Version of 14.5.01

On a question of Natkaniec

D.H.Fremlin

University of Essex, Colchester, England

Theorem Suppose that $\mathfrak{p} = \mathfrak{c}$. Then there is a family $\langle F_\xi \rangle_{\xi<\mathfrak{c}}$ of closed negligible subsets of $\mathbb{R}$ such that $\bigcup_{\xi<\mathfrak{c}} F_\xi = \mathbb{R}$ and $\bigcup_{\xi\in B} F_\xi$ is measurable and has the Baire property for every $B \subseteq \mathfrak{c}$.

proof (a) Let $\mathcal{U}$ be the set of finite unions of rational intervals in $\mathbb{R}$, together with $\emptyset$; note that $\mathcal{U}$ is countable and that $\overline{U} \setminus U \subseteq \mathbb{Q}$ for every $U \in \mathcal{U}$. Let $\langle A_\xi \rangle_{\xi<\mathfrak{c}}$ be an enumeration of $[\mathbb{R}]^{\le\omega}$, and $\langle x_\xi \rangle_{\xi<\mathfrak{c}}$ an enumeration of $\mathbb{R}$.

(b) For each $\xi < \mathfrak{c}$ we can find a closed negligible set $F_\xi \subseteq \mathbb{R}$ such that $x_\xi \in F_\xi$ and, for every $\eta < \xi$, either $A_\eta \cap F_\xi \neq \emptyset$ or there is a $U \in \mathcal{U}$ such that $A_\eta \subseteq U \subseteq \mathbb{R} \setminus F_\xi$. **P** Let P be the set of pairs (U, I) where $U \in \mathcal{U}$ and $I \in [\mathbb{R} \setminus U]^{<\omega}$, ordered by saying that $(U, I) \le (V, J)$ if $U \subseteq V$ and $I \subseteq J$. Then P is σ-centered upwards. For each $\eta < \xi$, write

$$Q_\eta = \{(U, I) : (U, I) \in P, \text{ either } A_\eta \subseteq U \text{ or } A_\eta \cap I \neq \emptyset\}.$$

Then Q_η is cofinal with P. For $n \in \mathbb{N}$, write $Q'_n = \{(U, I) : (U, I) \in P,\ \mu([-n, n] \setminus U) \le 2^{-n}\}$, where μ is Lebesgue measure; then Q'_n is cofinal with P. Because $\#(\xi) < \mathfrak{p}$, there is an upwards-directed set $R \subseteq P$ meeting every Q_η and every Q'_n and containing $(\emptyset, \{x_\xi\})$. Set $F_\xi = \mathbb{R} \setminus \bigcup\{U : (U, I) \in R\}$; this serves. **Q**

(c) Because $x_\xi \in F_\xi$ for every ξ, $\bigcup_{\xi\in\mathbb{R}} F_\xi = \mathbb{R}$. Now take any $B \subseteq \mathfrak{c}$ and consider $E = \bigcup_{\xi\in B} F_\xi$. Set $H = \mathbb{R} \setminus \operatorname{int} E$. Then $E \cap H$ is meager and negligible. **P** $H \setminus E$ is dense in H. Let $\eta < \mathfrak{c}$ be such that A_η is a dense subset of $H \setminus E$. Set

$$M = \mathbb{Q} \cup \bigcup_{\xi\le\eta} F_\xi.$$

Because $\#(\eta) < \mathfrak{p}$, M is meager and negligible (Fremlin 84, 22B and 22G). **?** If there is an $x \in E \cap H \setminus M$, let $\xi \in B$ be such that $x \in F_\xi$. Then $\xi > \eta$. Since $A_\eta \cap F_\xi = \emptyset$, there is a $U \in \mathcal{U}$ such that $A_\eta \subseteq U$ and $U \cap F_\xi = \emptyset$; but this means that $H \subseteq \overline{U}$ and

$$x \in F_\xi \cap H \subseteq \overline{U} \setminus U \subseteq \mathbb{Q} \subseteq M,$$

which is impossible. **X** Thus $E \cap H \subseteq M$ is meager and negligible. **Q**

So $E \setminus \operatorname{int} E = E \cap H$ is meager and negligible, and E is measurable and has the Baire property.

(d) As B is arbitrary, $\langle F_\xi \rangle_{\xi<\mathfrak{c}}$ has the properties declared.

Remark T.Natkaniec asked (March 2001) whether, if $\langle F_\xi \rangle_{\xi<\omega_1}$ is a cover of $\mathbb{R}$ by closed nowhere dense sets, there must be some $B \subseteq \omega_1$ such that $\bigcup_{\xi\in B} F_\xi$ does not have the Baire property. Subject to the continuum hypothesis, the construction above gives a negative answer.

Question In the theorem, we have ideals

$$\mathcal{I} = \{A : A \subseteq \mathfrak{c},\ \bigcup_{\xi\in A} F_\xi \text{ is negligible}\},$$

$$\mathcal{J} = \{A : A \subseteq \mathfrak{c},\ \bigcup_{\xi\in A} F_\xi \text{ is meager}\}$$

of $\mathcal{P}\mathfrak{c}$. What can be said about these? What do the quotient algebras $\mathcal{P}\mathfrak{c}/\mathcal{I}$, $\mathcal{P}\mathfrak{c}/\mathcal{J}$ look like?

References

Fremlin D.H. [84] *Consequences of Martin's Axiom.* Cambridge U.P., 1984.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX