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Freese-Nation numbers

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1. Basic theory

1A Definitions Let P be a partially ordered set. A function $f : P \to \mathcal{P}P$ is a **Freese-Nation function** if $[p, q] \cap f(p) \cap f(q) \neq 0$ whenever $p \leq q$ in P. The **Freese-Nation number** $\mathrm{FN}(P)$ of P is the smallest cardinal κ such that there is a Freese-Nation function from P to $[P]^{<\kappa}$. The **regular Freese-Nation number** $\mathrm{FN}^*(P)$ is the smallest regular infinite cardinal $\kappa \geq \mathrm{FN}(P)$.

Note that $p \in f(p)$ for any Freese-Nation function f on P and any $p \in P$. P has the 'Freese-Nation property' if $\mathrm{FN}(P) \leq \omega$, and the 'weak Freese-Nation property' if $\mathrm{FN}(P) \leq \omega_1$. In the language of Fuchino & Soukup 97, $\mathrm{FN}(P) \leq \kappa$ if P has the 'κ-Freese-Nation property'.

1B Proposition Let P be a partially ordered set and $f : P \to \mathcal{P}P$ a Freese-Nation function.

(a) If $Q \subseteq P$ is order-convex, then $q \mapsto Q \cap f(q)$ is a Freese-Nation function on Q.

(b) If $Q \subseteq P$ and $h : P \to Q$ is a retraction (that is, an order-preserving mapping such that $h(q) = q$ for $q \in Q$), then $q \mapsto h[f(q)]$ is a Freese-Nation function on Q.

1C Proposition Let P be a partially ordered set and Q a subset of P. Suppose that
either (i) Q is order-convex
or (ii) Q is a retract of P
or (iii) Q is, in its induced ordering, a Dedekind complete lattice.
Then $\mathrm{FN}(Q) \leq \mathrm{FN}(P)$.

proof (i) and (ii) are immediate from 1B. Now (iii) follows. **P** Set $P_1 = \bigcup_{q,q' \in \mathbb{Q}}[q, q']$. Then P_1 is an order-convex subset of P, so $\mathrm{FN}(P_1) \leq \mathrm{FN}(P)$. Now we can define $h : P_1 \to Q$ by setting $h(p) = \inf\{q : q \in Q,\ p \leq q\}$ for $p \in P_1$, the infimum being taken in Q. Then h is a retraction, so $\mathrm{FN}(Q) \leq \mathrm{FN}(P_1) \leq \mathrm{FN}(P)$. **Q**

1D Example Consider a partially ordered set $P = A \cup \{p^*\} \cup B$ where A and B are disjoint from each other and from p^*, and for p, $q \in P$ we have $p \leq q$ if

either $p = q$
or $p \in A \cup \{p^*\}$ and $q \in \{p^*\} \cup B$.

Then $\mathrm{FN}(P) \leq 3$, because $p \mapsto \{p, p^*\}$ is a Freese-Nation function. But there is no bound on $\mathrm{FN}(P \setminus \{p^*\})$.

Thus we can have $Q \subseteq P$ but $\mathrm{FN}(Q) > \mathrm{FN}(P)$.

1E Proposition Let P and Q be partially ordered sets and $h : P \to Q$ an order-preserving surjection.

(a) Suppose that whenever $q \leq q'$ in Q there are $p \leq p'$ in P such that $h(p) = q$ and $h(p') = q'$. (This is true if, for instance, P and Q are lattices and h is a lattice homomorphism.) Let κ be an infinite cardinal such that $\mathrm{FN}(P) \leq \kappa$ and $\#(\{p : h(p) = q\}) < \mathrm{cf}\,\kappa$ for every $q \in Q$. Then $\mathrm{FN}(Q) \leq \kappa$.

(b) Suppose that whenever $p \leq p'$ in P and $h(p) \leq q \leq h(p')$ there is a $p'' \in [p, p']$ such that $h(p'') = q$. (Again, this is true if, for instance, P and Q are lattices and h is a lattice homomorphism.) Let κ be an infinite cardinal such that $\mathrm{FN}(Q) \leq \mathrm{cf}\,\kappa$ and $\#(\{p : h(p) = q\}) < \kappa$ for every $q \in Q$. Then $\mathrm{FN}(P) \leq \kappa$.

proof (a) Let $f : P \to [P]^{<\kappa}$ be a Freese-Nation function. Set

$$g(q) = h[\bigcup\{f(p) : h(p) = q\}]$$

for each $q \in Q$; then $\#(g(q)) < \kappa$ for every q. If $q \leq q'$ in Q, let p, $p' \in P$ be such that $p \leq p'$ and $h(p) = q$ and $h(p') = q'$. Then there is an $r \in [p, p'] \cap f(p) \cap f(p')$, so that $h(r) \in [q, q'] \cap g(q) \cap g(q')$. Thus g is a Freese-Nation function on Q and $\mathrm{FN}(Q) \leq \kappa$.

(b) This time, let $g : Q \to [Q]^{<\kappa}$ be a Freese-Nation function, and set

$$f(p) = h^{-1}[g(h(p))]$$

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for $p \in P$. Then $f : P \to [P]^{<\kappa}$ is a function. If $p \le p'$ in P, take $q \in [f(p), f(p')] \cap g(f(p)) \cap g(f(p'))$. Then there is a $p'' \in [p, p']$ such that $h(p'') = q$, and $p'' \in [p, p'] \cap f(p) \cap f(p')$. So f is a Freese-Nation function on P and $\mathrm{FN}(P) \le \kappa$.

1F Definition Let P be a partially ordered set and Q any subset of P. The **Freese-Nation index** of Q in P is the least cardinal κ such that $\mathrm{cf}(Q \cap]-\infty, p]) < \kappa$ and $\mathrm{ci}(Q \cap [p, \infty[) < \kappa$ for every $p \in P$.

1G Lemma (Fuchino Koppelberg & Shelah 96) Let P be a partially ordered set, and suppose that we have a family $\langle P_\xi \rangle_{\xi \le \alpha}$ of subsets of P, where α is a non-zero limit ordinal, such that $P_\alpha = P$, $P_\xi \subseteq P_\eta$ whenever $\xi \le \eta \le \alpha$ and $P_\zeta = \bigcup_{\xi<\zeta} P_\xi$ for every non-zero limit ordinal $\zeta \le \alpha$. For $\xi < \alpha$ let κ_ξ be the Freese-Nation index of P_ξ in $P_{\xi+1}$. Suppose that $\kappa \ge \max(\sup_{\xi<\alpha} \mathrm{FN}(P_{\xi+1}), \sup_{\xi<\alpha} \kappa_\xi)$ is a regular cardinal. Then $\mathrm{FN}(P) \le \kappa$.

proof For each $\xi < \alpha$ choose a Freese-Nation function $f_\xi : P_{\xi+1} \to [P_{\xi+1}]^{<\kappa}$. For $p \in P$, let $\zeta(p)$ be that $\xi < \alpha$ such that $p \in P_{\xi+1} \setminus P_\xi$, and let $D_p \subseteq P_{\zeta(p)}$ be a set of size less than κ such that $D_p \cap]-\infty, p]$ is cofinal with $P_{\zeta(p)} \cap]-\infty, p]$ and $D_p \cap [p, \infty[$ is cofinal with $P_{\zeta(p)} \cap [p, \infty[$. Define g inductively by setting $g(p) = f_{\zeta(p)}(p) \cup \bigcup_{q \in D_p} g(q)$ for every $p \in P$. Because κ is regular, g is a function from P to $[P]^{<\kappa}$.

Now g is a Freese-Nation function on P. **P** I induce on ξ to show that if $p, q \in P$ and $\max(\zeta(p), \zeta(q)) = \xi$ then $g(p) \cap g(q) \cap [p, q] \ne \emptyset$. For the inductive step to $\xi < \alpha$, if $\zeta(p) = \zeta(q) = \xi$ then

$$g(p) \cap g(q) \cap [p, q] \supseteq f_\xi(p) \cap f_\xi(q) \cap [p, q] \ne \emptyset.$$

If $\zeta(p) < \zeta(q) = \xi$, then there is an $r \in D_q$ such that $p \le r \le q$; now if $\eta = \max(\zeta(p), \zeta(r))$,

$$g(p) \cap g(q) \cap [p, q] \supseteq g(p) \cap g(r) \cap [p, r] \ne \emptyset$$

by the inductive hypothesis. The same argument works if $\zeta(q) < \zeta(p)$. **Q**

(What if κ is not regular?)

1H Proposition Let P be a partially ordered set and κ a regular infinite cardinal. Let $g : P \to [P]^{<\kappa}$ be a function such that whenever $p \le q$ in P there are $r \in g(q)$ and $s \in g(p)$ such that $p \le r \le s \le q$. Then $\mathrm{FN}(P) \le \kappa$.

proof Induce on $\#(P)$. For $\#(P) \le \kappa$ this is elementary (see Proposition 2A below). For the inductive step to $\#(P) = \lambda > \kappa$, express P as $\bigcup_{\xi<\mathrm{cf}\,\lambda} P_\xi$ where (i) for each $\xi < \mathrm{cf}\,\lambda$, $\#(P_\xi) < \lambda$ and $g(p) \subseteq P_\xi$ for every $p \in P_\xi$ (ii) $P_\xi = \bigcup_{\eta<\xi} P_\eta$ for limit ordinals $\xi < \mathrm{cf}\,\lambda$. By the inductive hypothesis, $\mathrm{FN}(P_\xi) \le \kappa$ for every ξ. Also the Freese-Nation index of P_ξ in $P_{\xi+1}$ is at most κ for each $\xi < \mathrm{cf}\,\lambda$. **P** For $p \in P_{\xi+1}$, let $A \subseteq P_\xi$ be a set of size less than κ such that whenever $r \in g(p)$ and $[r, p] \cap P_\xi \ne \emptyset$ then $[r, p] \cap A \ne \emptyset$. Now if $q \in P_\xi$ and $q \le p$, there are $r \in g(p)$, $s \in g(q)$ such that $q \le r \le s \le p$. In this case $s \in P_\xi$ so $s \in [r, p] \cap P_\xi$ so there is an $s' \in A \cap [r, p]$ and $q \le s' \le p$. Thus A is cofinal in $P_\xi \cap]-\infty, p]$ and $\mathrm{cf}(P_\xi \cap]-\infty, p]) < \kappa$. Similarly, $\mathrm{ci}(P_\xi \cap [p, \infty[) < \kappa$. **Q**

By Lemma 1G, $\mathrm{FN}(P) \le \kappa$ and the induction continues.

1I Proposition Let P be a partially ordered set. Then $\mathrm{FN}^*(P)$ is the least regular cardinal κ such that there is an enumeration $\langle p_\xi \rangle_{\xi<\lambda}$ of P such that $\mathrm{cf}(\{p_\eta : \eta < \xi,\, p_\eta < p_\xi\})$ and $\mathrm{ci}(\{p_\eta : \eta < \xi,\, p_\eta > p_\xi\})$ are both less than κ for every $\xi < \lambda$.

proof (a) Suppose that $\kappa = \mathrm{FN}^*(P)$. Let $\langle q_\xi \rangle_{\xi<\lambda}$ be any enumeration of P. If $\lambda \le \kappa$ then this family already satisfies the condition. Otherwise, let $f : P \to [P]^{<\kappa}$ be a Freese-Nation function. For each $\xi < \lambda$ let A_ξ be the smallest subset of P containing q_ξ such that $f(p) \subseteq A_\xi$ for every $p \in A_\xi$. For $p \in P$ set $\zeta(p) = \min\{\xi : p \in A_\xi\}$. Note that $\#(A_\xi) \le \kappa$ for every ξ, so that $\#(\bigcup_{\eta<\xi} A_\eta) \le \max(\kappa, \#(\xi)) < \lambda$ for every $\xi < \kappa$. We can therefore enumerate P as $\langle p_\xi \rangle_{\xi<\lambda}$ in such a way that if $\xi \le \eta$ then $\zeta(p_\xi) \le \zeta(p_\eta)$, while for any ξ the set $D_\xi = \{\eta : \eta < \xi,\, \zeta(p_\eta) = \zeta(p_\xi)\}$ has cardinal less than κ.

If $\xi < \lambda$, set $B = \{p_\eta : \eta < \xi\}$. Then, writing $C = \{p : \zeta(p) < \zeta(p_\xi)$, we have $B = C \cup D_\xi$. Now $f(p) \subseteq C$ for every $p \in C$, so $f(p_\xi) \cap C \cap]-\infty, p_\xi]$ is cofinal with $C \cap]-\infty, p_\xi]$; accordingly $((f(p_\xi) \cap C) \cup D_\xi) \cap]-\infty, p_\xi]$ is cofinal with $B \cap]-\infty, p_\xi]$, and $\mathrm{cf}(B \cap]-\infty, p_\xi]) < \kappa$. Similarly, $\mathrm{ci}(B \cap [p_\xi, \infty[) < \kappa$. As ξ is arbitrary, $\langle p_\xi \rangle_{\xi<\lambda}$ satisfies the condition.

(b) Now suppose the condition is satisfied by a regular infinite cardinal κ and $\langle p_\xi \rangle_{\xi<\lambda}$. Set $P_\xi = \{p_\eta : \eta < \xi\}$ for $\xi \le \lambda$. Then the hypothesis is that the Freese-Nation index of P_ξ in $P_{\xi+1}$ is always at most κ.

Also $\mathrm{FN}(P_{\xi+1}) \le \max(\omega, \mathrm{FN}(P_\xi)$ for every ξ, because if $g : P_\xi \to \mathcal{P}P_\xi$ is a Freese-Nation function, then setting

$$g'(q) = g(q) \cup \{p_\xi\} \text{ if } q \in P_\xi,$$
$$= \{p_\xi\} \text{ if } q = p_\xi,$$

$g' : P_{\xi+1} \to \mathcal{P}_{\xi+1}$ is a Freese-Nation function. So a simple induction, using Lemma 1G at limit ordinals, shows us that $\mathrm{FN}(P_\xi) \le \kappa$ for every $\xi \le \lambda$, and in particular that $\mathrm{FN}(P) \le \kappa$.

1J Corollary Suppose that κ is a regular uncountable cardinal such that whenever $A \subseteq P$ and $p \in P$ then $\mathrm{cf}(A \cap]-\infty, p])$ and $\mathrm{ci}(A \cap [p, \infty[)$ are both less than κ. Then $\mathrm{FN}(P) \le \kappa$.

proof Apply Proposition 1I with any enumeration of P.

1K Proposition Let P be a partially ordered set and Q a subset of P. Let κ be a regular infinite cardinal such that both $\mathrm{FN}(P)$ and the Freese-Nation index of Q in P are at most κ. Then $\mathrm{FN}(Q) \le \kappa$.

proof Let $f : P \to [P]^{<\kappa}$ be a Freese-Nation function. For each $p \in P$, let $D_p \subseteq Q$ be a set of cardinal less than κ such that $D_p \cap]-\infty, p]$ is cofinal with $Q \cap]-\infty, p]$ and $D_p \cap [p, \infty[$ is coinitial with $Q \cap [p, \infty[$. For $q \in Q$ set $g(q) = \bigcup_{p \in f(q)} D_p$; then $g(q) \in [Q]^{<\kappa}$. Now if q, $q' \in Q$ and $q \le q'$, there is a $p \in [q, q'] \cap f(q) \cap f(q')$. Now there are r, $r' \in D_p$ such that $q \le r \le p \le r' \le q'$. In this case $r' \in g(q)$ and $r \in g(q')$. So g satisfies the conditions of Proposition 1H and $\mathrm{FN}(Q) \le \kappa$.

1L Lemma Let P be a partially ordered set and $f : P \to [P]^{<\kappa}$ a Freese-Nation function. If $Q \subseteq P$ is such that $f(q) \subseteq Q$ for every $q \in Q$, then the Freese-Nation index of Q in P is at most κ.

proof If $p \in P$, then $f(p) \cap Q \cap]-\infty, p]$ is cofinal with $Q \cap]-\infty, p]$ and $f(p) \cap Q \cap [p, \infty[$ is coinitial with $Q \cap [p, \infty[$.

1M Proposition If $\langle P_i \rangle_{i \in I}$ is a family of partially ordered sets, with product P, then $\mathrm{FN}(P) \le \#(\prod_{i \in I} \mathrm{FN}(P_i))$.

proof If $f_i : P_i \to \mathcal{P}_i$ is a Freese-Nation function for each i, then $p \mapsto \prod_{i \in I} f_i(p(i)) : P \to \mathcal{P}$ is a Freese-Nation function.

2. Cardinal bounds

2A Proposition (Fuchino Koppelberg & Shelah 96) If $\#(P) \ge 3$, $\mathrm{FN}(P) \le \#(P)$.

proof If $3 \le \#(P) < \omega$, use Proposition 2D of Fremlin & Penman n01. If P is infinite, enumerate it as $\langle p_\xi \rangle_{\xi < \kappa}$ where $\kappa = \#(P)$, and set $f(p_\xi) = \{p_\eta : \eta \le \xi\}$ for $\xi < \kappa$.

2B Proposition (Fuchino Koppelberg & Shelah 96) Let P be a partially ordered set. Then P has no totally ordered subset of cofinality greater than $\mathrm{FN}^*(P)$.

proof ? Suppose, if possible, otherwise. Set $\kappa = \mathrm{FN}^*(P)$, and let $f : P \to [P]^{<\kappa}$ be a Freese-Nation function and $\langle p_\xi \rangle_{\xi < \kappa^+}$ a strictly increasing family in P. For each $\xi < \kappa^+$ set $U(\xi) = \bigcup_{\xi \le \eta < \kappa^+} [p_\xi, p_\eta]$. Then for each $\xi < \kappa^+$ there is an $\alpha_\xi < \kappa^+$ such that $U(\xi) \cap f(p_\xi) \subseteq [p_\xi, p_{\alpha_\xi}]$, because $\#(f(p_\xi)) \le \kappa$. Let $\beta < \kappa^+$ be such that $\alpha_\xi < \beta$ for every $\xi < \beta$ and $\mathrm{cf}\,\beta = \kappa$. Then $\#(f(p_\beta)) < \kappa$, so there is a $\xi < \beta$ such that $f(p_\beta) \cap [p_\xi, p_\beta] \subseteq [p_\eta, p_\beta]$ for every $\eta < \beta$; in particular, $f(p_\beta) \cap [p_\xi, p_\beta] \subseteq [p_{\alpha_\xi+1}, p_\beta]$. But as $f(p_\xi)$ does not meet $[p_{\alpha_\xi+1}, p_\beta]$, f is not a Freese-Nation function. **X**

2C Corollary Let P be a partially ordered set and $f : P \to \mathcal{P}P$ a Freese-Nation function. Let $\kappa \ge \omega$ be a regular cardinal and set $Q = \{p : p \in P, \#(f(p)) < \kappa\}$. Then there is no subset of Q of order type κ^+.

proof If $R \subseteq Q$ is well-ordered, there is a retraction $h : P \to R$, and now $r \mapsto h[f(r)] : R \to [R]^{<\kappa}$ is a Freese-Nation function on R, by Proposition 1B(b). By Proposition 2B, $\mathrm{otp}\, R < \kappa^+$.

2D Theorem (Fuchino Koppelberg & Shelah 96) If κ is an infinite cardinal, $\kappa < \mathrm{FN}(\mathcal{P}\kappa) \le 2^\kappa$.

proof (a) By Proposition 2A, $\mathrm{FN}(\mathcal{P}\kappa) \le 2^\kappa$. So all I have to show is that $\mathrm{FN}(\mathcal{P}\kappa) > \kappa$.

(b) Suppose first that κ is regular. **?** Suppose, if possible, that $f : \mathcal{P}\kappa \to [\mathcal{P}\kappa]^{<\kappa}$ is a Freese-Nation function. Then $D = \bigcup_{\xi<\kappa} f(\{\xi\})$ has cardinal κ. Now there is a set $a \subseteq \kappa$, of cardinal κ, which does not include any member of D of cardinal κ. So if we look at $b = \bigcup(D \cap f(a))$, this has cardinal less than κ (because κ is regular), and there is a $\xi \in a \setminus b$; in which case $[\{\xi\}, a] \cap f(a) \cap f(\{\xi\}) = \emptyset$. **X**

(c) So suppose from now on that κ is singular, with cofinality λ. Let $\langle\kappa_\xi\rangle_{\xi<\lambda}$ be a strictly increasing family of regular cardinals with supremum κ, starting from $\kappa_0 > \lambda$, and let P be the set $\prod_{\xi<\lambda} \kappa_\xi$ with its usual ordering. We need to know that $\mathrm{FN}(P) \le \mathrm{FN}(\mathcal{P}\kappa)$; this is because we have a natural embedding of P in $\mathcal{P}(\lambda \times \kappa)$, identifying $p \in P$ with its ordinate set $\{(\xi, \eta) : \xi < \lambda, \eta \le p(\xi)\}$. Since P is Dedekind complete, this is enough to ensure that $\mathrm{FN}(P) \le \mathrm{FN}(\mathcal{P}(\lambda \times \kappa)) = \mathrm{FN}(\mathcal{P}\kappa)$, by Proposition 1C(iii).

(d) **?** Now suppose, if possible, that $f : P \to [P]^{<\kappa}$ is a Freese-Nation function. For each $\xi < \lambda$, set $A_\xi = \{p : \#(f(p)) < \kappa_\xi\}$, so that $P = \bigcup_{\xi<\lambda} A_\xi$. For $p \in P$, $\xi < \lambda$ set $Q(p,\xi) = \{q : q \in P,\ q \ge p,\ q\restriction\xi = p\restriction\xi\}$. Because $\mathrm{cf}\,\kappa_\xi > \lambda$ for every ξ, any family in $\mathcal{Q} = \{Q(p,\xi) : p \in P,\ \xi < \lambda\}$ with the finite intersection property and cardinal at most λ has non-empty intersection. There must therefore be a $Q_0 \in \mathcal{Q}$ and a $\xi_0 < \lambda$ such that $Q \cap A_{\xi_0} \ne \emptyset$ whenever $Q \in \mathcal{Q}$ and $Q \subseteq Q_0$.

Express Q_0 as $Q(p_0, \xi_0')$. Then $Q(p,\eta) \cap A_{\xi_0} \ne \emptyset$ whenever $p \ge p_0$ and $p\restriction\xi_0' = p_0\restriction\xi_0'$. Set $\zeta = \max(\xi_0 + 1, \xi_0')$. Then $\mathrm{cf}\,\kappa_\xi = \kappa_\xi \ge \kappa_{zeta}$ for every $\xi \ge \zeta$, so we can choose inductively a strictly increasing family $\langle q_\xi\rangle_{\xi<\kappa_\zeta}$ in A_{ξ_0} such that $q_\xi\restriction\zeta = p_0\restriction\zeta$ for every ξ. Thus A_{ξ_0} includes a set with order type $\kappa_\zeta > \kappa_{\xi_0}$. But this is impossible, by Corollary 2C. **X**

Thus $\mathrm{FN}(\mathcal{P}\kappa) \ge \mathrm{FN}(P) > \kappa$, as required.

2E Corollary If $\mathfrak{A}$ is a Dedekind complete Boolean algebra and $\omega \le \kappa < \mathrm{sat}(\mathfrak{A})$, then $\mathrm{FN}(\mathfrak{A}) > \kappa$.

proof $\mathcal{P}\kappa$ can be embedded in $\mathfrak{A}$.

2F Proposition (cf. Fuchino & Soukup 97, Theorem 17) Let $\mathfrak{A}$ be an infinite Boolean algebra and $\kappa \ge \mathrm{FN}(\mathfrak{A})$ a regular cardinal. Let $\langle a_\xi\rangle_{\xi<\kappa^+}$ be a disjoint family in $\mathfrak{A}$. Then there is a $d \in \mathfrak{A}$ such that $\{\xi : \xi < \kappa,\ a_\xi \subseteq d\}$ and $\{\xi : \xi < \kappa,\ a_\xi \cap d = 0\}$ both have cardinal κ^+.

proof Let $f : \mathfrak{A} \to [\mathfrak{A}]^{<\mathrm{FN}(\mathfrak{A})}$ be a Freese-Nation function; replacing f by $a \mapsto f(a) \cup f(1 \setminus a)$ if necessary, we may suppose that $f(a) = f(1 \setminus a)$ for every a. For $c \in \mathfrak{A}$ set $J_c = \{\xi : \xi < \kappa^+,\ a_\xi \subseteq c\}$ and $K_c = \{\xi : \xi < \kappa^+,\ a_\xi \cap c = 0\}$. Let $\delta < \kappa^+$ be such that $\mathrm{cf}\,\delta = \kappa$ and

whenever $c \in \bigcup_{\xi<\delta} f(a_\xi)$ and $\#(J_c) \le \kappa$, then $\sup J_c < \delta$,

whenever $c \in \bigcup_{\xi<\delta} f(a_\xi)$ and $\#(K_c) \le \kappa$, then $\sup K_c < \delta$.

