Sets determined by few coordinates

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Cartesian products are so familiar that it is easy to imagine that they can have no surprises for us. In this note I discuss an elementary concept which has surprisingly far-reaching implications for both general topology and measure theory.

1. Definition Let $\langle X_i\rangle_{i\in I}$ be any family of sets, with product X. For $E\subseteq X$ and $J\subseteq I$, I say that E is **determined by coordinates in** J if $x\restriction J\ne y\restriction J$ whenever $x\in E$ and $y\in X\setminus E$; that is, if E is expressible in the form $\pi_J^{-1}[F]$ for some $F\in\prod_{i\in J}X_i$, where $\pi_J(x)=x\restriction J$ for $x\in X$.

I write $\mathcal{F}_E$ for the family of sets $J\subseteq I$ such that E is determined by coordinates in J.

2. The most elementary facts (a) For any fixed $J\subseteq I$, the set $\{E:E\subseteq X$ is determined by coordinates in $J\}$ contains X and is closed under complementation and arbitrary unions; in particular, it is a σ-algebra of subsets of X.

Turning these round, we see that $\mathcal{F}_{X\setminus E}=\mathcal{F}_E$ and that $\mathcal{F}_{E\cup F}$ and $\mathcal{F}_{E\cap F}$ both include $\mathcal{F}_E\cap\mathcal{F}_F$ for all E, $F\subseteq X$; in fact, $\mathcal{F}_{\bigcup\mathcal{A}}$ and $\mathcal{F}_{\bigcap\mathcal{A}}$ include $\bigcap_{A\in\mathcal{A}}\mathcal{F}_A$ for any non-empty $\mathcal{A}\subseteq\mathcal{P}X$.

(b) For any $E\subseteq X$, $\mathcal{F}_E$ is either $\mathcal{P}I$ (if $E=\emptyset$ or $E=X$) or a filter on I. **P** It is immediate from the definition in §1 that $K\in\mathcal{F}_E$ whenever $J\in\mathcal{F}_E$ and $J\subseteq K\subseteq I$, and that $\emptyset\in\mathcal{F}_E$ iff either E or $X\setminus E$ is empty. If J, $K\in\mathcal{F}_E$ and $x\in E$ and $y\in X\setminus E$, set $z(i)=x(i)$ if $i\in J$, $y(i)$ if $i\in I\setminus J$. Then $z\restriction J=x\restriction J$, so $z\in E$; accordingly $z\restriction K\ne y\restriction K$; but as $z\restriction I\setminus J=y\restriction I\setminus J$,

$$y\restriction J\cap K\ne z\restriction J\cap K=x\restriction J\cap K.$$

As x and y are arbitrary, $J\cap K\in\mathcal{F}_E$; as J and K are arbitrary, $\mathcal{F}_E$ is a filter on I, except in the trivial cases in which it contains $\emptyset$. **Q**

3. Principal filters When using the concept 'E is determined by coordinates in J' it is particularly helpful to know when there is a minimal set J determining E; that is, when $\mathcal{F}_E$ has a least element. An important case in which this is true is the following.

Proposition Suppose that $\langle X_i\rangle_{i\in I}$ is a family of topological spaces, with topological product X, and that $E\subseteq X$ belongs to the algebra generated by the open subsets of X. Then $\mathcal{F}_E$ has a least element; that is, it is of the form $\{J:J^*\subseteq J\subseteq I\}$ for some $J^*\subseteq I$.

proof Set $J^*=\bigcap\mathcal{F}_E$. Let $\mathcal{G}$ be a finite family of open subsets of X such that E belongs to the algebra of subsets of X generated by $\mathcal{G}$. Set

$$Q_0=(E\times(X\setminus E))\cup((X\setminus E)\times E),\quad Q=\{(x,y):(x,y)\in Q_0,\,x\restriction J^*=y\restriction J^*\}.$$

? Suppose, if possible, that Q is not empty. For $(x,y)\in Q$ set

$$\mathcal{G}_{xy}=\{G:G\in\mathcal{G},\,G\supseteq\{z:z\in X,\,z\supseteq x\cap y\}\},$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX where in the formula $z \supseteq x \cap y$ I am interpreting x, y and z as functions with domain I, so that $z \supseteq x \cap y$ iff z agrees with x and y at any coordinate where they are equal. Set $n = \sup_{(x,y)\in Q} \#(\mathcal{G}_{xy})$ and fix $(x, y) \in Q$ such that $\#(\mathcal{G}_{xy}) = n$.

Since just one of x, y belongs to E, there must be some $G \in \mathcal{G}$ such that just one of x, y belongs to G; as the structure so far declared is symmetric between x and y, there is no loss of generality in supposing that $x \in G$ and $y \notin G$. Now G is an open set, so there is a finite set $K \subseteq I$ such that $z \in G$ whenever $z \in X$ and $z\restriction K = x\restriction K$. Because $\mathcal{F}_E$ is downwards-directed, there is a $J \in \mathcal{F}_E$ such that $J \cap K = J^* \cap K$.

