, then $q \mapsto Q \cap f(q)$ is a Freese-Nation function on Q.

(b) If $Q \subseteq P$ and $h : P \to Q$ is a retraction (that is, an order-preserving mapping such that $h(q) = q$ for $q \in Q$), then $q \mapsto h[f(q)]$ is a Freese-Nation function on Q.

1C Proposition Let P be a partially ordered set and Q a subset of P. Suppose that

either (i) Q is order-convex

or (ii) Q is a retract of P

or (iii) Q is, in its induced ordering, a Dedekind complete lattice.

Then $\mathrm{FN}(Q) \leq \mathrm{FN}(P)$.

proof (i) and (ii) are immediate from 1B. Now (iii) follows. **P** Set $P_1 = \bigcup_{q,q' \in \mathbb{Q}}[q,q']$. Then P_1 is an order-convex subset of P, so $\mathrm{FN}(P_1) \leq \mathrm{FN}(P)$. Now we can define $h : P_1 \to Q$ by setting $h(p) = \inf\{q : q \in Q,\, p \leq q\}$ for $p \in P_1$, the infimum being taken in Q. Then h is a retraction, so $\mathrm{FN}(Q) \leq \mathrm{FN}(P_1) \leq \mathrm{FN}(P)$. **Q**

1D Example Consider a partially ordered set $P = A \cup \{p^*\} \cup B$ where A and B are disjoint from each other and from p^*, and for p, $q \in P$ we have $p \leq q$ if

either $p = q$

or $p \in A \cup \{p^*\}$ and $q \in \{p^*\} \cup B$.

Then $\mathrm{FN}(P) \leq 3$, because $p \mapsto \{p, p^*\}$ is a Freese-Nation function. But there is no bound on $\mathrm{FN}(P \setminus \{p^*\})$.

Thus we can have $Q \subseteq P$ but $\mathrm{FN}(Q) > \mathrm{FN}(P)$.

1E Proposition Let P and Q be finite partially ordered sets and $h : P \to Q$ an order-preserving surjection.

(a) Suppose that whenever $q \leq q'$ in Q there are $p \leq p'$ in P such that $h(p) = q$ and $h(p') = q'$. (This is true if, for instance, P and Q are lattices and h is a lattice homomorphism.) Let κ be an infinite cardinal such that $\mathrm{FN}(P) \leq \kappa$ and $\#(\{p : h(p) = q\}) < \mathrm{cf}\,\kappa$ for every $q \in Q$. Then $\mathrm{FN}(Q) \leq 1 + (\mathrm{FN}(P) - 1)\max_{q \in P} \#(h^{-1}[\{q\}]$.

(b) Suppose that whenever $p \leq p'$ in P and $h(p) \leq q \leq h(p')$ there is a $p'' \in [p, p']$ such that $h(p'') = q$. (Again, this is true if, for instance, P and Q are lattices and h is a lattice homomorphism.) Let κ be an infinite cardinal such that $\mathrm{FN}(Q) \leq \mathrm{cf}\,\kappa$ and $\#(\{p : h(p) = q\}) < \kappa$ for every $q \in Q$. Then $\mathrm{FN}(P) \leq 1 + 1 + (\mathrm{FN}(Q) - 1)\max_{q \in P} \#(h^{-1}[\{q\}]$.

proof (a) Let $f : P \to [P]^{<\mathrm{FN}(P)}$ be a Freese-Nation function. Set

$$g(q) = h[\bigcup\{f(p) : h(p) = q\}]$$

for each $q \in Q$; then $\#(g(q)) \leq (\mathrm{FN}(P) - 1)\max_{q \in P} \#(h^{-1}[\{q\}]$ for every q. Now g is a Freese-Nation function on Q. **P** If $q \leq q'$ in Q, let p, $p' \in P$ be such that $p \leq p'$ and $h(p) = q$ and $h(p') = q'$. Then there is an $r \in [p, p'] \cap f(p) \cap f(p')$, so that $h(r) \in [q, q'] \cap g(q) \cap g(q')$. **Q** So we have the result.