For $\xi < \delta$, let $c_\xi \in f(a_\xi) \cap f(1 \setminus a_\delta)$ be such that $a_\xi \subseteq c_\xi \subseteq 1 \setminus a_\delta$. Because $\#(f(1 \setminus a_\delta)) < \mathrm{cf}(\delta)$, there is a d such that $\{\xi : \xi < \delta,\ c_\xi = d\}$ is cofinal with δ. Note that $d \in \bigcup_{\xi<\delta} f(a_\xi)$.

Because J_d is cofinal with δ, it must have cardinal κ^+. Because $\delta \in K_d$, $\#(K_d) = \kappa^+$. So we have a suitable d.

3. Totally ordered sets

3A Proposition If P is an infinite totally ordered set and κ is a regular uncountable cardinal such that there is neither a well-ordered nor an inverse-well-ordered subset of P of size κ, then $\mathrm{FN}(P) \le \kappa$.

proof Use Corollary 1J.

3B Corollary If P is an infinite totally ordered set and $\mathrm{sat}(P)$ is its saturation in its order topology, then $\mathrm{FN}(P) \le \mathrm{sat}(P)$.

3C Theorem If P is an infinite totally ordered set, then $\#(P) \le 2^{<\mathrm{FN}(P)}$.

proof Set $\kappa = 2^{<\mathrm{FN}(P)}$, and let $f : P \to [P]^{<\mathrm{FN}(P)}$ be a Freese-Nation function.

? Suppose, if possible, that $\#(P) \ge \kappa^+$. Since $\mathrm{FN}(P) \le \kappa$, there is an $\alpha < \mathrm{FN}(P)$ such that $A = \{p : p \in P,\ \#(f(p)) = \alpha\}$ has cardinal at least κ^+. By the Δ-system lemma, there is a set $B \subseteq A$ of cardinal κ^+ such that $\langle f(p)\rangle_{p\in B}$ is a Δ-system with root I say. Next, because $2^{\#(I)} < \#(B)$, there are p, $q \in B$ such that $p < q$ and

$$I \cap \,]-\infty, p[\, = I \cap \,]-\infty, q[, \quad I \cap \,]-\infty, p] = I \cap \,]-\infty, q].$$

But this means that there can be no $r \in I$ such that $p \le r \le q$; while $f(p) \cap f(q) \subseteq I$. So f cannot be a Freese-Nation function. **X**

So $\#(P) \le \kappa$, as claimed.

Remark In fact we can say: if P is a partially ordered set and $\kappa \ge 2^{<\mathrm{FN}(P)}$ is such that P has no weak antichains of size κ^+, then $\#(P) \le \kappa$.

3D Proposition For any infinite cardinal κ, $\mathrm{FN}(\kappa) = \kappa$.

proof By Proposition 2A, $\mathrm{FN}(\kappa) \le \kappa$. By Theorem 2B of Fremlin & Penman n01, $\mathrm{FN}(\kappa)$ is infinite. By Proposition 2B, $\lambda^+ \le \mathrm{FN}(\kappa)$ for every infinite cardinal $\lambda < \kappa$, so $\kappa \le \mathrm{FN}(\kappa)$.

4. $\mathcal{P}\mathbb{N}$

4A Proposition Let $\mathfrak{A}$ be a complete Boolean algebra with linking number κ. Then $\mathrm{FN}(\mathfrak{A}) \le \mathrm{FN}(\mathcal{P}\kappa)$.

proof Let $\langle A_\xi\rangle_{\xi<\kappa}$ be a family of linked subsets of $\mathfrak{A} \setminus \{0\}$ covering $\mathfrak{A} \setminus \{0\}$. For each $\xi < \kappa$ set $B_\xi = \{b : \exists\, a \in A_\xi,\ a \subseteq b\}$, so that B_ξ is also linked. Define $h : \mathfrak{A} \to \mathcal{P}\kappa$ by setting $h(a) = \{\xi : \xi < \kappa,\ a \in B_\xi\}$ for each $a \in \mathfrak{A}$. Then h is injective and is an order-isomorphism between $\mathfrak{A}$ and $h[\mathfrak{A}]$. **P** h is order-preserving just because $b' \in B_\xi$ whenever $b' \supseteq b \in B_\xi$. If a, $b \in \mathfrak{A}$ and $a \not\subseteq b$, that is, $a \setminus b \ne 0$, let $\xi \in \kappa$ be such that $a \setminus b \in A_\xi$. Then $a \in B_\xi$, but $b \notin B_\xi$, because B_ξ is linked. So $\xi \in h(a) \setminus h(b)$ and $h(a) \not\subseteq h(b)$. **Q**

By Proposition 1C(iii), $\mathrm{FN}(\mathfrak{A}) = \mathrm{FN}(h[\mathfrak{A}]) \le \mathrm{FN}(\mathcal{P}\kappa)$.

4B Theorem (Fuchino Geschke Shelah & Soukup 01) The following sets all have the same regular Freese-Nation number:

(i) $\mathcal{P}\mathbb{N}$;
(ii) $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$;
(iii) $\mathbb{N}^\mathbb{N}$, with its usual ordering $\le$;
(iv) $\mathbb{N}^\mathbb{N}$ with the ordering $\le^*$, defined by setting $x \le^* y$ if either $x \le y$ or $\{n : y(n) \le x(n)\}$ is finite;
(v) any infinite σ-linked Dedekind complete Boolean algebra;
(vi) any infinite measurable algebra of Maharam type at most $\mathfrak{c}$;
(vii) the regular open algebra of $\mathbb{R}$.

proof (a) $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \mathrm{FN}(\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega})$ by Proposition 1E.

(b) $\mathcal{P}\mathbb{N} \equiv \{0,1\}^\mathbb{N}$ can be identified with an order-convex subset of $\mathbb{N}^\mathbb{N}$, so $\mathrm{FN}(\mathcal{P}\mathbb{N}) \le \mathrm{FN}(\mathbb{N}^\mathbb{N})$ by Proposition 1C(i).

Also $\mathbb{N}^\mathbb{N}$ is order-isomorphic to a subset of $\mathcal{P}(\mathbb{N}\times\mathbb{N}) \equiv \mathcal{P}\mathbb{N}$, if we identify $x \in \mathbb{N}^\mathbb{N}$ with its ordinate set $\{(n,i) : i \le x(n)\}$; since $\mathbb{N}^\mathbb{N}$ is a Dedekind complete lattice, this is enough to show that $\mathrm{FN}(\mathbb{N}^\mathbb{N}) \le \mathrm{FN}(\mathcal{P}\mathbb{N})$.

(c) If we define an equivalence relation $\sim$ on $\mathbb{N}^\mathbb{N}$ by saying that $x \sim y$ if $\{n : x(n) \ne y(n)\}$ is finite, then we have a quotient partially ordered set P. Now 1E tells us that $\mathrm{FN}(P) = \mathrm{FN}(\mathbb{N}^\mathbb{N}, \le) = \mathrm{FN}(\mathbb{N}^\mathbb{N}, \le^*)$.

(d) If $\mathfrak{A}$ is a σ-linked Dedekind complete Boolean algebra then $\mathrm{FN}(\mathfrak{A}) \le \mathrm{FN}(\mathcal{P}\mathbb{N})$, by 4A. On the other hand, if $\mathfrak{A}$ is any infinite Dedekind σ-complete Boolean algebra, then $\mathcal{P}\mathbb{N}$ is isomorphic to a subalgebra of $\mathfrak{A}$, so $\mathrm{FN}(\mathcal{P}\mathbb{N}) \le \mathrm{FN}(\mathfrak{A})$.

(e) Now any measurable algebra of Maharam type at most $\mathfrak{c}$, and the regular open algebra of $\mathbb{R}$, are all σ-linked.

4C Corollary (Fuchino Koppelberg & Shelah 96) $\mathrm{FN}^*(\mathcal{P}\mathbb{N}) \ge \mathfrak{b}$.

proof There is a family $\langle x_\xi\rangle_{\xi<\mathfrak{b}}$ in $\mathbb{N}^\mathbb{N}$ such that $x_\xi <^* x_\eta$ whenever $\xi < \eta$; now use Proposition 2B.

4D Proposition (see Fuchino Geschke & Soukup 01, Theorem 2.2) Let λ be a cardinal and $\kappa \ge \mathrm{FN}(\mathcal{P}\lambda)$ a regular cardinal. Suppose that a topological space X has π-weight at most λ and is not the union of fewer than κ nowhere dense sets. Then there is a set $A \subseteq X$ of size at most κ which is not the union of fewer than κ nowhere dense sets.

proof Let $\mathcal{U}$ be a π-base for the topology of X with $\#(\mathcal{U}) \le \lambda$, and $f : \mathcal{P}\mathcal{U} \to [\mathcal{P}\mathcal{U}]^{<\kappa}$ a Freese-Nation function. Let $\mathbb{V}$ be the set of those $\mathcal{V} \subseteq \mathcal{U}$ such that $\bigcup\mathcal{V}$ is dense. Because κ is regular and X is not the union of fewer than κ nowhere dense sets, there is a set $M \subseteq X \cup \mathcal{P}\mathcal{U}$, of cardinal at most κ, such that

$f(\mathcal{V}) \subseteq M$ for every $\mathcal{V} \in M \cap \mathcal{PU}$,

for every $x \in X \cap M$, $\{U : U \in \mathcal{U}, x \notin U\} \in M$,

whenever $\mathbb{W} \subseteq M$ and $\#(\mathbb{W}) < \kappa$, there is an $x \in M$ such that $x \in \bigcap_{\mathcal{W} \in \mathbb{W} \cap \mathbb{V}} \bigcup \mathcal{W}$.

Set $A = M \cap X$.

? Suppose, if possible, that $A = \bigcup_{\xi<\theta} A_\xi$ where $\theta < \kappa$ and every A_ξ is nowhere dense. For each $\xi < \theta$, set $\mathcal{V}_\xi = \{U : U \in \mathcal{U}, U \cap A_\xi = \emptyset\}$, and set $\mathbb{W} = M \cap \mathbb{V} \cap \bigcup_{\xi<\theta} f(\mathcal{V}_\xi)$. Then $\#(\mathbb{W}) < \kappa$, so there is an $x \in A \cap \bigcap_{\mathcal{V} \in \mathbb{W}} \bigcup \mathcal{V}$. Let $\xi < \theta$ be such that $x \in A_\xi$. Set $\mathcal{V} = \{U : U \in \mathcal{U}, x \notin U\}$. Then $\mathcal{V}_\xi \subseteq \mathcal{V}$, so there is a $\mathcal{W} \in f(\mathcal{V}_\xi) \cap f(\mathcal{V})$ such that $\mathcal{V}_\xi \subseteq \mathcal{W} \subseteq \mathcal{V}$. Now $\mathcal{V} \in M$, so $\mathcal{W} \in M$; also $\bigcup \mathcal{V}_\xi$ is dense and is included in $\bigcup \mathcal{W}$, so $\mathcal{W} \in \mathbb{V}$ and $\mathcal{W} \in \mathbb{W}$. But this means that $x \in \bigcup \mathcal{W}$, which is impossible, because $\mathcal{W} \subseteq \mathcal{V}$. **X**

4E Corollary Let $\mathcal{M}$ be the ideal of meager subsets of $\mathbb{R}$.

(a) If $\mathrm{FN}^*(\mathcal{P}\mathbb{N}) \le \mathrm{cov}(\mathcal{M})$, then $\mathrm{non}(\mathcal{M}) \le \mathrm{FN}^*(\mathcal{P}\mathbb{N})$.

(b) If $\mathrm{FN}^*(\mathcal{P}\mathbb{N}) = \omega_1$, then $\mathrm{shr}(\mathcal{M}) = \omega_1$.

proof (a) Apply 4D with $X = \mathbb{R}$, $\lambda = \omega$ and $\kappa = \mathrm{FN}^*(\mathcal{P}\mathbb{N})$.

(b) Apply 4D with X any non-meager subset of $\mathbb{R}$, $\lambda = \omega$ and $\kappa = \omega_1$.