Define $w \in X$ by setting $w(i) = y(i)$ if $i \in J$, $x(i)$ if $i \in I \setminus J$. Then $w \in E$ iff $y \in E$, so $(x, w) \in Q_0$; moreover,

$$w \supseteq x \cap y \supseteq x\restriction J^* = y\restriction J^*,$$

so $(x, w) \in Q$. Because $x\cap w \supseteq x\cap y$, $\mathcal{G}_{xw} \supseteq \mathcal{G}_{xy}$. On the other hand, if $i \in K$, either $i \in J^*$ and $x(i) = y(i) = w(i)$, or $i \notin J^*$ and $i \notin J$ and again $x(i) = w(i)$. So $w\restriction K = x\restriction K$ and $G \in \mathcal{G}_{xw}$; while of course $G \notin \mathcal{G}_{xy}$ because $y \notin G$. But this means that $\mathcal{G}_{xw} \supseteq \mathcal{G}_{xy} \cup \{G\}$ is a proper superset of $\mathcal{G}_{xy}$, which contradicts the choice of x and y. **X**

This contradiction shows that $Q = \emptyset$, that is, that if $x \in E$ and $y \in X \setminus E$ then $x\restriction J^* \neq y\restriction J^*$. Accordingly J^* belongs to $\mathcal{F}_E$ and must be the required smallest element of $\mathcal{F}_E$.

4. Remarks (a) As soon as we leave the *algebra* generated by the open sets, the filters $\mathcal{F}_E$ can become much more complex. For instance, if we take $I = \mathbb{N}$ and $X_i = \{0, 1\}$ (with its usual discrete topology) for $i \in \mathbb{N}$, and set

$$\begin{aligned} H &= \{x : x \in X, \limsup_{n\to\infty} x(np) = 1 \text{ for every prime number } p\} \\ &= \bigcap_{m\in\mathbb{N}, p \text{ prime}} \bigcup_{n\ge m} \{x : x(np) = 1\}, \end{aligned}$$

then H is a G_δ set, while $\mathcal{F}_H$ becomes

$$\{J : J \subseteq \mathbb{N}, \{n : np \notin J\} \text{ is finite for every prime number } p\}.$$

(The point is that if $\{n : np \notin J\}$ is infinite, we can set

$$\begin{aligned} x(i) &= 0 \text{ if } i \in J \cap p\mathbb{N}, \\ &= 1 \text{ otherwise}, \\ y(i) &= 0 \text{ if } i \in p\mathbb{N}, \\ &= 1 \text{ otherwise}; \end{aligned}$$

then $x \in H$ and $y \notin H$ and $x\restriction J = y\restriction J$, so $J \notin \mathcal{F}_H$.) In this case, $\mathcal{F}_H$ certainly does not have a least member.

(b) I note however that

$$\mathcal{F}_H = \bigcap_{p \text{ prime}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} \{J : mp \in J\}$$

is an $\mathrm{F}_{\sigma\delta}$ set in the usual topology of $\mathcal{P}\mathbb{N}$. I do not know whether $\mathcal{F}_E$ is a Borel subset of $\mathcal{P}\mathbb{N}$ whenever E is a Borel subset of $X = \{0, 1\}^{\mathbb{N}}$. What is true is that $\mathcal{F}_E$ is coanalytic whenever E is Borel. **P** Consider the set

$$A = \{(x, y, J) : x \in E,\ y \notin E,\ x{\restriction}J = y{\restriction}J\} \subseteq X \times X \times \mathcal{P}\mathbb{N}.$$

Because E and $X \setminus E$ are Borel, and $\{(x, y, J) : x{\restriction}J = y{\restriction}J\}$ is closed, A is a Borel subset of $X \times X \times \mathcal{P}\mathbb{N}$, and its projection $\mathcal{P}\mathbb{N} \setminus \mathcal{F}_E$ is an analytic subset of $\mathcal{P}\mathbb{N}$. **Q**

(c) Continuing the argument of (b), we see that if $E \subseteq \{0,1\}^{\mathbb{N}}$ is a Borel set then $\mathcal{F}_E$ has the strong property of Baire ([7], §11.VIII). Now suppose that all cofinite sets belong to $\mathcal{F}_E$. By a theorem of Talagrand ([10], Th. 21), there is a strictly increasing sequence $\langle n_k \rangle_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\{k : J \cap n_{k+1} \setminus n_k = \emptyset\}$ is finite for every $J \in \mathcal{F}_E$. Put another way, if we take $n_0 = 0$ and think of E as a subset of the product $\prod_{k\in\mathbb{N}} \{0,1\}^{n_{k+1}\setminus n_k}$, then the filter of determining sets for E becomes just the Fréchet filter.

(d) In the other direction, if we start with a filter $\mathcal{F}$ on $\mathbb{N}$, there is always a set $F \subseteq X$ such that $\mathcal{F} = \mathcal{F}_F$; just take $F = \{\chi J : J \in \mathcal{F}\}$. (If $J \notin \mathcal{F}$, then $x = \chi\mathbb{N} \in F$ and $y = \chi J \notin F$ and $x{\restriction}J = y{\restriction}J$.) Of course F will be Borel, or analytic, or coanalytic iff $\mathcal{F}$ is.

5. Closures and interiors Because projections onto subproducts are open maps, we have the following simple result. If $\langle X_i \rangle_{i\in I}$ is any family of topological spaces with product X, and $J \subseteq I$, and E any subset of X determined by coordinates in J, then $\overline{E}$ and $\operatorname{int} E$ are also determined by coordinates in J. **P** Writing $\pi_J : X \to \prod_{i\in J} X_i$ for the canonical map, $\pi_J[\operatorname{int} E]$ is open, so must be $\operatorname{int} \pi_J[E]$, and $\operatorname{int} E = \pi_J^{-1}[\operatorname{int} \pi_J[E]]$ is determined by coordinates in J. Similarly, $\overline{E} = \pi_J^{-1}[\overline{\pi_J[E]}]$ is determined by coordinates in J. **Q** Thus the family of sets determined by coordinates in J is closed under the operations of closure and interior as well as complementation and arbitrary unions and intersections. Put another way, what is happening here is that the principal filters $\mathcal{F}_{\operatorname{int} E}$ and $\mathcal{F}_{\overline{E}}$ are both included in $\mathcal{F}_E$, for any set $E \subseteq X$.

6. Continuous functions We have a result similar to that of Proposition 3 concerning continuous functions on product spaces. If $\langle X_i \rangle_{i\in I}$ is any family of sets with product X, and f is a function from X to any set Y, we can say that f is **determined by coordinates in** $J \subseteq I$ if $f(x) = f(x')$ whenever $x, x' \in X$ and $x{\restriction}J = x'{\restriction}J$. Now it is easy to see that

$$\begin{aligned} &f \text{ is determined by coordinates in } J \\ &\qquad\iff f = g\pi_J \text{ for some } g : \prod_{i\in J} X_i \to Y \\ &\qquad\iff f^{-1}[\{y\}] \text{ is determined by coordinates in } J \text{ for every } y \in Y \\ &\qquad\iff J \in \bigcap_{B\subseteq Y} \mathcal{F}_{f^{-1}[B]}. \end{aligned}$$

In particular, $\{J : f \text{ is determined by coordinates in } J\}$ is $\mathcal{P}I$ if f is constant and otherwise is a filter on I.

Now observe that if all the X_i are topological spaces, X is given the product topology, and Y is a T_0 space, then f is determined by coordinates in J iff $f^{-1}[H]$ is determined by coordinates in J for every open set $H \subseteq Y$; that is, iff $J \supseteq \bigcup_{H\subseteq Y \text{ is open}} J^*_{f^{-1}[H]} = M$ say, where for any open set $G \subseteq X$ I write J^*_G for the smallest subset of I such that G is determined by coordinates in J^*_G. So here also we have a smallest set M such that f is determined by coordinates in M.

7. In memory of Arthur Stone Once we know that a set, or a function, is determined by coordinates in some minimal set, the obvious next step is to ask whether that minimal set has to be of any particular type; in particular, whether it has to be 'small'. In this context we have the following result.

Proposition ([9]) Suppose that $\langle X_i \rangle_{i \in I}$ is a family of topological spaces such that $X = \prod_{i \in I} X_i$ is ccc. Then

(a) for any open set $G \subseteq X$, there is an open set $H \subseteq G$ such that $\overline{H} = \overline{G}$ and H is determined by coordinates in a countable set;

(b) any regular open subset of X is determined by coordinates in a countable set;

(c) any continuous function from X to a second-countable Hausdorff space Y is determined by coordinates in a countable set;

(d) any set in the Baire σ-algebra of X is determined by coordinates in a countable set.

If $\prod_{i \in K} X_i$ is perfectly normal for every finite $K \subseteq I$ (for instance, if every X_i is metrizable), then we can add

(e) a subset of X is a zero set iff it is closed and determined by a countable set of coordinates;

(f) a subset of X is in the Baire σ-algebra of X iff it is a Borel set and is determined by a countable set of coordinates.

proof (a) Write $\mathcal{H}$ for the family of open sets $H \subseteq X$ such that J^*_H is countable, where J^*_H is the minimal set such that H is determined by coordinates in J^*_H. Then $\mathcal{H}$ includes a base for the topology of X and is closed under countable unions. If we set $\mathcal{H}_1 = \{H : H \in \mathcal{H}, H \subseteq G\}$ and take $\mathcal{H}_0$ to be a maximal disjoint subset of $\mathcal{H}_1$, then $H = \bigcup \mathcal{H}_0$ is a dense open subset of G. Because X is ccc, $\mathcal{H}_0$ is countable, so $H \in \mathcal{H}$, as required.

(b) Continuing from (a), $\overline{G} = \overline{H}$ is also determined by coordinates in the countable set J^*_H; so if $G = \operatorname{int} \overline{G}$, then $J^*_G = J^*_H$ is countable.

(c) Let $\mathcal{V}$ be a countable base for the topology of Y. For each $V \in \mathcal{V}$, set $G_V = \operatorname{int} \overline{f^{-1}[V]}$, so that G_V is determined by a countable set J_V of coordinates. Set $J = \bigcup_{V \in \mathcal{V}} J_V$, so that $J \subseteq I$ is countable. If $x, x' \in X$ and $f(x) \neq f(x')$, then there is a $V \in \mathcal{V}$ such that $f(x) \in V$ and $f(x') \notin \overline{V}$. In this case, $x \in G_V$ but $x' \notin G_V$, so $x \restriction J \neq x' \restriction J$. As x and x' are arbitrary, f is determined by coordinates in the countable set J.

(d) From (c) we see at once that any zero set must be determined by coordinates in some countable set. Since the family $\{W : W \subseteq X$ is determined by coordinates in a countable set$\}$ is a σ-algebra of subsets of X containing every zero set, it includes the Baire σ-algebra.

(e) Now suppose that every X_i is perfectly normal and that $F \subseteq X$ is a closed set determined by coordinates in a countable subset J of I. Express J as $\bigcup_{n \in \mathbb{N}} K_n$ where $\langle K_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence of finite sets. For each $n \in \mathbb{N}$ let G_n be the union of all the open sets determined by coordinates in K_n disjoint from F; then $\langle G_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of open sets with union $X\setminus F$, and each G_n is of the form $\pi_{J_n}^{-1}[H_n]$ where H_n is an open subset of $\prod_{i\in K_n} X_i$, so is a cozero set, because we are supposing that $\prod_{i\in K_n} X_i$ is perfectly normal. But this means that $X\setminus F$ is the union of a sequence of cozero sets, so is itself a cozero set, and F is a zero set.

(f) Finally, let $E\subseteq X$ be a Borel set determined by coordinates in a countable set $J\subseteq I$. Identifying X with $X_J\times X_{I\setminus J}$, where $X_J=\prod_{i\in I} X_i$, we have $E=F\times X_{I\setminus J}$ where F is a Borel subset of X_J. By the argument of (d), every open subset of X_J is a cozero set, so every Borel subset of X_J is in the Baire σ-algebra of X_J, and E is in the Baire σ-algebra of X.

Remark From (b) we see that if Y is a regular Hausdorff space of countable character and $f:X\to Y$ is continuous, then for every $x\in X$ there is a countable set $J\subseteq I$ such that $f(x')=f(x)$ whenever $x'\in X$ and $x'\restriction J=x\restriction J$. This fact is the basis of a theorem of Gleason (see [6], chapter 7, Theorem 23, and also [5]): if $X=\prod_{i\in I} X_i$ is a family of separable spaces, Y is a regular Hausdorff space of countable character, and $f:X\to Y$ is continuous, then f is determined by coordinates in a countable set, so that $f[X]$ is separable.

8. Products of probability spaces If the factors X_i carry probability measures, then we can look at the filters $\mathcal{F}_E$ for measurable subsets E of their product. As the example in 4a above already shows, we cannot expect these filters to be principal except in very special cases. However we have the following very important result.

Theorem Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, and λ the completed product probability measure on $X=\prod_{i\in I} X_i$; write Λ for its domain. For any set $E\in\Lambda$ set

$$\mathcal{F}_E^*=\bigcup\{\mathcal{F}_F: F\in\Lambda,\ \lambda(E\triangle F)=0\}.$$

Then there is a countable set $J^*\subseteq I$ such that $\mathcal{F}_E^*=\{J: J^*\subseteq J\subseteq I\}$.

proof Because $\mathcal{F}_E^*$ is not changed if we alter E by a negligible set, we may suppose throughout that E belongs to the σ-algebra $\widehat{\bigotimes}_{i\in I}\Sigma_i$ generated by sets of the form $\{x: x(i)\in V\}$ where $i\in I$ and $V\in\Sigma_i$.

(a) If J, $K\in\mathcal{F}_E^*$, then $J\cap K\in\mathcal{F}_E^*$. **P** There are F, $H\in\Lambda$, both differing from E by negligible sets, such that F is determined by coordinates in J and H is determined by coordinates in J. Set

$$G=\{x: x\in X,\ (x\restriction J)\cup t\in H \text{ for almost every } t\in X_{I\setminus J}\},$$

where $X_{I\setminus J}=\prod_{i\in I\setminus J} X_i$ is given its product probability measure $\lambda_{I\setminus J}$, and for $s\in X_J$, $t\in X_{I\setminus J}$ I write $s\cup t$ for their common extension to a member of X. Now, because $\lambda(F\triangle H)=0$, Fubini's theorem applied to $X\cong X_J\times X_{I\setminus J}$ shows that $G\triangle G'$ is negligible, where

$$G'=\{x: x\in X,\ (x\restriction J)\cup t\in F \text{ for almost every } t\in X_{I\setminus J}\}.$$

But as F is determined by coordinates in J, $G'=F$. So $G\triangle E$ is negligible. Next, if x, $x'\in X$ and $x\restriction J\cap K=x'\restriction J\cap K$, then $\{t:(x\restriction J)\cup t\in H\}=\{t:(x'\restriction J)\cup t\in H\}$, because H is determined by coordinates in K. So $x \in G$ iff $x' \in G$; which means that G is determined by coordinates in $J \cap K$, and $J \cap K \in \mathcal{F}_G \subseteq \mathcal{F}_E^*$. **Q**

(b) If $\langle J_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{F}_E^*$ with intersection J, then $J \in \mathcal{F}_E^*$. **P** Write $\mathcal{L}^\infty$ for the space of bounded $\widehat{\bigotimes}_{i\in I}\Sigma_i$-measurable real-valued functions on X. For $f \in \mathcal{L}^\infty$ define $P_n f : X \to \mathbb{R}$, for $n \in \mathbb{N}$, and $Pf : X \to \mathbb{R}$ by setting

$$(P_n f)(x) = \int f((x \restriction J_n) \cup t)\lambda_{I\setminus J_n}(dt), \quad (Pf)(x) = \int f((x \restriction J) \cup t)\lambda_{I\setminus J}(dt)$$

for each $n \in \mathbb{N}$, $x \in X$. Then P_n and P are linear, and $\|P_n f\|_1 \le \|f\|_1$, $\|Pf\|_1 \le \|f\|_1$ for every $f \in \mathcal{L}^\infty$. If we write $\mathcal{S}$ for the linear subspace of $\mathcal{L}^\infty$ consisting of functions which are determined by coordinates in finite sets, then $\lim_{n\to\infty} P_n f = Pf$ for any $f \in \mathcal{S}$, because if f is determined by coordinates in a finite set K, and n is so large that $J_n \cap K = J \cap K$, then $P_n f = Pf$. But as $\mathcal{S}$ is $\| \|_1$-dense in $\mathcal{L}^\infty$, this means that $\lim_{n\to\infty} \|P_n f - Pf\|_1 = 0$ for every $f \in \mathcal{L}^\infty$.

For each $n \in \mathbb{N}$, there is a set F_n, determined by coordinates in J_n, such that $F_n \triangle E$ is negligible; adjusting F_n by a negligible set if need be, we may suppose that $F_n \in \widehat{\bigotimes}_{i\in I}\Sigma_i$. Now

$$\chi E = \chi F_n = P_n(\chi F_n) = P_n(\chi E)$$

almost everywhere. But this means that $\chi E = P(\chi E)$ a.e., because $\lim_{n\to\infty} \|P_n(\chi E) - P(\chi E)\|_1 = 0$. Setting $F = \{x : P(\chi E)(x) = 1\}$, F is determined by coordinates in J and $F \triangle E$ is negligible; so $J \in \mathcal{F}_E^*$, as required. **Q**

(c) Of course every set in $\widehat{\bigotimes}_{i\in I}\Sigma_i$ is determined by coordinates in some countable set, so $\mathcal{F}_E^*$ contains a countable set.

(d) By (c), $J^* = \bigcap \mathcal{F}_E^*$ is countable and is the intersection of a sequence in $\mathcal{F}_E^*$; by (a) and (b), $J^* \in \mathcal{F}_E^*$; also $K \in \mathcal{F}_E^*$ whenever $J \in \mathcal{F}_E^*$ and $J \subseteq K \subseteq I$; so $\mathcal{F}_E^*$ must be precisely $\{K : J^* \subseteq K \subseteq I\}$.

9. Measurable functions Suppose, in the context of Theorem 8, that we have a measurable function $f : X \to \mathbb{R}$. Just as in §6, there is a least set $J^* \subseteq I$ for which there is a $g : X \to \mathbb{R}$ determined by coordinates in J^* and equal almost everywhere to f; $J^* = \bigcup_{q\in\mathbb{Q}} J_q^*$, where, for each $q \in \mathbb{Q}$, J_q^* is the smallest set such that there is a set F depending on coordinates in J_q^* such that $\lambda(F \triangle \{x : f(x) \le q\}) = 0$.

10. Topological product measures The product measure considered in Theorem 8 is the 'ordinary' product measure λ, defined from the outer measure

$$\theta A = \inf\{\textstyle\sum_{n=0}^\infty \lambda C_n : \langle C_n \rangle_{n\in\mathbb{N}} \text{ is a sequence of basic cylinder sets covering } A\}.$$

If we start from τ-additive topological measures on the factors X_i, there is an alternative product measure $\tilde\lambda$ on X, extending λ, which is again a τ-additive topological measure. (See [4], §417. When each X_i is compact and Hausdorff and each μ_i is a Radon measure, $\tilde\lambda$ is the unique Radon measure extending λ.) But the results of §§8-9 are not affected by moving to $\tilde\lambda$ in place of λ, because for any set E measured by $\tilde\lambda$ there is still a set $F \in \widehat{\bigotimes}_{i\in I}\Sigma_i$ such that $\tilde\lambda(E \triangle F) = 0$.

For any measurable set $F \subseteq X$ we have $\mathcal{F}^*_F \supseteq \mathcal{F}_F$, by the definition of $\mathcal{F}^*_F$. If every μ_i is strictly positive, and F is closed and self-supporting in the sense that $\tilde\lambda(F \cap G) > 0$ whenever $G \subseteq X$ is open and $F \cap G$ is not empty, then $\mathcal{F}^*_F = \mathcal{F}_F$. **P** Take any $J \in \mathcal{F}^*_F$. If $x \in X \setminus F$ and $y \in F$, there is a basic open set G such that $x \in G \subseteq X \setminus F$. Express G as $G' \cap G''$ where G' is determined by coordinates in J and G'' is determined by coordinates in $I \setminus J$. There is an F' such that $\tilde\lambda(F' \triangle F) = 0$ and F' is determined by coordinates in J, so that

$$0 = \tilde\lambda(F \cap G) = \tilde\lambda(F' \cap G' \cap G'') = \tilde\lambda(F' \cap G') \cdot \tilde\lambda G''.$$

But as every μ_i is strictly positive, so is $\tilde\lambda$, and $\tilde\lambda G'' > 0$. This means that

$$\tilde\lambda(F \cap G') = \tilde\lambda(F' \cap G') = 0.$$

As F is self-supporting, $F \cap G'$ must be empty, and $y \notin G'$. So $x{\restriction}J \neq y{\restriction}J$. As x and y are arbitrary, $J \in \mathcal{F}_F$ and $\mathcal{F}_F = \mathcal{F}^*_F$. **Q**

Thus in the special case of a self-supporting closed set F and strictly positive measures μ_i, the minimal sets defined by Proposition 3 and Theorem 8 will be the same. In particular, $\mathcal{F}_F$ contains a countable set.

11. Interpretation in measure algebras If we start from a family $\langle(\mathfrak{A}_i, \bar\mu_i)\rangle_{i \in I}$ of probability algebras, we know from the Loomis-Sikorski theorem that we can represent each $(\mathfrak{A}_i, \bar\mu_i)$ as the measure algebra of a probability space (X_i, μ_i) ([4], §321). We can now consider the product probability measure λ on $X = \prod_{i \in I} X_i$ and the corresponding measure algebra $(\mathfrak{A}, \bar\lambda)$. The X_i and μ_i are not uniquely determined, but the algebra $(\mathfrak{A}, \bar\lambda)$ can be specified, up to isomorphism, by the following universal mapping theorem ([4], §325).

> **Theorem** Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i \in I}$ be a family of probability spaces, and λ the product probability measure on $X = \prod_{i \in I} X_i$. For each $i \in I$ write $(\mathfrak{A}_i, \bar\mu_i)$ for the measure algebra of (X_i, Σ_i, μ_i), and let $(\mathfrak{A}, \bar\lambda)$ be the measure algebra of (X, λ). For each $i \in I$ let $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}$ be the measure-preserving Boolean homomorphism defined by setting $\varepsilon_i(E^\bullet) = \{x : x(i) \in E\}^\bullet$ for every $E \in \Sigma_i$. Suppose that $(\mathfrak{B}, \bar\nu)$ is a probability algebra and $\pi_i : \mathfrak{A}_i \to \mathfrak{B}$ is a Boolean homomorphism for each i and $\bar\nu(\inf_{i \in I} \pi_i a_i) = \prod_{i \in I} \bar\mu a_i$ whenever $a_i \in \mathfrak{A}_i$ for every $i \in I$. Then there is a unique measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $\pi_i = \pi\varepsilon_i$ for every $i \in I$.

(In [4], the structure $(\mathfrak{A}, \bar\lambda, \langle\varepsilon_i\rangle_{i \in I})$ is called 'the' probability algebra free product of $\langle(\mathfrak{A}_i, \bar\mu_i)\rangle_{i \in I}$.) In terms of this structure, Theorem 8 can be regarded as saying that for every $a \in \mathfrak{A}$ there is a (countable) smallest $J \subseteq I$ such that a belongs to the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i \in J} \varepsilon_i[\mathfrak{A}_i]$. Let me emphasise that what is remarkable here is that there is a *smallest* countable set with this property. That there is *some* countable set is just part (c) of the proof of Theorem 8, which is not exactly trivial, but is an easily proved from the outer-measure definition of the product measure indicated in §10.

12. Interpretation in stochastic processes Now suppose that (X, Σ, μ) is a probability space, that $\langle\Sigma_i\rangle_{i \in I}$ is an independent family of σ-subalgebras of Σ, and that Σ is the σ-algebra generated by $\bigcup_{i \in I} \Sigma_i$. Then the measure algebra $(\mathfrak{A}, \bar\mu)$ of (X, Σ, μ) also represents the probability algebra free product of the measure algebras of $(X, \Sigma_i, \mu{\restriction}\Sigma_i)$.

So Theorem 8 becomes: for every $E \in \Sigma$ there is a (countable) smallest set $J_E^* \subseteq I$ such that $E \triangle F$ is negligible for some set F in the σ-algebra generated by $\bigcup_{i \in J_E^*} \Sigma_i$.

In this form, part (b) of the proof of Theorem 8 amounts to saying that $\langle \chi F_n \rangle_{n \in \mathbb{N}}$ is a reverse martingale adapted to a decreasing sequence $\langle \mathrm{T}_n \rangle_{n \in \mathbb{N}}$ of σ-algebras, where T_n is the σ-algebra generated by $\bigcup_{i \in J_n} \Sigma_i$. But the ordinary theorem that reverse martingales converge a.e. ([4], §275) assures us only that the limit can be taken to be $\bigcap_{n \in \mathbb{N}} \mathrm{T}_n$-measurable; and the σ-algebra $\tilde{\Sigma}_J$ generated by $\bigcup_{i \in J} \Sigma_i$ is normally strictly smaller than $\bigcap_{n \in \mathbb{N}} \mathrm{T}_n$. The extra element needed is the fact that for every $H \in \bigcap_{n \in \mathbb{N}} \mathrm{T}_n$ there is an $H' \in \tilde{\Sigma}_J$ such that $H \triangle H'$ is negligible. When J is empty, this is just the zero-one law: a random variable which is independent of any finite set of coordinates must be essentially constant.

It does not seem to be a trivial matter to identify the set J_E^* from a given set E. Remember that if, for instance, we set

$$E = \{x : \textstyle\sum_{i=0}^{k} x(i) \text{ is even}\} \subseteq \{0,1\}^{\mathbb{N}},$$

then E is *stochastically* independent of the subalgebra generated by any set of k or fewer coordinates, while $J_E^* = \mathbb{N}$. What is true is that $j \in J_E^*$ iff there are a $V \in \Sigma_j$ and a W in the algebra Σ_j^* generated by $\bigcup_{i \neq j} \Sigma_i$ such that $\mu(E \cap V \cap W) \neq \mu(E \cap W) \cdot \mu V$. But we cannot determine whether $j \in J_E^*$ from the interaction of E and Σ_j alone.

13. Interpretation in Hilbert space Continuing the ideas of §12, we observe that if we take U_i to be the closed linear subspace of $L^2 = L^2(\mu)$ generated by $\{\chi F^\bullet : F \in \Sigma_i\}$, then the U_i are orthogonal and $\bigoplus_{i \in I} U_i$ is dense in L^2. So we can find an orthonormal basis $\langle e_\alpha \rangle_{\alpha \in A}$ for L^2 consisting of members of $\bigcup_{i \in I} U_i$. For any square-integrable function f, the sets $A_f = \{\alpha : (f^\bullet | e_\alpha) \neq 0\}$ and $J_f = \{i : a_\alpha \in U_i \text{ for some } \alpha \in A_f\}$ are countable; and now J_f is the smallest subset of I such that f is equal almost everywhere to some function determined by coordinates in J_f. With appropriate translations this approach provides an alternative proof of Theorem 8.