(b) This time, let $g : Q \to [Q]^{<\mathrm{FN}(Q)}$ be a Freese-Nation function, and set

$$f(p) = h^{-1}[g(h(p))]$$

for $p \in P$. Then $f : P \to \mathcal{P}P$ is a Freese-Nation function. **P** If $p \le p'$ in P, take $q \in [f(p), f(p')] \cap g(f(p)) \cap g(f(p'))$. Then there is a $p'' \in [p, p']$ such that $h(p'') = q$, and $p'' \in [p, p'] \cap f(p) \cap f(p')$. **Q** So again we have the result.

1F Definition Let P be a partially ordered set and Q any subset of P. The **Freese-Nation index** of Q in P is the least cardinal κ such that $\mathrm{cf}(Q \cap \,]-\infty, p]) < \kappa$ and $\mathrm{ci}(Q \cap [p, \infty[) < \kappa$ for every $p \in P$.

2A Remark Note that if we have a partially ordered set P such that $\#(P) = m$ and $\mathrm{FN}(P) = n$, then we can describe P entirely by two $m \times (n-2)$ arrays, coding $\{(p, r) : p < r, r \in f(p)\}$ and $\{(p, r) : r < p, r \in f(p)\}$; and that for any $p, q \in P$ we shall then be able to determine whether $p \le q$ by checking at most $(n-2)^2$ points in our arrays.

2B Theorem If $\mathrm{FN}(P) = n \in \mathbb{N} \setminus \{0\}$ then P has no totally ordered subset with 2^{n-1} members.

proof Induce on n. Let $f : P \to [P]^{<n}$ be a Freese-Nation function. If $n = 1$ then $f(p) = \emptyset$ for every $p \in P$ so P itself is empty and has no totally ordered set with one member. If $n = 2$ then $f(p) = \{p\}$ for every $p \in P$ so there are no p, q with $p < q$ and no totally ordered set in P has two members. Thus the induction starts. For the inductive step to $n + 1 \ge 3$, **?** suppose, if possible, that $\mathrm{FN}(P) = n + 1$ and that P has a totally ordered subset Q with 2^n members. Then we can express Q as $Q_0 \cup Q_1$ where $\#(Q_0) = \#(Q_1) = 2^{n-1}$ and $p \le q$ for every $p \in Q_0$, $q \in Q_1$. Let $f : P \to [P]^{\le n}$ be a Freese-Nation function. Let p_1 be the greatest member of Q_0 and q_0 the smallest member of Q_1. Set $f_0(p) = f(p) \cap \,]-\infty, p_1]$ for $p \in \,]-\infty, p_1]$; then f_0 is a Freese-Nation function on $]-\infty, p_1]$. Because $]-\infty, p_1]$ has a chain Q_0 with 2^{n-1} members, the inductive hypothesis tells us that there is an $r_0 \in \,]-\infty, p_1]$ such that $\#(f_0(r_0)) \ge n$, in which case $f_0(r_0)$ must be equal to $f(r_0)$, that is, $f(r_0) \subseteq \,]-\infty, p_1]$. Similarly, there is an $r_1 \in [q_0, \infty[$ such that $f(r_1) \subseteq [q_0, \infty[$. But this means that $r_0 \le p_1 \le q_0 \le r_1$ and $f(r_0) \cap f(r_1) = \emptyset$, which is impossible. **X**

So the induction proceeds.

Remark What the proof here is saying is that if $p < q$ in P then

$$\mathrm{FN}(P) > \min(\mathrm{FN}(]-\infty, p]), \mathrm{FN}([q, \infty[))$$

(if the right-hand side is finite).

2C Theorem If P is a finite totally ordered set and $2^n \le \#(P) < 2^{n+1}$ then $\mathrm{FN}(P) = n + 2$.

proof By Proposition 2, if $\mathrm{FN}(P) \le n + 1$ then $\#(P) < 2^n$; so if $2^n \le \#(P)$ then $\mathrm{FN}(P) \ge n + 2$.

Induce on n. If $1 \le \#(P) < 2$ then $\mathrm{FN}(P) = 2$, so the induction starts with $n = 0$. For the inductive step to $n \ge 1$, if $2^n \le \#(P) < 2^{n+1}$, let $p^* \in P$ be such that P_0 and P_1 both have fewer than 2^n members, where $P_0 = \,]-\infty, p^*[$ and $P_1 = \,]p^*, \infty[$. By the inductive hypothesis, there are Freese-Nation functions $f_0 : P_0 \to [P_0]^{\le n}$ and $f_1 : P_1 \to [P_1]^{\le n}$. Set

$$\begin{aligned} f(p) &= \{p^*\} \text{ if } p = p^*, \\ &= f_0(p) \cup \{p^*\} \text{ if } p \in P_0, \\ &= f_1(p) \cup \{p^*\} \text{ if } p \in P_0; \end{aligned}$$

then $f : P \to [P]^{\le n+1}$ is a Freese-Nation function and $\mathrm{FN}(P) \le n + 2$, so the induction continues.