4F Proposition (see FUCHINO GESCHKE & SOUKUP 01, Theorem 2.2) Suppose that $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \mathfrak{p}$. Then $\mathfrak{a} = \mathfrak{p}$.

proof Write $\mathfrak{A} = \mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$. Then $\mathrm{FN}^*(\mathfrak{A}) = \mathfrak{p}$ (Theorem 4B). Let $f : \mathfrak{A} \to [\mathfrak{A}]^{<\mathfrak{p}}$ be a Freese-Nation function. Because $\mathfrak{p}$ is regular (FREMLIN 84, 21K), we can choose inductively families $\langle a_\xi \rangle_{\xi<\mathfrak{p}}$ in $\mathfrak{A}$, $\langle A_\xi \rangle_{\xi<\mathfrak{p}}$ in $[\mathfrak{A}]^{<\mathfrak{p}}$ such that, for each $\xi < \mathfrak{p}$,

A_ξ is the smallest subset of $\mathfrak{A}$ including $\{1\} \cup \{a_\eta : \eta < \xi\}$ and such that $f(a) \subseteq A_\xi$ whenever $a \in A_\xi$,

$a_\xi \cap a_\eta = 0$ for every $\eta < \xi$,

whenever $a \in A_\xi$ and $1 \setminus a \not\subseteq \sup_{\eta \in I} a_\eta$ for every finite $I \subseteq \xi$, then $a_\xi \setminus a \ne 0$;

to see that the last two can be achieved together, see FREMLIN 84, 21A. (We need to take care at the beginning, with finite ξ, to ensure that $\sup_{\xi<n} a_\xi$ is never 1.) Now $\sup_{\xi<\mathfrak{p}} a_\xi = 1$. **P?** Suppose, if possible, that $a \in \mathfrak{A} \setminus \{1\}$ includes every a_ξ. Because $\#(f(a)) < \mathfrak{p}$, there is a $\xi < \mathfrak{p}$ such that $f(a) \cap A_\eta \subseteq f(a) \cap A_\xi$ for every $\eta < \mathfrak{p}$. Now there is a $b \in f(a_\xi) \cap f(a)$ such that $a_\xi \subseteq b \subseteq a$, and $b \in A_{\xi+1}$, so $b \in A_\xi$. But in this case we surely cannot have $1 \setminus b \subseteq \sup_{\eta \in I} a_\eta$ for any finite $I \subseteq \xi$, because $1 \setminus b \supseteq 1 \setminus a$. So $a_\xi \setminus b \ne 0$, which is impossible. **XQ**

So $\langle a_\xi \rangle_{\xi<\mathfrak{p}}$ witnesses that $\mathfrak{a} \le \mathfrak{p}$; as certainly $\mathfrak{p} \le \mathfrak{a}$, $\mathfrak{a} = \mathfrak{p}$.

4G Lemma Suppose that $\mathcal{A}$ is a family of infinite subsets of $\mathbb{R}$ such that $\#(\mathcal{A}) < \mathfrak{p}$. Then there is a family $\langle G_\xi \rangle_{\xi<\mathrm{cov}(\mathcal{M})}$ of open sets such that $G_\xi \cap A_n \ne \emptyset$ for every $n \in \mathbb{N}$ and $\bigcap_{\xi<\mathrm{cov}(\mathcal{M})} G_\xi = \emptyset$.

proof (a) There is a double sequence $\langle H_{mi} \rangle_{m,i\in\mathbb{N}}$ of open sets in $\mathbb{R}$ such that $\langle H_{mi} \rangle_{i\in\mathbb{N}}$ is disjoint for each m and for every $A \in \mathcal{A}$ there is an $m \in \mathbb{N}$ such that $A \cap H_{mi} \ne \emptyset$ for every $i \in \mathbb{N}$. **P** Let $\mathcal{U}$ be a countable base for the topology of $\mathbb{R}$, closed under finite unions. Let P be the partially ordered set of pairs $p = (\langle H^{(p)} \rangle_{m,i\in I_p}, J_p)$ where

$I_p \subseteq \mathbb{N} \times \mathbb{N}$ is finite,

$J_p \subseteq \mathcal{A} \times \mathbb{N}$ is finite,

$H^{(p)}_{mi} \in \mathcal{U}$ for every $(m,i) \in I_p$,

$A \cap H^{(p)}_{mi} \ne \emptyset$ whenever $(A,m) \in J_p$ and $(m,i) \in I_p$,

$A \setminus \overline{\bigcup\{U_{mi} : i \in I_p[\{m\}]\}}$ is infinite whenever $(A,m) \in J_p$.

Order P by saying that $p \le q$ if $I_p \subseteq I_q$, $J_p \subseteq J_q$ and $H^{(p)}_{mi} = H^{(q)}_{mi}$ for every $(m,i) \in I_p$. Then P is σ-centered upwards, because if p and q agree on their first coordinate we can just take the union of their second coordinates to get a common upper bound in P.

For $A \in \mathcal{A}$ and $(m,i) \in \mathbb{N}\times\mathbb{N}$ set $Q_A = \{p : p \in P, \exists\ m \in \mathbb{N}, (A,m) \in J_p\}$ and $Q'_{mi} = \{p : p \in P, (m,i) \in I_p\}$. Then every Q_A is cofinal with P (because if $(m,i) \notin I_p$ for any i, we can just add (A,m) to J_p) and every Q'_{mi} is cofinal with P (because if $(m,i) \notin I_p$, we can add a new $H^{(q)}_{mi}$ by just taking a bit of care over finitely many sets in $\mathcal{A}$). Because $\#(\mathcal{A}) < \mathfrak{p}$, there is an upwards-directed $R \subseteq P$ meeting every Q_A and every Q'_{mi}. If we now set $H_{mi}=H^{(p)}_{mi}$ whenever $p\in R$ and $(m,i)\in I_p$, we shall have an appropriate family $\langle H_{mi}\rangle_{m,i\in\mathbb{N}}$. **Q**

(b) Now there is a family $\langle g_\xi\rangle_{\xi<\operatorname{cov}(\mathcal{M})}$ in $\mathbb{N}^{\mathbb{N}}$ such that for every $h\in\mathbb{N}^{\mathbb{N}}$ there is a $\xi<\kappa$ such that $h(i)\ne g_\xi(i)$ for every i (Bartoszyński 87, or Fremlin 05?, 521Q). Let $r:\mathbb{N}\to\mathbb{N}$ be any function such that $\{j:r(j)=m\}$ is infinite for every $m\in\mathbb{N}$. For $n\in\mathbb{N}$ and $\xi<\operatorname{cov}(\mathcal{M})$ set $G_{n\xi}=\bigcup_{j\ge n}H_{r(j),g_\xi(j)}$. Then for any $A\in\mathcal{A}$ there is an $m\in\mathbb{N}$ such that $A\cap H_{mi}\ne\emptyset$ for every $i\in\mathbb{N}$, and now there is a $j\ge n$ such that $r(j)=m$, so that $A\cap G_{n\xi}\supseteq A\cap H_{m,g_\xi(j)}\ne\emptyset$.

Take any $x\in\mathbb{R}$. For each $j\in\mathbb{N}$ let $h(j)\in\mathbb{N}$ be such that $x\notin H_{r(j),i}$ for any $i\ne h(j)$. Then there is a $\xi<\kappa$ such that $\{j:h(j)=g_\xi(j)\}$ is finite; suppose that $n\in\mathbb{N}$ is such that $h(j)\ne g_\xi(j)$ for any $j\ge n$. Then $x\notin H_{r(j),g_\xi(j)}$ for any $j\ge n$, so $x\notin G_{n\xi}$. As x is arbitrary, $\bigcap_{n\in\mathbb{N},\xi<\operatorname{cov}(\mathcal{M})}G_{n\xi}$ is empty; so that $\langle G_{n\xi}\rangle_{n\in\mathbb{N},\xi<\operatorname{cov}(\mathcal{M})}$, suitably re-indexed, will serve.

4H Proposition (Fuchino Geschke & Soukup 01, Proposition 2.3.) Suppose that $\mathrm{FN}(\mathcal{P}\mathbb{N})=\mathfrak{p}$. Then there is no cardinal κ such that $\operatorname{cov}(\mathcal{M})\le\kappa<\mathfrak{c}$ and $\operatorname{cf}([\kappa]^{<\mathfrak{p}})\le\kappa$.

proof ? Suppose, if possible, otherwise. Let $\mathcal{U}$ be a countable base for the topology of $\mathbb{R}$, and $f:\mathcal{P}\mathcal{U}\to[\mathcal{P}\mathcal{U}]^{<\mathfrak{p}}$ a Freese-Nation function. For $\mathcal{V}\subseteq\mathcal{U}$ set $F_{\mathcal{V}}=\mathbb{R}\setminus\bigcup\mathcal{V}$. For any set N, write $\mathcal{F}_N$ for the family of infinite subsets of $\mathbb{R}$ which belong to N. Then we can find a set M, of cardinal κ, such that

whenever $\mathcal{V}\in M\cap\mathcal{P}\mathcal{U}$ then $F_{\mathcal{V}}\in M$,
whenever $A\in[M]^{<\mathfrak{p}}$ there is an $N\in M\cap[M]^{<\mathfrak{p}}$ such that $A\subseteq N$,
whenever $G\in M$ is an open subset of $\mathbb{R}$, then $\{U:U\in\mathcal{U},\,U\subseteq G\}\in M$,
whenever $\mathcal{V}\in M\cap\mathcal{P}\mathcal{U}$ then $f(\mathcal{V})\subseteq M$,
whenever $N\in M$ and $\#(N)<\mathfrak{p}$ then

$$\bigcap\{G:G\in M\text{ is an open subset of }\mathbb{R},\,G\cap F\ne\emptyset\text{ for every }F\in\mathcal{F}_N\}=\emptyset.$$

(Use Lemma 4G to see that the last is achievable.)

Because $\kappa<\mathfrak{c}$, there is an $x\in\mathbb{R}\setminus M$. Set $\mathcal{V}=\{V:V\in\mathcal{U},\,x\notin V\}$. Set $\mathcal{F}=M\cap\{F_{\mathcal{W}}:\mathcal{W}\in f(\mathcal{V})\}$. Then $\mathcal{F}$ is a family of closed subsets of $\mathbb{R}$ belonging to M, and $\#(\mathcal{F})<\mathfrak{p}$, so there is an $N\in M\cap[M]^{<\mathfrak{p}}$ such that $\mathcal{F}\subseteq N$, and there must be an open set $G\subseteq\mathbb{R}$ such that $G\cap F\ne\emptyset$ for every $F\in\mathcal{F}_N$ and $G\in\mathcal{M}$ and $x\notin G$. Set $\mathcal{V}'=\{U:U\in\mathcal{U},\,V'\subseteq G\}$, so that $\mathcal{V}'\in M$ and $\mathcal{V}'\subseteq\mathcal{V}$. There is therefore a $\mathcal{W}\in f(\mathcal{V})\cap f(\mathcal{V}')$ such that $\mathcal{V}'\subseteq\mathcal{W}\subseteq\mathcal{V}$. Now $\mathcal{W}\in M$ and $x\in F_{\mathcal{W}}$. Since $x\notin M$, $F_{\mathcal{W}}$ must be infinite, while $F_{\mathcal{W}}\in\mathcal{F}\subseteq N$. But this means that $F_{\mathcal{W}}\in\mathcal{F}_N$, while $F_{\mathcal{W}}\cap G=\emptyset$, contrary to the choice of G. **X**

5. Liftings

5A Theorem Let P and Q be partially ordered sets. Set $\kappa=\mathrm{FN}^*(Q)$. Suppose that $h:P\to Q$ is a function with the property that whenever $q\in Q$ and $A\in[P]^{<\kappa}$ there is a $p^*\in P$ such that $h(p^*)=q$ and $p\le p^*$ whenever $p\in A$ and $h(p)\le q$, while $p\ge p^*$ whenever $p\in A$ and $h(p)\ge q$.

Then there is an order-preserving function $g:Q\to P$ such that hg is the identity on Q.

proof By Proposition 1I, there is an enumeration $\langle q_\xi\rangle_{\xi<\lambda}$ of Q such that, for each $\xi<\lambda$, there are sets A_ξ, $A'_\xi\subseteq\xi$ such that $\#(A_\xi\cup A'_\xi)<\kappa$ and $\{q_\eta:\eta\in A_\xi\}$ is cofinal with $\{q_\eta:\eta<\xi,\,q_\eta<q_\xi\}$), while $\{q_\eta:\eta\in A'_\xi\}$ is coinitial with $\{q_\eta:\eta<\xi,\,q_\eta>q_\xi\}$. Choose $\langle p_\xi\rangle_{\xi<\lambda}$ inductively in such a way that, for each ξ, $h(p_\xi)=q_\xi$ and $p_\eta\le p_\xi$ for every $\eta\in A_\xi$, while $p_\eta\ge p_\xi$ for every $\eta\in A'_\xi$.

Setting $g(q_\xi)=p_\xi$ for every $\xi<\lambda$, $g:Q\to P$ is such that hg is the identity. Now g is order-preserving. **P** I show by induction on $\max(\xi,\eta)$ that if ξ, $\eta<\lambda$ and $q_\xi\le q_\eta$ then $p_\xi\le p_\eta$. For $\xi=\eta=0$ this is trivial. For the inductive step to ζ, the case $\xi=\eta=\zeta$ is again trivial. If $\xi<\eta=\zeta$, then there must be a $\xi'\in A_\eta$ such that $q_\xi\le q_{\xi'}\le q_\zeta$. By the inductive hypothesis, $p_\xi\le p_{\xi'}$, while $p_{\xi'}\le p_\eta$ by the construction of p_η. So $p_\xi\le p_\eta$. The same argument works if $\eta<\xi=\zeta$. **Q**

So we have a suitable function g.