14. Completion regular measures and Kakutani's theorem Suppose that $\langle X_i \rangle_{i \in I}$ is a family of separable metrizable spaces and that μ is a finite measure on X which is inner regular with respect to the family $\mathcal{E}$ of closed sets. Then μ is completion regular, that is, is inner regular with respect to the zero sets, iff it is inner regular with respect to the family $\mathcal{W}$ of sets determined by coordinates in countable sets. **P** If μ is completion regular, then of course it is inner regular with respect to $\mathcal{W}$, because X (being a product of separable spaces) is ccc, so zero sets belong to $\mathcal{W}$ (see 7c above). On the other hand, if μ is inner regular with respect to $\mathcal{W}$, then it is inner regular with respect to $\mathcal{W} \cap \mathcal{E}$, because both $\mathcal{W}$ and $\mathcal{E}$ are closed under countable intersections (see [4], 412A). But 7e tells us that every member of $\mathcal{W} \cap \mathcal{E}$ is a zero set. So μ is completion regular. **Q**

If μ is a quasi-Radon measure, then it is inner regular with respect to the family $\mathcal{K}$ of closed subsets K of X which are self-supporting ([4], §415). In this case, μ is completion regular iff $\mathcal{K} \subseteq \mathcal{W}$. **P** ([1]) If $\mathcal{K} \subseteq \mathcal{W}$ then μ is inner regular with respect to $\mathcal{W}$, therefore completion regular. If μ is completion regular and $K \in \mathcal{K}$, then there is a subset F of K which is a countable union of zero sets and has the same measure as K. Now $F \in \mathcal{W}$ so $\overline{F} \in \mathcal{W}$. On the other hand, $\overline{F} \subseteq K$ and $K \setminus \overline{F}$ is negligible, so $K = \overline{F} \in \mathcal{W}$, as required. **Q**

Now suppose that μ_i is a strictly positive Borel measure on X_i for each i. Kakutani's celebrated theorem can be regarded as stating *either* that the ordinary completed product measure λ on X is a τ-additive topological measure *or* that the τ-additive product measure $\tilde{\lambda}$ is completion regular *or* that λ and $\tilde{\lambda}$ coincide. (The point is that it is relatively easy to show that λ is inner regular with respect to the closed sets, therefore completion regular.) But $\tilde{\lambda}$, being a τ-additive measure on a regular space, is inner regular with respect to the self-supporting closed sets, which belong to $\mathcal{W}$, as noted in §10; so $\tilde{\lambda}$ is completion regular.

15. Two applications I close by mentioning two applications of the ideas here, to show how they can be used in problems which may not appear to be related to the structures I have discussed.

(a) In [2], I solved an old problem of R.G.Haydon by showing that if $\omega_1 < \mathfrak{m}$ (that is, if $\mathrm{MA}(\omega_1)$ is true), and (X, μ) is a Radon probability space of uncountable Maharam type, then there is a continuous surjection from X onto $[0,1]^{\omega_1}$. It was already clear from the work of [8] that the key to the problem was a proper understanding of the measure algebra $\mathfrak{A}$ of the usual measure on $\{0,1\}^{\omega_1}$, and it had long been known that probability algebras provided ccc partially ordered sets to which Martin's axiom might be applied. The key idea in my solution was to examine the set

$$\{(a, K) : a \in \mathfrak{A} \setminus \{0\},\ K \in [\omega_1]^{<\omega},\ a \text{ is determined by coordinates in } \omega_1 \setminus K\},$$

ordered by saying that $(a, K) \le (b, L)$ if $b \subseteq a$ and $K \subseteq L$. This is ccc; the argument is not trivial, but the obvious approach leads straight back to a lemma which turns out to be true. And similarly standard arguments using the hypothesis that $\omega_1 < \mathfrak{m}$ lead to enough Boolean-independent families in $\mathfrak{A}$ for one to be consistent with the structure of the family of compact subsets of X and provide a continuous surjection from a closed subset of X onto $\{0,1\}^{\omega_1}$. The point I want to make here is that it was essential to look at the structure on $\mathfrak{A}$ defined by its representation as a measure algebra of a product space, and that the idea of being 'determined by coordinates in J' was the key addition.

(b) The second application was of a very different kind, in [3], Theorem 3G. Here, in the course of proving that a weakly α-favourable measure of a certain type was necessarily α-favourable, I found myself depending on the following fact: we can choose, for each zero set $F \subseteq \{0,1\}^I$, a countable set $M_F \subseteq I$ such that (i) a zero set F is always determined by coordinates in M_F (ii) if $\langle F_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence of zero sets with intersection F, then $[M_F]^{<\omega} \subseteq \bigcup_{n \in \mathbb{N}} [M_{F_n}]^{<\omega}$. I do not see any way of achieving this except by taking M_F to be the smallest set such that F is determined by coordinates in M_F, as in Proposition 3, and then using 2a to observe that if $\langle F_n \rangle_{n \in \mathbb{N}} \downarrow F$ then $M_F \subseteq \liminf_{n \to \infty} M_{F_n}$.

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