Remark For a generalization of this idea see **Penman n01**, 2.2.

2D Proposition If P is finite, then $\mathrm{FN}(P) \le 2 + \frac{1}{2}\#(P)$.

proof Enumerate P as $\{p_i : i < n\}$ and set

$$f(p_i) = \{p_{i+2k} : k \in \mathbb{N},\ 2k < n - i\} \cup \{p_{i-2k-1} : k \in \mathbb{N},\ 2k < i + 2\}$$

for each $i < n$. Then f is a Freese-Nation function (in fact, $f(p) \cap f(q) \cap \{p, q\} \ne \emptyset$ for all $p, q \in P$) and

$$\#(f(p_i)) \le \tfrac{1}{2}(n - i) + \tfrac{1}{2}(i + 2) \le 1 + \tfrac{1}{2}n$$

for each i, so $\mathrm{FN}(P) \leq 2 + \frac{1}{2}n$.

Remark If $\#(P) = 0$, $\mathrm{FN}(P) = 0$ (the empty function is a Freese-Nation function, and can be interpreted as a function from P to $[P]^{<0} = \{Q : Q \subseteq P,\ \#(Q) < 0\}$); if $\#(P) = 1$, $\mathrm{FN}(P) = 2$; if $\#(P) = 2$, $\mathrm{FN}(P)$ is either 2 or 3.

2E Proposition For any $n \geq 4$, there is a partially ordered set P of size $4n - 11$ such that $\mathrm{FN}(P) = n$.

proof Let P_0, P_1 be disjoint sets of size $2n - 5$ and $2n - 6$ respectively, and set $P = P_0 \cup P_1$, with the partial order defined by saying that $p \leq q$ iff either $p = q$ or $p \in P_0$ and $q \in P_1$. **?** Suppose, if possible, that $\mathrm{FN}(P) \leq n - 1$. Then we have a Freese-Nation function $f : P \to [P]^{\leq n-2}$. For each $p \in P_0$ there are at most $n - 3$ points in $P_1 \cap f(p)$, and for each $p \in P_1$ there are at most $n - 3$ points in $P_0 \cap f(p)$. So there are at most $(4n - 11)(n - 3) = 42n - 1 - 23n + 33$ pairs (p, q) such that $p \in P_0$, $q \in P_1$ and either $p \in f(q)$ or $q \in f(p)$. But there are $(2n - 5)(2n - 6) = 42n - 1 - 22n + 30$ pairs (p, q) where $p \in P_0$ and $q \in P_1$, so for one of these we have $[p, q] \cap f(p) \cap f(q) = \emptyset$. **X**

To see that $\mathrm{FN}(P)$ is exactly n, enumerate P_0 as $\langle p_i \rangle_{i<2n-5}$ and P_1 as $\langle q_i \rangle_{i<2n-6}$. Set $f(p_i) = \{p_i\} \cup \{q_{i+_{2n-6}j} : j < n - 2\}$ for each $i < 2n - 5$, where $+_{2n-6}$ is addition mod $2n - 6$. Then for each $j < 2n - 6$, $\{i : q_j \in f(p_i)\}$ has at least $n - 3$ members, so if we set $f(q_j) = \{q_j\} \cup \{p_i : i < 2n - 5,\ q_j \notin f(p_j)\}$ we get a Freese-Nation function $f : P \to [P]^{<n}$.

2F Two examples (a) If $P = \mathcal{P}\{0, 1\}$ then $\#(P) = 4$, $\mathrm{FN}(P) = 4$.

(b) $\mathrm{FN}(\mathcal{P}3) = 5$.([1]) **P** The delicate part is to show that $\mathrm{FN}(\mathcal{P}3) > 4$. Let $g : \mathcal{P}3 \to [\mathcal{P}3]^{\leq 2}$ be a function such that $p \notin g(p)$ for every p; we have to show that $p \mapsto g(p) \cup \{p\}$ is not a Freese-Nation function. For $p \subset q \subseteq 3$ let $C(p, q)$ be the statement 'either $p \in g(q)$ or $q \in g(p)$ or $]p, q[\cap g(p) \cap g(q) \neq \emptyset$'; we wish to show that there is at least one pair $p \subset q$ for which $C(p, q)$ is false.

Note first that we can suppose that $g(\emptyset) \cap g(\{0, 1, 2\}) \neq \emptyset$. For we can certainly suppose that $\hat{q} \in g(\emptyset)$ for some $\hat{q} \neq \{0, 1, 2\}$ and that $\hat{p} \in g(\{0, 1, 2\})$ for some $\hat{p} \neq \emptyset$. So if, for instance, we had $g(\emptyset) = \{\hat{q}, \{0, 1, 2\}\}$ and/or $g(\{0, 1, 2\}) = \{\emptyset, \hat{p}\}$, we could replace it by $\{\hat{p}, \hat{q}\}$ and $C(\emptyset, \{0, 1, 2\})$ would be true, while no other $C(p, q)$ previously witnessed would be rendered false. By symmetry, it is enough to consider the case $\{0\} \in g(\emptyset) \cap g(\{0, 1, 2\})$.

At this point we seem to need to examine cases.

case 1 $g(\{0, 1, 2\}) = \{\{0\}, \{1, 2\}\}$. If $C(\{0, 2\}, \{0, 1, 2\})$ and $C(\{0, 1\}, \{0, 1, 2\})$ are to be true, we must have $\{0, 1, 2\} \in g(\{0, 2\})$ and $\{0, 1, 2\} \in g(\{0, 1\})$. Now $\{0\}$ and $\{1\}$ cannot both belong to $g(\emptyset)$; there is still enough symmetry so that we need consider only the case $\{1\} \notin g(\emptyset)$, in which case $\emptyset \in g(\{1\})$. Consider $C(\{1\}, \{0, 1, 2\})$. If this is to be true, one of $\{1, 2\}$, $\{0, 1, 2\}$ must belong to $g(\{1\})$; in particular, $\{0, 1\} \notin g(\{1\})$; if $C(\{1\}, \{0, 1\})$ is to be true, we must have $\{1\} \in g(\{0, 1\})$, so that $g(\{0, 1\}) = \{\{1\}, \{0, 1, 2\}\}$. At this point we see that, since we are supposing that $\{1\} \notin g(\emptyset)$, the only way to witness $C(\emptyset, \{0, 1\})$ is to put $\{0, 1\}$ into $g(\emptyset)$. But this means that $g(\emptyset) = \{\{0\}, \{0, 1\}\}$ and $\{2\} \notin g(\emptyset)$. Repeating the argument of the last two sentences, we find that the only way to witness $C(\emptyset, \{0, 2\})$ will be to put $\{0, 2\}$ into $g(\emptyset)$, and there isn't room. So this route fails.

case 2 $g(\{0, 1, 2\}) = \{\{0\}, \{0, 1\}\}$. In this case, $\{0, 1, 2\} \in g(\{1, 2\}) \cap g(\{0, 2\})$. $C(\{2\}, \{0, 1, 2\})$ can be true only if $\{0, 1, 2\} \in g(\{2\})$. Once again, at most one of $\{1\}$, $\{2\}$ can belong to $g(\emptyset)$; this time we look at them separately.

case 2a $\{2\} \notin g(\emptyset)$, so $\emptyset \in g(\{2\})$ and $g(\{2\}) = \{\emptyset, \{0, 1, 2\}\}$. To witness $C(\{2\}, \{1, 2\})$ and $C(\{2\}, \{0, 2\})$ we must have $\{2\} \in g(\{1, 2\}) \cap g(\{0, 2\})$; but now there is no way to achieve both $C(\emptyset, \{1, 2\})$ and $C(\emptyset, \{0, 2\})$ with the one remaining free place in $g(\emptyset)$.

case 2b $\{1\} \notin g(\emptyset)$, so $\emptyset \in g(\{1\})$. If $C(\{1\}, \{0, 1, 2\})$ is true, one of $\{0, 1, 2\}$, $\{0, 1\}$ belongs to $g(\{1\})$ so $\{1, 2\} \notin g(\{1\})$; if $C(\{1\}, \{1, 2\})$ is true, $\{1\} \in g(\{1, 2\})$. Now $C(\emptyset, \{1, 2\})$ can be true only if $\{1, 2\} \in g(\emptyset)$, in which case $\{2\} \notin g(\emptyset)$ and we are returned to case 2a.