6. Boolean algebras

6A Proposition (Koppelberg?) Let $\mathfrak{A}$ be a Boolean algebra such that $\#(\mathfrak{A})=\lambda\le\mathrm{FN}^*(\mathfrak{A})^+$. Then there is an enumeration $\langle a_\xi\rangle_{\xi<\lambda}$ in $\mathfrak{A}$ such that, writing $\mathfrak{A}_\xi$ for the subalgebra of $\mathfrak{A}$ generated by $\{a_\eta:\eta<\xi\}$, the Freese-Nation index of $\mathfrak{A}_\xi$ in $\mathfrak{A}$ is never greater than $\mathrm{FN}^*(\mathfrak{A})$.

proof Set $\kappa = \mathrm{FN}^*(\mathfrak{A})$. If $\lambda \le \kappa$ we can take any enumeration of $\mathfrak{A}$. So suppose that $\lambda = \kappa^+$. Let $f : \mathfrak{A} \to [\mathfrak{A}]^{<\mathrm{FN}(\mathfrak{A})}$ be a Freese-Nation function. Let $\langle b_\xi \rangle_{\xi<\lambda}$ be any enumeration of $\mathfrak{A}$ and for each $\xi < \lambda$ let $\mathfrak{B}_\xi$ be the smallest subalgebra of $\mathfrak{A}$ including $\{a_\eta : \eta < \xi\}$ and such that $f(b) \subseteq \mathfrak{B}_\xi$ for every $b \in B_\xi$. Then $\#(\mathfrak{B}_\xi) \le \kappa < \lambda$ for every ξ. So we can enumerate $\mathfrak{A}$ as $\langle a_\xi \rangle_{\xi<\lambda}$ in such a way that every $\mathfrak{B}_\theta$ is of the form $\{a_\eta : \eta < \zeta_\theta\}$ for some $\zeta_\theta < \lambda$, and $\zeta_{\theta+1} \le \zeta_\theta + \kappa$ for every θ.

Now take any $\xi < \lambda$ and $d \in \mathfrak{A}$. Then $a_\xi \in \mathfrak{B}_{\theta+1} \setminus \mathfrak{B}_\theta$ for some $\theta < \lambda$. Then $\#(\xi \setminus \zeta_\theta) < \kappa$. Consider $A = \{a : a \in \mathfrak{A}_\xi,\, a \subseteq d\}$. For $I \in [\xi \setminus \zeta_\theta]^{<\omega}$, let $\mathfrak{C}_I$ be the algebra generated by $\mathfrak{B}_\theta \cup \{a_\eta : \eta \in I\}$, and set $A_I = \{a : a \in \mathfrak{C}_I,\, a \subseteq d\}$. Then Then $A = \bigcup\{A_I : I \in [\xi \setminus \zeta_\theta]^{<\omega}\}$. Now $\mathrm{cf}\, A_I < \kappa$. **P** Let $c_0, \dots, c_n$ be the atoms of the algebra generated by $\{a_\eta : \eta \in I\}$. For $i \le n$, set $B_i = \{b : b \in \mathfrak{B}_\theta,\, b \subseteq d \cup (1 \setminus c_i)\}$. Then every member of $\mathfrak{C}_I$ is of the form $\sup_{i \le n} b_i \cap c_i$ where $b_i \in \mathfrak{B}_\theta$ for each i. So every member of A_I is of the form $\sup_{i \le n} b_i \cap c_i$ where $b_i \in B_i$ for each i. But the Freese-Nation index of $\mathfrak{B}_\theta$ in $\mathfrak{A}$ is at most κ, by Lemma 1L, so $\mathrm{cf}\, B_i < \kappa$ for each i and $\mathrm{cf}\, A_I < \kappa$. It follows at once that $\mathrm{cf}\, A < \kappa$.

Similarly, $\mathrm{ci}\{a : a \in \mathfrak{A}_\xi,\, a \supseteq d\} < \kappa$; as d is arbitrary, the Freese-Nation index of $\mathfrak{A}_\xi$ in $\mathfrak{A}$ is at most κ. As ξ is arbitrary, we have a suitable family $\langle a_\xi \rangle_{\xi<\lambda}$.

6B Proposition (Koppelberg) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\kappa = \mathrm{FN}^*(\mathfrak{A})$. Suppose that $\#(\mathfrak{A}) \le \kappa^+$, and that whenever $B, B' \in [\mathfrak{B}]^\kappa$ are such that $b \subseteq b'$ for every $b \in B$ and $b' \in B'$, there is a $c \in \mathfrak{B}$ such that $b \subseteq c \subseteq c'$ for all $b \in B$, $b' \in B'$. Let $\pi : \mathfrak{A} \to \mathfrak{B}$ be a Boolean homomorphism such that $\mathrm{add}\, I \ge \kappa$, where $I = \{b : \pi b = 0\}$. Then there is a Boolean homomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ such that $\pi\phi$ is the identity on $\mathfrak{A}$.

proof Let I be the ideal $\{b : \pi b = 0\}$. Set $\lambda = \#(\mathfrak{A})$ and let $\langle a_\xi \rangle_{\xi<\lambda}$ be an enumeration of $\mathfrak{A}$ as in Proposition 6A. Let $\mathfrak{A}_\xi$ be the subalgebra of $\mathfrak{A}$ generated by $\{a_\eta : \eta < \xi\}$ for each $\xi \le \lambda$, and define $\phi_\xi : \mathfrak{A}_\xi \to \mathfrak{B}$ inductively. At the inductive step to $\xi + 1$ we need to choose $b_\xi = \phi(a_\xi)$ in such a way that $\pi(b_\xi) = a_\xi$ and $\phi(a) \subseteq b_\xi \subseteq \phi(a')$ whenever $a, a' \in \mathfrak{A}_\xi$ and $a \subseteq a_\xi \subseteq a'$. Let A be a cofinal subset of $\mathfrak{A}_\xi \cap [0, a_\xi]$ and A' a coinitial subset of $\mathfrak{A}_\xi \cap [a, 1]$ both of cardinals less than κ. Take $d \in \mathfrak{B}$ such that $\pi(d) = a_\xi$; then $\phi_\xi(a) \setminus d$ and $d \setminus \phi_\xi(a')$ belong to I for every $a \in A$ and $a' \in A'$. So there is an $e \in I$ such that $\phi_\xi(a) \setminus d \subseteq e$ and $d \setminus \phi_\xi(a') \subseteq e$ for every $a \in A$, $a' \in A'$. There is also a $d' \in \mathfrak{B}$ such that $\phi_\xi(a) \subseteq d' \subseteq \phi_\xi(a')$ for every $a \in A$ and $a' \in A'$. If we set

$$b_\xi = (d \setminus e) \cup (e \cap d'),$$

this works, and the induction proceeds.

6C The square principle For an infinite cardinal κ, let $\square_\kappa$ be the statement

there is a family $\langle C_\alpha \rangle_{\alpha<\kappa^+}$ of sets such that

(i) for every $\alpha < \kappa^+$, $C_\alpha \subseteq \alpha$ is a club set in α of order type at most κ;

(ii) whenever $\alpha < \kappa^+$ and $\delta < \alpha$ is such that $\delta = \sup(\delta \cap C_\alpha)$, then $C_\delta = \delta \cap C_\alpha$.

(See Devlin 84, §IV.5.)

6D Lemma Suppose that κ is an uncountable cardinal such that $\square_\kappa$ is true and $\mathrm{cf}([\kappa]^{\le\omega}) \le \kappa^+$ and there is a strictly increasing sequence $\langle \kappa_n \rangle_{n\in\mathbb{N}}$ of infinite cardinals, with supremum κ, such that $\mathrm{cf}([\kappa_n]^{\le\omega}) \le \kappa_n$ for every $n \in \mathbb{N}$. Then there are families $\langle M_{\alpha n} \rangle_{\alpha<\kappa^+, n\in\mathbb{N}}$, $\langle M_\alpha \rangle_{\alpha<\kappa^+}$ of sets and a function sk such that

(i) $\#(M_{\alpha n}) < \kappa$ for every $\alpha < \kappa^+$, $n \in \mathbb{N}$;

(ii) $\langle M_{\alpha n} \rangle_{n\in\mathbb{N}}$ is non-decreasing for each $\alpha < \kappa^+$;

(iii) $\langle M_\alpha \rangle_{\alpha<\kappa^+}$ is a non-decreasing family, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for every limit ordinal $\alpha < \kappa^+$, and $\kappa^+ \subseteq \bigcup_{\alpha<\kappa^+} M_\alpha$;

(iv) if $\alpha < \kappa^+$ has uncountable cofinality, $M_\alpha = \bigcup_{n\in\mathbb{N}} M_{\alpha n}$;

(v) $X \subseteq \mathrm{sk}(X)$ for every set X;

(vi) $\mathrm{sk}(X)$ is countable whenever X is countable;

(vii) $A \subseteq \mathrm{sk}(X)$ whenever $A \in \mathrm{sk}(X)$ is countable;

(viii) for every $\alpha < \kappa^+$ of uncountable cofinality there is an $m \in \mathbb{N}$ such that whenever $n \ge m$ and $A \subseteq M_{\alpha n}$ is countable there is a countable set $D \in M_{\alpha n}$ such that $A \subseteq \mathrm{sk}(D)$;

(ix) $\bigcup_{\alpha<\kappa^+} M_\alpha \cap [\kappa]^{\le\omega}$ is cofinal with $[\kappa]^{\le\omega}$.

proof (a) Take $\langle C_\alpha \rangle_{\alpha<\kappa^+}$ from 6C. For each $\alpha < \kappa^+$ such that $\mathrm{cf}\,\alpha > \omega$, take $C'_\alpha \subseteq C_\alpha$ to be a club set in α of order type $\mathrm{cf}\,\alpha$ such that $\delta = \sup(C_\alpha \cap \delta)$ for every $\delta \in C'_\alpha$; for $\alpha < \kappa^+$ such that $\#(C_\alpha) = \kappa$, set

$C'_\alpha = \emptyset$; for other $\alpha < \kappa^+$ take $C'_\alpha \supseteq C_\alpha$ to be a set of cardinal κ_n for some n. Then whenever $\alpha < \kappa^+$ has uncountable cofinality, there is an $m \in \mathbb{N}$ such that

$$[C'_\alpha]^{\le\omega} \subseteq \bigcup\{[C'_\delta]^{\le\omega} : \delta \in C'_\alpha,\ \#(C'_\delta) = \kappa_m\}.$$

P For each $\delta \in C'_\alpha$ of countable cofinality,

$$\mathrm{otp}(C_\delta) = \mathrm{otp}(\delta \cap C_\alpha) < \mathrm{otp}(C_\alpha) \le \kappa,$$

so $\#(C_\delta) < \kappa$ and there is some $n \in \mathbb{N}$ such that $\#(C'_\delta) = \kappa_n$. Because $\{\delta : \delta \in C'_\alpha, \mathrm{cf}(\delta) = \omega\}$ is cofinal with α, there is some $m \in \mathbb{N}$ such that

$$D = \{\delta : \delta \in C'_\alpha,\ \mathrm{cf}\,\delta = \omega,\ \#(C'_\delta) = \kappa_m\}$$

is cofinal with α. If $A \in [C'_\alpha]^{\le\omega}$, take $\delta \in D$ such that $A \subseteq \delta$. Then

$$A \subseteq \delta \cap C_\alpha = C_\delta \subseteq C'_\delta.\ \mathbf{Q}$$

(b) For each non-zero $\alpha < \kappa^+$, fix on a surjective function $f_\alpha : \kappa \to \alpha$. For each $n \in \mathbb{N}$, let $g_n : \kappa_n \to [\kappa_n]^{\le\omega}$ be such that $g_n[\kappa_n]$ is cofinal with $[\kappa_n]^{\le\omega}$. For each $\delta < \kappa^+$, let $h_\delta : \#(C'_\delta) \to C'_\delta$ be a bijection. Finally, let $e : \kappa^+ \to [\kappa]^{\le\omega}$ be such that $e[\kappa^+]$ is cofinal with $[\kappa]^{\le\omega}$. Now, for any set X, write $\mathrm{sk}(X)$ for the smallest set including X and such that

$f_\alpha(\xi) \in \mathrm{sk}(X)$ whenever $0 < \alpha < \kappa^+$, $\xi < \kappa$ and $\alpha, \xi \in \mathrm{sk}(X)$,
$g_n(\xi) \in \mathrm{sk}(X)$ whenever $n \in \mathbb{N}$ and $\xi \in \kappa_n \cap \mathrm{sk}(X)$,
$h_\delta[A] \in \mathrm{sk}(X)$ whenever $\delta \in \mathrm{sk}(X) \cap \kappa^+$ and $A \in \mathrm{sk}(X)$,
$e(\alpha) \in \mathrm{sk}(X)$ whenever $\alpha \in \mathrm{sk}(X) \cap \kappa^+$,
$A \cup B \in \mathrm{sk}(X)$ whenever $A, B \in \mathrm{sk}(X)$,
$A \subseteq \mathrm{sk}(X)$ whenever $A \in \mathrm{sk}(X)$ is countable.

Of course we always have $\mathrm{sk}(\mathrm{sk}(X)) = \mathrm{sk}(X)$ and $\#(\mathrm{sk}(X)) \le \max(\omega, \#(X))$.

(c) For each $\alpha < \kappa$ and $n \in \mathbb{N}$, set $M_{\alpha n} = \mathrm{sk}(\kappa_n \cup C'_\alpha)$ and $M_\alpha = \mathrm{sk}(\kappa \cup \alpha)$. Then of course

$$\#(M_{\alpha n}) \le \max(\omega, \kappa_n, \#(C'_\alpha)) < \kappa.$$

Also $\langle M_\alpha\rangle_{\alpha<\kappa^+}$ is non-decreasing and $M_\alpha = \bigcup_{\beta<\kappa} M_\beta$ whenever $\alpha < \kappa$ is a non-zero limit ordinal, while $\kappa^+ \subseteq \bigcup_{\alpha<\kappa^+} M_\alpha$.

(d) $M_\alpha = \bigcup_{n\in\mathbb{N}} M_{\alpha n}$ whenever $\alpha < \kappa^+$ has uncountable cofinality. **P** Of course $M_{\alpha n} \subseteq M_\alpha$ for every n just because $\mathrm{sk}(X) \subseteq \mathrm{sk}(Y)$ whenever $X \subseteq Y$. On the other hand, if $\beta < \alpha$, let $\delta \in C'_\alpha$ be such that $\beta < \delta$, and $\xi < \kappa$ such that $f_\delta(\xi) = \beta$; then if $n \in \mathbb{N}$ is such that $\xi < \kappa_n$, $\beta \in M_{\alpha n}$. So $\bigcup_{n\in\mathbb{N}} M_{\alpha_n} \supseteq \alpha$. Moreover,

$$\kappa = \bigcup_{n\in\mathbb{N}} \kappa_n \subseteq \bigcup_{n\in\mathbb{N}} M_{\alpha n}.$$

So $\alpha \cup \kappa \subseteq \bigcup_{n\in\mathbb{N}} M_{\alpha n}$ and M_α must be exactly $\bigcup_{n\in\mathbb{N}} M_{\alpha n}$. **Q**

(e) Now suppose that $\alpha < \kappa^+$ has uncountable cofinality. Let $m \in \mathbb{N}$ be such that

$$[C'_\alpha]^{\le\omega} \subseteq \bigcup\{[C'_\delta]^{\le\omega} : \delta \in C'_\alpha,\ \#(C'_\delta) = \kappa_m\}.$$

((a) above). Suppose that $n \ge m$ and $A \subseteq M_{\alpha n}$ is countable. Then there must be a countable set $C \subseteq \kappa_n \cup C'_\alpha$ such that $A \subseteq \mathrm{sk}(C)$. Let $\delta \in C'_\alpha$ be such that $\mathrm{cf}\,\delta = \omega$ and $\#(C'_\delta) = \kappa_m$ and $C \cap C'_\alpha \subseteq C'_\delta$. Then there is a $\xi < \kappa_n$ such that

$$g_n(\xi) \supseteq (C \cap \kappa_n) \cup h_\delta^{-1}[C \cap C'_\alpha].$$

Now ξ and δ belong to $M_{\alpha n}$, so $g_n(\xi)$ and $h_\delta[g_n(\xi)]$ and $D = g_n(\xi) \cup h_\delta[g_n(\xi)]$ all belong to $M_{\alpha n}$, and are countable. But $C \subseteq D$, so $A \subseteq \mathrm{sk}(D)$, as required by (viii).

(f) Finally, if $A \subseteq \kappa$ is countable, there is a $\beta < \kappa^+$ such that $e(\beta) \supseteq A$, and now $e(\beta) \in M_{\beta+1}$.

6E Theorem (Fuchino & Soukup 97) Suppose that

(α) $\mathrm{cf}([\lambda]^{\le\omega}) \le \lambda$ for every cardinal λ of uncountable cofinality,
(β) $\square_\lambda$ is true for every uncountable cardinal λ of countable cofinality.

Let $\mathfrak{A}$ be a ccc Dedekind complete Boolean algebra, and κ a regular infinite cardinal such that $\mathrm{FN}(\mathfrak{B}) \le \kappa$ for every countably generated order-closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$. Then $\mathrm{FN}(\mathfrak{A}) \le \kappa$.

proof Induce on the Maharam type $\tau(\mathfrak{A})$ of $\mathfrak{A}$.

(a) If $\tau(\mathfrak{A}) \le \omega$ the result is trivial.

(b) For the inductive step to $\tau(\mathfrak{A}) = \lambda$, where λ is an infinite cardinal of uncountable cofinality, let $\langle a_\xi \rangle_{\xi<\lambda}$ enumerate a τ-generating subset of $\mathfrak{A}$. For each $\beta < \lambda$, let $\mathfrak{B}_\beta$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $\{a_\xi : \xi < \beta\}$, and for $\alpha \le \lambda$ set $\mathfrak{A}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{B}_\beta$. By the inductive hypothesis, $\mathrm{FN}^*(\mathfrak{B}_\beta) \le \kappa$ for every $\beta < \lambda$. Also, for $\alpha < \kappa$, either α has uncountable cofinality, in which case (because $\mathfrak{A}$ is ccc) $\mathfrak{A}_\alpha = \mathfrak{B}_\alpha$ is order-closed, or α has countable cofinality, in which case $\mathfrak{A}_\alpha$ is a countable union of order-closed subalgebras. In either case, the Freese-Nation index of $\mathfrak{A}_\alpha$ in $\mathfrak{A}_{\alpha+1}$ is countable. By Lemma 1G, $\mathrm{FN}(\mathfrak{A}) \le \kappa$.

(c)(i) For the inductive step to $\tau(\mathfrak{A}) = \lambda$, where λ is an uncountable cardinal of countable cofinality, let $\langle \lambda_n \rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence of uncountable successor cardinals with supremum λ. By hypothesis (α), $\mathrm{cf}([\lambda_n]^{\le\omega}) \le \lambda_n$ for each n, and also $\mathrm{cf}([\lambda]^{\le\omega}) \le \mathrm{cf}([\lambda^+]^{\le\omega} \le \lambda^+$. So we may use the method of Lemma 6D to construct families $\langle M_{\alpha n} \rangle_{\alpha<\lambda^+, n\in\mathbb{N}}$, $\langle M_\alpha \rangle_{\alpha<\lambda^+}$ and sk as described there. Enumerate a τ-generating set in $\mathfrak{A}$ as $\langle a_\xi \rangle_{\xi<\lambda}$, and for any set X write $\mathfrak{B}_X$ for the order-closed subalgebra of $\mathfrak{A}$ generated by $\{a_\xi : \xi \in X \cap \lambda\}$. For each $\alpha < \lambda^+$ set $Q_\alpha = \bigcup\{\mathfrak{B}_{\mathrm{sk}(D)} : D \in M_\alpha \text{ is countable}\}$.

(ii) If $\alpha < \lambda^+$ has uncountable cofinality, then Q_α is the union of a non-decreasing sequence of order-closed subalgebras of $\mathfrak{A}$ of Maharam type less than λ. **P** By (iv) and (viii) of 6D, $M_\alpha = \bigcup_{n\in\mathbb{N}} M_{\alpha n}$, and there is an $m \in \mathbb{N}$ such that whenever $n \ge m$ and $\mathcal{D} \subseteq M_{\alpha n}$ is countable there is a countable set $F \in M_{\alpha n}$ such that $\mathcal{D} \subseteq \mathrm{sk}(F)$. For each $n \ge m$, set $\mathfrak{C}_n = \bigcup\{\mathfrak{B}_{\mathrm{sk}(D)} : D \in M_{\alpha n} \text{ is countable}\}$. Then for any countable set $C \subseteq \mathfrak{C}_n$, there is a countable set $\mathcal{D}$ of countable sets belonging to $M_{\alpha n}$ such that $C \subseteq \bigcup_{D\in\mathcal{D}} \mathfrak{B}_{\mathrm{sk}(D)}$. But now there is a countable set $F \in M_{\alpha n}$ such that $\mathcal{D} \subseteq \mathrm{sk}(F)$; by (vii) of 6D, $\bigcup \mathcal{D} \subseteq \mathrm{sk}(F)$ and $\mathfrak{B}_{\mathrm{sk}(D)} \subseteq \mathfrak{B}_{\mathrm{sk}(F)}$ for every $D \in \mathcal{D}$. But this means that $C \subseteq \mathfrak{B}_{\mathrm{sk}}(F) \subseteq \mathfrak{C}_n$, while $\mathfrak{B}_{\mathrm{sk}}(F)$ is an order-closed subalgebra of $\mathfrak{A}$. Because $\mathfrak{A}$ is ccc, this is enough to show that $\mathfrak{C}_n$ is an order-closed subalgebra of $\mathfrak{A}$; and because $M_\alpha = \bigcup_{n\in\mathbb{N}} M_{\alpha n}$ (6D(iv)), $Q_\alpha = \bigcup_{n\in\mathbb{N}} \mathfrak{C}_n$. Also $\mathfrak{C}_n$ is τ-generated by

$$\{a_\eta : \text{there is a countable } D \in M_{\alpha n} \text{ such that } \eta \in \mathrm{sk}(D) \cap \lambda^+\},$$

so $\tau(\mathfrak{C}_n) \le \max(\omega, \#(M_{\alpha n})) < \lambda$. **Q**

It follows at once that the Freese-Nation index of Q_α in $\mathfrak{A}$ is countable, and from the inductive hypothesis we see also that $\mathrm{FN}^*(\mathfrak{C}_n) \le \kappa$ for every $n \ge m$, so that (using Lemma 1G, as usual) $\mathrm{FN}^*(Q_\alpha) \le \kappa$.

(iii) If $\alpha < \lambda^+$ is the union of a sequence $\langle \alpha_n \rangle_{n\in\mathbb{N}}$ of ordinals of uncountable cofinality, then $M_\alpha = \bigcup_{n\in\mathbb{N}} M_{\alpha_n}$ so $Q_\alpha = \bigcup_{n\in\mathbb{N}} Q_{\alpha_n}$ is again a countable union of order-closed subalgebras of $\mathfrak{A}$, and the Freese-Nation index of Q_α in $\mathfrak{A}$ is at most ω. Moreover, because $\mathrm{FN}(Q_{\alpha_n}) \le \kappa$ for each n, $\mathrm{FN}(Q_\alpha) \le \kappa$.

(iv) If $a \in \mathfrak{A}$, there is a countable set $D \subseteq \lambda$ such that $a \in \mathfrak{B}_D$. But now there is an $\alpha < \lambda^+$, of uncountable cofinality, such that $D \subseteq D'$ for some countable $D' \in M_\alpha$, and $a \in Q_\alpha$.

(v) Let $F \subseteq \lambda^+$ be the set of ordinals which are either of uncountable cofinality, or the union of a sequence of such ordinals; so that F is a club set in λ^+, and Q_α has countable Freese-Nation index in $\mathfrak{A}$ for every $\alpha \in F$. By (iv), $\bigcup_{\alpha\in F} Q_\alpha = \mathfrak{A}$. So if we enumerate F in ascending order as $\langle \alpha_\xi \rangle_{\xi<\lambda^+}$ and set $P_\xi = Q_{\alpha_\xi}$ for each ξ, $P_{\lambda^+} = \mathfrak{A}$ then $\langle P_\xi \rangle_{\xi\le\lambda^+}$ satisfies the conditions of Lemma 1G, so $\mathrm{FN}(\mathfrak{A}) \le \kappa$, and the induction proceeds.