$g(\{0, 1, 2\}) = \{\{0\}, \{0, 2\}\}$ is similar.

case 3 $g(\{0, 1, 2\}) = \{\{0\}, \{1\}\}$. In this case, $\{0, 1, 2\}$ belongs to all of $g(\{1, 2\})$, $g(\{0, 2\})$, $g(\{0, 1\})$ and $g(\{2\})$.

case 3a $\{2\} \notin g(\emptyset)$, $g(\{2\}) = \{\emptyset, \{0, 1, 2\}\}$. Then $\{2\} \in g(\{1, 2\}) \cap g(\{0, 2\})$ to witness $C(\{2\}, \{1, 2\})$ and $C(\{2\}, \{0, 2\})$; and we have no way to witness both $C(\emptyset, \{1, 2\})$ and $C(\emptyset, \{0, 2\})$.

[1] We are grateful to G.Brightwell for noticing an error in the 2001 version of this note.

case 3b $\{2\} \in g(\emptyset)$, $g(\{2\}) = \{\emptyset, \{0,1,2\}\}$, $\emptyset \in g(\{1\})$. To witness $C(\emptyset, \{0,1\})$, one of $\emptyset$, $\{0\}$ must belong to $g(\{0,1\})$; to witness $C(\emptyset, \{0,2\})$, one of $\emptyset$, $\{2\}$ must belong to $g(\{0,2\})$; but this leaves no way to witness both $C(\{1\}, \{1,2\})$ and $C(\{1\}, \{0,1\})$.

$g(\{0,1,2\}) = \{\{0\}, \{2\}\}$ is similar.

Thus there is no such function g, and $\mathrm{FN}(\mathcal{P}3) > 4$. The analysis above shows that allowing $\#(g(p)) = 3$ for almost any p will break the blockage, so $\mathrm{FN}(\mathcal{P}3)$ is 5 exactly. **Q**

2G Remarks From 2C we see that the Freese-Nation number of a partially ordered set can be large compared with the degree of its covering graph (Penman n01). In 2E, the constraining factor is the size of the covering graph; see Penman n01, 2.4. From 8D below we see that the dimension can be large compared with the Freese-Nation number. By the example in 2E (which has dimension 2 and height 2) we see that the Freese-Nation number can be large compared with the dimension and the height.

2H Proposition (G.Brightwell, private communication) For any finite partially ordered set P, $\mathrm{FN}(P) \leq \mathrm{height}(P) + \mathrm{width}(P)$.

proof Set $m = \mathrm{width}(P)$, and express P as $\bigcup_{i<m} P_i$ where each P_i is totally ordered. For $p \in P$ choose $k(p) < m$ such that $p \in P_{k(p)}$ and set $J = \{i : i < m,\ i \neq k(p),\ \exists\, q \in P_i,\ p < q\}$. Set

$$f(p) = (P_{k(p)} \cap \,]-\infty, p]) \cup \{\min(P_i \cap \,]p, \infty[: i \in J\}.$$

Then $\#(f(p)) \leq \#(P_{k(p)}) + \#(J) < \mathrm{height}(P) + \mathrm{width}(P)$.

2K Proposition If P and Q are partially ordered sets and $\mathrm{FN}(P)$, $\mathrm{FN}(Q)$ are finite and not zero, then $\mathrm{FN}(P \times Q) - 1 \leq (\mathrm{FN}(P) - 1)(\mathrm{FN}(Q) - 1)$.

proof Let $f : P \to [P]^{<\mathrm{FN}(P)}$ and $g : Q \to [Q]^{<\mathrm{FN}(Q)}$ be Freese-Nation functions. Then $(p,q) \mapsto f(p) \times g(q)$ is a Freese-Nation function on $P \times Q$.

2L Corollary For $n \in \mathbb{N}$, set $\phi(n) = \mathrm{FN}(\mathcal{P}n) - 1$. Then $\phi(m+n) \leq \phi(m)\phi(n)$, so

$$\lim_{n\to\infty} \mathrm{FN}(\mathcal{P}n)^{1/n} = \lim_{n\to\infty} \phi(n)^{1/n} = \inf_{n\geq 1} \phi(n)^{1/n}.$$

proof $\mathcal{P}(m+n) \cong \mathcal{P}m \times \mathcal{P}n$.

2M Theorem $(\frac{2}{\sqrt{3}})^n + 1 \leq \mathrm{FN}(\mathcal{P}n) \leq 4^{(n+1)/3} + 1$ for every $n \in \mathbb{N}$.

proof (a) Take any $r \geq 1$ and consider $n = 6r$. Set $m_r = \mathrm{FN}(\mathcal{P}n) - 1$ and let $f : \mathcal{P}n \to [\mathcal{P}n]^{\leq m_r}$ be a Freese-Nation function. Set $P_0 = [n]^{2r}$ and $P_1 = [n]^{4r}$, so that $\#(P_0) = \#(P_1) = \frac{(6r)!}{(2r)!(4r)!}$ and $\#(\{(a,b) : a \in P_0,\ b \in P_1,\ a \subseteq b\}) = \frac{(4r)!}{(2r)!(2r)!}\#(P_0)$. Now if $a \in P_0$ and $b \in P_1$ and $a \subseteq b$, there is a $c \in f(a) \cap f(b)$ such that $a \subseteq c \subseteq b$, and one of $c \setminus a$, $b \setminus c$ has at most r members. But, for $0 \leq k \leq r$,

$$\{(a,c,b) : a \in P_0,\ b \in P_1,\ a \subseteq c \subseteq b,\ c \in f(a),\ \#(b \setminus c) = k\}$$

has at most

$$\#(P_0) \cdot m_r \cdot \frac{(2r+k)!}{k!(2r)!} \leq m_r \#(P_0) \cdot \frac{(3r)!}{r!(2r)!}$$

members, and similarly

$$\{(a,c,b) : a \in P_0,\ b \in P_1,\ a \subseteq c \subseteq b,\ c \in f(b),\ \#(c \setminus a) = k\}$$

has at most $m_r \#(P_0) \cdot \frac{(3r)!}{r!(2r)!}$ members. So we get $\frac{(4r)!}{(2r)!(2r)!} \leq (2r+1) m_r \frac{(3r)!}{r!(2r)!}$.

Now by Stirling's formula, or otherwise, $\lim_{r\to\infty} (\frac{(4r)!}{(2r)!(2r)!})^{1/r} = 16$ and $\lim_{r\to\infty} (\frac{(3r)!}{r!(2r)!})^{1/r} = \frac{27}{4}$. So $\lim_{r\to\infty} m_r^{1/r} \geq \frac{64}{27}$ and

$$\lim_{n\to\infty} \mathrm{FN}(\mathcal{P}n)^{1/n} = \lim_{r\to\infty} m_r^{1/6r} \geq \frac{2}{\sqrt{3}}.$$

So $\mathrm{FN}(\mathcal{P}n) \ge (\frac{2}{\sqrt{3}})^n + 1$ for every n.

(b) On the other side, we know that, setting $\phi(n) = \mathrm{FN}(\mathcal{P}n) - 1$, $\phi(0) = 1$, $\phi(1) = 2$, $\phi(2) = 3$ and $\phi(3) = 4$. Since ϕ is submultiplicative, $\phi(3r) \le 4^r$, $\phi(3r+1) \le 2 \cdot 4^r$ and $\phi(3r+2) \le 3 \cdot 4^r$ for every r. Accordingly $\phi(n) \le 4^{(n+1)/3}$ and $\mathrm{FN}(\mathcal{P}n) \le 4^{(n+1)/3} + 1$ for every n.

8. Freese-Nation numbers and dimension

8A Lemma If X is any set, $I \subseteq \mathcal{P}X$ and $\#(I) = n \ge 1$ there is an $a \subseteq X$ such that $\#(a) \le 2n - 1$ and $a \cap b \ne a \cap c$ for all distinct b, $c \in I$.

proof Induce on n.

Remark Take $X = 2n - 1$, $I = \{\{i\} : i < n\}$, $J = \{\emptyset\} \cup \{\{i, i+n\} : i < n-1\}$. Then there is no proper subset a of X such that $a \cap b \ne a \cap c$ for all $b \in I$, $c \in J$.