6F Proposition Suppose that λ is an infinite cardinal and that $\mathfrak{A}_\lambda$ is the measure algebra of the usual measure on $[0,1]^\lambda$. Set $\kappa = \mathrm{FN}(\mathfrak{A}_\lambda)$. Then $\kappa \le \mathfrak{c}^+$ and we have a family $\mathcal{V} \subseteq [\lambda]^{\le\mathfrak{c}}$, cofinal with $[\lambda]^{\le\mathfrak{c}}$, such that $\#(\{A \cap V : V \in \mathcal{V}\}) < \kappa$ for every countable set $A \subseteq \lambda$.

proof (a) For $I \subseteq \lambda$, let $\mathfrak{B}_I$ be the closed subalgebra of $\mathfrak{A}_\lambda$ consisting of those $a \in \mathfrak{A}_\lambda$ expressible in the form $E^\bullet$ for some $E \subseteq \{0,1\}^\lambda$ determined by coordinates in I. For $a \in \mathfrak{A}_\lambda$, let I_a be the smallest subset of λ such that $a \in \mathfrak{B}_I$ (Fremlin 02, 325M); of course I_a is always countable.

(b) For each $a \in \mathfrak{A}_\lambda$, set

$$g(a) = \{b : I_b \subseteq I_a\}.$$

Then $\#(g(a)) \le \mathfrak{c}$. If $a \subseteq b$, then there is a $c \in \mathfrak{A}_\lambda$ such that $a \subseteq c \subseteq b$ and $I_c \subseteq I_a \cap I_b$. (Take $c = \mathrm{upr}(a, \mathfrak{B}_{I_b})$.) So g is a Freese-Nation function. This shows that $\kappa \le \mathfrak{c}^+$.

(c) Now let $f : \mathfrak{A}_\lambda \to [\mathfrak{A}_\lambda]^{<\kappa}$ be a Freese-Nation function. Let $\mathcal{V}$ be the family of those sets $V \in [\lambda]^{\le\mathfrak{c}}$ such that $f(a) \subseteq \mathfrak{B}_V$ for every $a \in \mathfrak{B}_V$; because $\#(f(a)) \le \mathfrak{c}$ for every a, $\mathcal{V}$ is cofinal with $[\lambda]^{\le\mathfrak{c}}$.

Take any countable $A \subseteq \lambda$. If A is finite, then of course $\{V \cap A : V \in \mathcal{V}\}$ has cardinal less than κ. Otherwise, enumerate A as $\langle\xi_n\rangle_{n\in\mathbb{N}}$. For each n, let $e_n \in \mathfrak{B}_{\{\xi_n\}}$ be such that $\bar\mu e_n = 2^{-n-2}$, and consider $a = \sup_{n\in\mathbb{N}} e_n$. Then $\#(f(a)) < \kappa$. If $V \in \mathcal{V}$, set $b_V = \sup_{n\in\mathbb{N},\xi_n\in V} e_n$, $b'_V = \sup_{n\in\mathbb{N},\xi_n\notin V} e_n$; then $I_{b_V} = A \cap V$ and $I_{b'_V} = A \setminus V$ and $a = b_V \cup b'_V$, so b_V is the largest member of $\mathfrak{B}_V$ included in a. (If $c \in \mathfrak{B}_V$ and $c \subseteq a$, then $c \setminus b_V$ is included in b'_V, while they are stochastically independent and $b'_V \ne 1$, so $c \setminus b_V = 0$.) There must be a $c \in f(b_V) \cap f(a)$ such that $b_V \subseteq c \subseteq a$; but in this case $c \in \mathfrak{B}_V$ so $c = b_V$. So $\{V \cap A : V \in \mathcal{V}\} \subseteq \{I_b : b \in f(a)\}$ has cardinal less than κ, as required.

6G Theorem (compare Fuchino Geschke Shelah & Soukup 01, Theorem 6.2) Suppose that GCH and CC are true, where CC is $(\omega_{\omega+1}, \omega_\omega) \twoheadrightarrow (\omega_1, \omega)$. Let $\mathfrak{A}$ be a probability algebra with a homogeneous principal ideal with Maharam type at least ω_ω. Then $\mathrm{FN}(\mathfrak{A}) = \mathfrak{c}^+ = \omega_2$.

proof (a) Because $\mathfrak{A}$ is certainly embeddable as a closed subalgebra of the measure algebra of the usual measure on some power of $[0,1]$, $\mathrm{FN}(\mathfrak{A}) \le \mathfrak{c}^+ = \omega_2$. Set $\lambda = \omega_\omega$ and let $\mathfrak{A}_\lambda$ be the measure algebra of the usual measure on $[0,1]^\lambda$, and $\kappa = \mathrm{FN}(\mathfrak{A}_\lambda)$. Then $\mathfrak{A}_\lambda$ is embeddable as an order-closed sublattice of $\mathrm{FN}(\mathfrak{A})$ so $\mathrm{FN}(\mathfrak{A}) \ge \kappa$.

(b) ? Suppose, if possible, that $\kappa \le \omega_1$. Let $\mathcal{V} \subseteq [\lambda]^{\le\mathfrak{c}}$ be as in 6F. Because GCH is true, $[\lambda]^{\le\omega}$ and $[\lambda]^{\le\mathfrak{c}}$ both have cardinal $\lambda^+ = \omega_{\omega+1}$; let $\langle I_\xi\rangle_{\xi<\lambda^+}$ and $\langle V_\xi\rangle_{\xi<\lambda^+}$ be enumerations of $[\lambda]^{\le\omega}$ and $\mathcal{V}$. Interpret these as subsets R, S of $\lambda^+ \times \lambda$. There is also a subset T of $\lambda^+ \times \lambda^+ \times \lambda$ such that whenever $\xi < \lambda^+$ the set $\{(\eta,\zeta) : (\xi,\eta,\zeta) \in T\}$ is an injection from ξ to λ. Because $(\lambda^+, \lambda) \twoheadrightarrow (\omega_1, \omega)$, we have a pair (Ω_1, Ω) of subsets of λ^+, of cardinals ω_1 and ω respectively, such that

$$(\Omega_1, \Omega, R \cap (\Omega_1 \times \Omega), S \cap (\Omega_1 \times \Omega), T \cap (\Omega_1 \times \Omega_1 \times \Omega)$$

is elementarily equivalent to $(\lambda^+, \lambda, R, S, T)$.

(c) The order type of Ω_1 is ω_1. **P** Note that T tells us what the ordering to λ^+ is; $\eta < \xi$ iff there is a ζ such that (ξ,η,ζ) belongs to T. Now for any $\xi \in \Omega_1$, $\{(\eta,\zeta) : (\xi,\eta,\zeta) \in T\}$ is an injection from ξ to λ, so $\{(\eta,\zeta) : (\xi,\eta,\zeta) \in T \cap (\Omega_1 \times \Omega_1 \times \Omega)\}$ must be an injection from $\xi \cap \Omega_1$ to Ω. Thus $\xi \cap \Omega_1$ is countable for every $\xi \in \Omega_1$ and $\mathrm{otp}(\Omega_1) \le \omega_1$; as $\#(\Omega_1) = \omega_1$, $\mathrm{otp}(\Omega_1)$ is ω_1 exactly. **Q**

Observe that for $\beta \in \Omega$, $\alpha \in \Omega_1$ we must have $I_\beta \cap \Omega \subseteq V_\alpha \cap \Omega_1$ iff there is no $\gamma \in \Omega$ such that $(\beta,\gamma) \in R$ and $(\alpha,\gamma) \notin S$ iff $I_\beta \subseteq V_\alpha$. Next, if $\xi \in \lambda^+$, there must be an $\eta \in \lambda^+$ such that $I_\eta \not\subseteq V_\zeta$ for any $\zeta < \xi$. **P** Let $\langle C_n\rangle_{n\in\mathbb{N}}$ be a sequence of subsets of ξ, with union ξ, such that $\#(C_n) \le \omega_n$ for each n. For each n, $\bigcup_{\zeta\in C_n} V_\zeta$ has cardinal at most $\max(\omega_1, \omega_n)$, so there is an $\alpha_n \in \omega_{n+2} \setminus (\omega_{n+1} \cup \bigcup_{\zeta\in C_n} V_\zeta)$; now there is an $\eta \in \lambda^+$ such that $I_\eta = \{\alpha_n : n \in \mathbb{N}\}$, so that $I_\eta \not\subseteq V_\zeta$ for any $\zeta < \xi$. **Q** It follows that if $\alpha \in \Omega_1$ there is a $\beta \in \Omega_1$ such that $I_\beta \cap \Omega \not\subseteq V_\gamma$ for any $\gamma \in \Omega_1 \cap \alpha$.

Finally, note that because $\mathcal{V}$ is cofinal with $[\lambda]^{\le\mathfrak{c}}$, every I_ξ is included in some V_η; so that for every $\alpha \in \Omega_1$ there is a $\beta \in \Omega_1$ such that $I_\alpha \cap \Omega_1 \subseteq V_\beta$.

(d) Set $A = \bigcup_{\xi\in\Omega} I_\xi$. Then A is countable. So $\{V \cap A : V \in \mathcal{V}\}$ is countable, because $\kappa = \omega_1$. So there must be an $\alpha \in \Omega_1$ such that

$$\{V_\gamma \cap A : \gamma \in \Omega_1\} = \{V_\gamma \cap A : \gamma \in \Omega_1 \cap \alpha\}.$$

But now there are a $\beta \in \Omega$ such that $I_\beta \cap \Omega \not\subseteq V_\xi$ whenever $\xi \in \Omega_1 \cap \alpha$, and a $\gamma \in \Omega_1$ such that $I_\beta \cap \Omega \subseteq V_\gamma$; so that $V_\gamma \cap A$ cannot be equal to $V_\xi \cap A$ for any $\xi \in \Omega_1 \cap \alpha$. **X**

This contradiction shows that $\kappa \ge \omega_2$, as claimed.

Remark Compare Fuchino & Soukup 97, Theorem 12, where it is shown that under the same hypotheses of GCH + CC the Freese-Nation number of $[\omega_\omega]^{\le\omega}$ is greater than ω_1, and also Fuchino Geschke Shelah & Soukup 01, Theorem 4.2, where a different special axiom is used to find a ccc complete Boolean algebra of size $\omega_{\omega+1}$ without the weak Freese-Nation property. For a model of GCH + CC, see Levinski Magidor & Shelah 90.

6H Theorem (Fuchino Koppelberg & Shelah 96) Let $\mathfrak{A}$ be a Boolean algebra and $\kappa = \max(\omega, \mathrm{FN}(\mathfrak{A}))$. Let λ be an infinite cardinal such that $\lambda^{<\kappa} = \lambda$. Then for any set $A \subseteq \mathfrak{A}$ of cardinal λ^+ there is a Boolean-independent set $B \subseteq A$ of cardinal λ^+.

proof Let $f : \mathfrak{A} \to [\mathfrak{A}]^{<\kappa}$ be a Freese-Nation function; enlarging f if necessary, we may suppose that $f(a) = f(1 \setminus a)$ for every $a \in \mathfrak{A}$. Choose $\langle a_\xi \rangle_{\xi<\lambda^+}$ inductively in A so that $a_\xi \notin \mathfrak{B}_\xi$ for every ξ, where $\mathfrak{B}_\xi$ is the smallest subalgebra of $\mathfrak{A}$ containing a_η for every $\eta < \xi$ and such that $f(b) \subseteq \mathfrak{B}_\xi$ for every $b \in \mathfrak{B}_\xi$; this is possible because $\#(\mathfrak{B}_\xi) \le \max(\#(\xi), \omega, \kappa) \le \lambda$ for every $\xi < \lambda^+$. Set $S = \{\xi : \xi < \lambda^+, \operatorname{cf}\xi \ge \kappa\}$, so that S is stationary in λ^+. For each $\xi \in S$, there is an $h(\xi) < \xi$ such that $f(a_\xi) \cap \mathfrak{B}_\xi \subseteq \mathfrak{B}_{h(\xi)}$. Let $\eta^* < \lambda^+$ be such that $S_1 = \{\xi : \xi \in S,\ h(\xi) = \eta^*\}$ is stationary in λ^+. Because $\#(\mathfrak{B}_{\eta^*}) \le \lambda$, $\mathfrak{B}_{\eta^*}$ has fewer than λ^+ sets of cardinal less than κ, so there must be I, $J \subseteq \mathfrak{B}_{\eta^*}$ such that

$$S_2 = \{\xi : \xi \in S_1,\ I = f(a_\xi) \cap \mathfrak{B}_{\eta^*} \cap [0, a_\xi],\ J = f(a_\xi) \cap \mathfrak{B}_{\eta^*} \cap [0, 1 \setminus a_\xi]\}$$

is stationary in λ^+. Note that I must be cofinal with $\mathfrak{B}_{\eta^*} \cap [0, a_\xi]$ and J with $\mathfrak{B}_{\eta^*} \cap [0, 1 \setminus a_\xi]$, so that I and J are both upwards-directed. Set $B = \{a_\xi : \xi \in S_2\}$, so that $B \subseteq A$ and $\#(B) = \lambda^+$.

Set

$$D = \{d : d \in \mathfrak{B}_{\eta^*},\ d \not\subseteq (b \cup b') \ne 0 \text{ for all } b \in I \text{ and } b' \in J\}.$$

Because $a_\xi \notin \mathfrak{B}_{\eta^*}$ for any $\xi \in S_2$, $1 \in D$. Because I and J are upwards-directed, $d \setminus b \in D$ whenever $d \in D$ and $b \in I \cup J$. Now if $d \in D$ and $K \subseteq S_2$ is finite and $c_\xi \in \{a_\xi, 1 \setminus a_\xi\}$ for $\xi \in K$, $d \cap \inf\{c_\xi : \xi \in K\} \ne 0$. **P** Induce on $\#(K)$. If $K = \emptyset$, this is trivial, because $0 \notin D$. If $K = \{\xi\}$ has one element, then $d \not\subseteq b$ for any $b \in I$, while $d \in \mathfrak{B}_{\eta^*}$, so $d \not\subseteq a_\xi$, and similarly $d \not\subseteq 1 \setminus a_\xi$; that is, $d \cap a_\xi$ and $d \cap (1 \setminus a_\xi)$ are both non-zero. For the inductive step to $\#(K) = n + 1 \ge 2$, set $\zeta = \max K$ and $L = K \setminus \{\zeta\}$. **?** If $d \in D$ and $c_\xi \in \{a_\xi, 1 \setminus a_\xi\}$ for $\xi \in K$ and $d \cap \inf_{\xi \in K} c_\xi = 0$, then $c = d \cap \inf_{\xi \in L} c_\xi$ is included in $1 \setminus c_\zeta$. So there is a $b \in f(1 \setminus c_\zeta) \cap f(c) \cap [c, 1 \setminus c_\zeta]$. But $f(1 \setminus c_\zeta) = f(a_\zeta)$ and $c \in \mathfrak{B}_\zeta$, so $b \in f(a_\zeta) \cap \mathfrak{B}_\zeta) \subseteq \mathfrak{B}_{\eta^*}$ and $b \in I \cup J$. So $d \setminus b \in D$. But $d \setminus b \subseteq \inf_{\xi \in L} c_\xi$, and taking any $\xi \in L$ we have $(d \setminus b) \cap (1 \setminus c_\xi) = 0$, which is impossible. **X** Thus the induction proceeds. **Q**

In particular, $\inf\{c_\xi : \xi \in K\} \ne 0$ whenever $K \subseteq S_2$ is finite and $c_\xi \in \{a_\xi, 1 \setminus a_\xi\}$ for $\xi \in K$; that is, B is Boolean-independent.

6I Corollary (Fuchino Koppelberg & Shelah 96) If $\mathfrak{A}$ is a Boolean algebra with more than four elements and the weak Freese-Nation property, then $\#(\mathfrak{A}) \le \operatorname{Ind}(\mathfrak{A})^\omega$, where $\operatorname{Ind}(\mathfrak{A})$ is $\sup\{\#(A) : A \subseteq \mathfrak{A}$ is Boolean-independent$\}$.

proof Set $\lambda = \operatorname{Ind}(\mathfrak{A})^\omega$ in 6H.

6J Corollary (Fuchino Koppelberg & Shelah 96) $c(\mathfrak{A}) \le 2^{<\mathrm{FN}(\mathfrak{A})}$ for any Boolean algebra $\mathfrak{A}$.

proof If $\mathrm{FN}(\mathfrak{A}) = n \in \mathbb{N} \setminus \{0\}$ then $\mathfrak{A}$ has no totally ordered set with 2^{n-1} elements, so $c(\mathfrak{A}) + 1 < 2^{n-1}$ and $c(\mathfrak{A}) < 2^{<\mathrm{FN}(\mathfrak{A})}$. If $\mathrm{FN}(\mathfrak{A}) = \kappa \ge \omega$, set $\lambda = 2^{<\kappa}$; then $\lambda^{<\kappa} = \lambda$. Now if $A \subseteq \mathfrak{A}$ is disjoint it has no Boolean-independent subset with more than one element, so $\#(A) \le \lambda$. Thus $c(\mathfrak{A}) \le 2^{<\mathrm{FN}(\mathfrak{A})}$ in this case also.

7. Freese-Nation numbers and dimension

7A Theorem (Nation & Schmerl 90) If λ is an infinite cardinal and P is a partially ordered set with $\mathrm{FN}(P) \le \omega$ and $\#(P) \le 2^\lambda$, then $\dim P \le \lambda$.

proof Let $X \subseteq \{0,1\}^P$ be a dense subset of cardinal at most λ (Engelking 89, 3.6.F). Let $f : P \to [P]^{<\omega}$ be a Freese-Nation function. For $x \in X$ set

$$A_x = \{p : x(q) = 1 \text{ whenever } q \in f(p) \cap [p, \infty[\},$$

$$B_x = \{p : x(q) = 0 \text{ whenever } q \in f(p) \cap]-\infty, p]\}.$$

Then $p \not\le q$ whenever $p \in A_x$ and $q \in B_x$. So there is a total ordering $\preccurlyeq_x$ extending $\le$ such that $q \preccurlyeq_x p$ whenever $p \in A_x$ and $q \in B_x$. Next, whenever p, $q \in P$ and $p \not\le q$ there is a $\xi < \lambda$ such that $p \in A_x$ and $q \in B_x$, just because $f(p) \cap [p\infty[$ and $f(q) \cap]-\infty, q]$ are disjoint and X is dense. So $\langle \preccurlyeq_x \rangle_{x \in X}$ witnesses that $\dim P \le \lambda$.

7B Remark The same method shows that if $\mathrm{FN}(P) \le \kappa$ and $\#(P) \le \lambda$ then $\dim P \le d(Z)$, where $Z = \{0,1\}^\lambda$ with the topology generated by sets of the form $\{x : x{\restriction}I = z\}$ for some $I \in [\lambda]^{<\max(\kappa,\omega)}$ and $z \in \{0,1\}^I$. In particular, for instance, $\dim P \le \mathfrak{c}$ if $\mathrm{FN}(P) \le \omega_1$ and $\#(P) \le 2^{\mathfrak{c}}$.

7D Example Let κ be a cardinal. Then there is a partially ordered set P such that $\mathrm{FN}(P) \le 5$ but $\dim P > \kappa$. If κ is finite, P can be taken to be finite.

proof Let λ be so large that if $g : [\lambda]^3 \to \kappa$ is any function then there is an $A \in [\lambda]^4$ such that g is constant on $[A]^3$. (For infinite κ, the Erdős-Rado theorem tells us that $\lambda = (2^{2^\kappa})^+$ will do. For finite κ we need a suitable Ramsey number.) Set $P = [\lambda]^{\le 2}$. Writing $f(p) = \{q : q \le p\}$ for each $p \in P$, $f : P \to [P]^{\le 4}$ is a Freese-Nation function, so $\mathrm{FN}(P) \le 5$.

? Now suppose, if possible, that $\langle \preccurlyeq_\xi \rangle_{\xi<\kappa}$ is a family of total orderings on P with intersection $\le$. For each $I \in [\lambda]^3$ choose $g(I) < \kappa$ such that $\{\xi, \zeta\} \preccurlyeq_{g(I)} \{\eta\}$ where $I = \{\xi, \eta, \zeta\}$ and $\xi < \eta < \zeta$. Let $A \in [\lambda]^4$ be such that g is constant on $[A]^3$; write α for its constant value. Express A as $\{\xi, \eta, \zeta, \theta\}$ where $\xi < \eta < \zeta < \theta$. Then

$$\{\xi, \zeta\} \preccurlyeq_\alpha \{\eta\} \le \{\eta, \theta\} \preccurlyeq_\alpha \{\zeta\} \le \{\xi, \zeta\},$$

and $\preccurlyeq_\alpha$ is not a total order extending $\le$. **X**

8. Cohen reals

8A Theorem If we add fewer than ω_ω Cohen reals to a model of ZFC+CH, we get a model in which $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$.

proof (a) Let $\mathfrak{A}$ be the regular open algebra of $\{0,1\}^\kappa$ where $\kappa < \omega_\omega$. For $I \subseteq \kappa$ let $\mathfrak{A}_I$ be the set of members of $\mathfrak{A}$ determined by coordinates in I. In $V^{\mathfrak{A}}$, let Q_I be the algebra of subsets $\dot{A}$ of $\mathcal{P}\mathbb{N}$ such that $[\![n \in \dot{A}]\!] \in \mathfrak{A}_I$ for every n.

(b) In $V^{\mathfrak{A}}$, the Freese-Nation index of Q_I in $\mathcal{P}\mathbb{N}$ is at most ω_1, for any ground-model set $I \subseteq \kappa$.

P Suppose that $\dot{A}$ is a $\mathfrak{A}$-name for a member of $\mathcal{P}\mathbb{N}$. Then there is a countable set $J \subseteq \kappa$ such that $[\![n \in \dot{A}]\!] \in \mathfrak{A}_J$ for every $n \in \mathbb{N}$. Let $\mathfrak{C}$ be a countable order-dense subalgebra of $\mathfrak{A}_J$. For $c \in \mathfrak{C}$ and $n \in \mathbb{N}$, set

$$d_{nc} = \sup\{b : b \in \mathfrak{A}_I,\, b \cap c \subseteq [\![n \in \dot{A}]\!]\},$$

and let $\dot{D}_c$ be the $\mathfrak{A}_I$-name for a subset of $\mathbb{N}$ defined by saying that $[\![n \in \dot{D}_c]\!] = d_{nc}$. Now

$$[\![\forall\, \dot{B} \in Q_I,\, \dot{B} \subseteq \dot{A} \Rightarrow \exists\, c \in \mathfrak{C},\, \dot{B} \subseteq \dot{D}_c \subseteq \dot{A}]\!] = 1.$$

To see this, take any name $\dot{B}$ for a member of Q_I, and suppose that $d = [\![\dot{B} \subseteq \dot{A}]\!] \in \mathfrak{A}_{I\cup J}$. Set

$$K = \{b \cap c : b \in \mathfrak{A}_I,\, c \in \mathfrak{C},\, b \cap c \subseteq d\}.$$

Then $\sup K = d$ in $\mathfrak{A}$. Now if $b \in \mathfrak{A}_I$, $c \in \mathfrak{C}$ and $b \cap c \subseteq d$, then $b \cap c \cap [\![n \in \dot{B}]\!] \subseteq [\![n \in \dot{A}]\!]$ for every n, so $b \cap [\![n \in \dot{B}]\!] \subseteq d_{nc}$, while $c \cap d_{nc} \subseteq [\![n \in \dot{A}]\!]$. But this means that $b \cap c \subseteq [\![\dot{B} \subseteq \dot{D}_c \subseteq \dot{A}]\!]$. As b and c are arbitrary,

$$d \subseteq [\![\exists\, c \in \mathfrak{C},\, \dot{B} \subseteq \dot{D}_c \subseteq \dot{A}]\!].$$

As this is true for every $\dot{B}$, we have the result.

So (in $V^{\mathfrak{A}}$) $\{\dot{D}_c : c \in \mathfrak{C}\}$ is a countable family in Q_I cofinal with $Q_I \cap [\emptyset, \dot{A}]$. **Q**

(c) Henceforth work in $V^{\mathfrak{A}}$. For $\xi \le \kappa$, set $P_\xi = \bigcup_{\eta<\kappa} Q_\eta$. Because $P_\xi = Q_\xi$ whenever $\mathrm{cf}\,\xi > \omega$, the Freese-Nation index of P_ξ in $\mathcal{P}\mathbb{N}$ is never greater than ω_1. We always have $P_\xi \equiv P_\eta$ if $\#(\xi) = \#(\eta)$. Because CH is true in the ground model, $\#(P_\xi) = \omega_1$ for $\xi < \omega_1$, so $\mathrm{FN}(P_\xi) \le \omega_1$ for $\xi < \omega_1$. Now an easy induction on ξ, using 1G at the inductive step, shows that $\mathrm{FN}(P_\xi) \le \omega_1$ for every $\xi \le \kappa$, so that $\mathrm{FN}(\mathcal{P}\omega) = \omega_1$.

8B Theorem If we add any number of Cohen reals to a model of ZFC+V=L, we obtain a model in which $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$.

proof Put the ideas of 8A and 6E together.

Remark See Geschke p00 for an alternative method of getting $\mathfrak{c} > \mathrm{FN}(\mathcal{P}\mathbb{N})$.

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