8B Proposition Let P be a finite partially ordered set and suppose that $\#(P) \le 2^k$ and $\mathrm{FN}(P) \le n+1$, where $n \ge 1$ and $k \ge 2n - 1$. Then

$$\dim P \le \frac{k!(2^{2n-1})!}{(2n-1)!(k-2n+1)!(n!)^2(2^{2n-1}-2n)!}.$$

proof (Compare FREMLIN N01, 8A, and NATION & SCHMERL 90.) Let $\phi : P \to \mathcal{P}k$ be an injective function and $f : P \to [P]^{\le n}$ a Freese-Nation function. Set

$$X = \{(a, I, J) : a \in [k]^{2n-1},\ I,\ J \in [\mathcal{P}a]^n,\ I \cap J = \emptyset\}.$$

Then

$$\begin{aligned}
\#(X) &\le \#([k]^{2n-1})\#([2^{2n-1}]^n)\#([2^{2n-1} - n]^n) \\
&= \frac{k!}{(2n-1)!(k-2n+1)!}\frac{(2^{2n-1})!}{n!(2^{2n-1}-n)!}\frac{(2^{2n-1}-n)!}{n!(2^{2n-1}-2n)!} \\
&= \frac{k!(2^{2n-1})!}{(2n-1)!(k-2n+1)!(n!)^2(2^{2n-1}-2n)!}.
\end{aligned}$$

For $x = (a, I, J) \in X$, set

$$A_x = \{p : \phi(q) \cap a \in I \text{ for every } q \in f(p) \cap [p, \infty[\},$$

$$B_x = \{p : \phi(q) \cap a \in J \text{ for every } q \in f(p) \cap]-\infty, p]\}.$$

If $x = (a, I, J) \in X$, then $p \not\le q$ for any $p \in A_x$ and $q \in B_x$, so there is a total order $\preccurlyeq_x$ on P, extending $\le$, such that $q \preccurlyeq_x p$ whenever $p \in A_x$ and $q \in B_x$.

If p, $q \in P$ and $p \not\le q$, then $I_0 = \{\phi(p') : p' \in f(p) \cap [p, \infty[\}$ and $J_0 = \{\phi(q') : q' \in f(q) \cap]-\infty, q]\}$ are disjoint subsets of $\mathcal{P}k$ both of size at most n, so that there must be an $a \in [k]^{2n-1}$ such that $b \cap a \ne c \cap a$ whenever $b \in I_0$ and $c \in J_0$. Next, $I_1 = \{b \cap a : b \in I_0\}$ and $J_1 = \{c \cap a : c \in J_0\}$ are disjoint subsets of $\mathcal{P}a$ both of size at most n, so there must be an $x = (a, I, J) \in X$ such that $I_1 \subseteq I$ and $J_1 \subseteq J$. In this case, $p \in A_x$ and $q \in B_x$ so $q \preccurlyeq_x p$ and $p \not\preccurlyeq_x q$.

So $\langle \preccurlyeq_x \rangle_{x \in X}$ witnesses that $\dim P \le \#(X)$.

8C Corollary For partially ordered sets P such that $\mathrm{FN}(P) \le n$, for fixed $n \ge 3$, $\dim P = O(\ln \#(P))^{2n-3}$ as $\#(P) \to \infty$.

Problems (a) For finite $n \ge 2$, what is the smallest partially ordered set P such that $\mathrm{FN}(P) \ge n$?

By Propositions 2D and 2E, $2n - 4 \le \#(P) \le 4n - 11$ if $n \ge 4$.

(b) For $n \in \mathbb{N}$, what is the exact Freese-Nation number of the partially ordered set $\mathcal{P}n$? (See 2K-2L.) What is the exact value of $\lim_{n\to\infty} \mathrm{FN}(\mathcal{P}n)^{1/n}$? By 2L it is between $\frac{2}{\sqrt{3}} \simeq 1{\cdot}15$ and $3^{1/3} \simeq 1{\cdot}45$.

(c) Let us say that a partially ordered set P **has simple levels** if we have a totally ordered set Q and a function $h : P \to Q$ such that, for p, $p' \in P$, $p \le p'$ iff either $p = p'$ or $h(p) < h(p')$. Now, suppose that P

is a finite partially ordered set. Is there necessarily a partially ordered set P', with simple levels, such that $\#(P') \le \#(P)$ and $\mathrm{FN}(P') \ge \mathrm{FN}(P)$? That is, in (a) above, is it enough to examine partially ordered sets with simple levels?

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