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Universally Kuratowski-Ulam spaces

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This note is a supplement to Fremlin Natkaniec & Recław 00.

1. Definition A **uKU′-space** is a topological space Y such that whenever X is a topological space and $E \subseteq X \times Y$ is nowhere dense, then $\{x : E[\{x\}]$ is not nowhere dense$\}$ is meager in X; that is, (X, Y) is a K-U′-pair in the sense of Fremlin Natkaniec & Recław 00 for every X.

2. Lemma A topological space Y is a uK-U space in the sense of Fremlin Natkaniec & Recław 00 iff its Baire part is uKU′, where the 'Baire part' of Y is $Y \setminus \bigcup\{V : V \subseteq Y$ is open and meager$\}$.

3. Lemma Suppose that Y_1, Y_2 are topological spaces with π-bases $\mathcal{V}_1$, $\mathcal{V}_2$, neither containing $\emptyset$, and that $\theta : \mathcal{V}_1 \to \mathcal{V}_2$ is a function such that

(α) $\theta[\mathcal{V}_1]$ is coinitial with $\mathcal{V}_2$,

(β) θ is order-preserving,

(γ) if V, $V' \in \mathcal{V}_1$ and $V \cap V' = \emptyset$ then $\theta V_1 \cap \theta V_2 = \emptyset$.

Then Y_1 is uKU′ iff Y_2 is.

proof (a) It will simplify the formulae below if we observe that θ has an extension to a function $\tilde{\theta}$ from the set of non-empty open subsets of Y_1 to the set of non-empty open subsets of Y_2 with the same properties. **P** Set $\tilde{\theta}G = \bigcup\{\theta V : V \in \mathcal{V}_1,\ V \subseteq G\}$ for every non-empty open set $G \subseteq Y_1$. **Q** Henceforth, therefore, I will suppose that $\mathcal{V}_1$, $\mathcal{V}_2$ are actually the families of non-empty open sets.

(b) Suppose that Y_1 is uKU′ and that X is a topological space, $E_2 \subseteq X \times Y_2$ a nowhere dense closed set. Set

$$E_1 = (X \times Y_1) \setminus \bigcup\{U \times V : U \subseteq X \text{ is open},\ V \in \mathcal{V}_1,\ (U \times \theta V) \cap E_2 = \emptyset\}.$$

Then E_1 is nowhere dense. **P** If $U \subseteq X$, $V_1 \subseteq Y_1$ are non-empty open sets there are a non-empty open $U' \subseteq U$ and a $V_2 \in \mathcal{V}_2$ such that $V_2 \subseteq \theta V_1$ and $U' \times V_2$ is disjoint from E_2, because E_2 is nowhere dense. There is a $V_1' \in \mathcal{V}_1$ such that $\theta V_1' \subseteq V_2$, by (i); since $\theta V_1' \cap \theta V_1$ is not empty, nor is $V_1'' = V_1' \cap V_1$, by (iii), and $\theta V_1'' \subseteq V_2$, so $U' \times \theta V_1''$ is disjoint from E_2 and $U' \times V_1''$ is disjoint from E_1. But $U' \times V_1''$ is a non-empty open subset of $U \times V$. **Q**

Because Y_1 is uKU′, $A = \{x : \operatorname{int} E_1[\{x\}] = \emptyset\}$ is comeager in X. **?** Suppose, if possible, that $x \in A$ and that $E_2[\{x\}]$ has non-empty interior. Then there is a $V_1 \in \mathcal{V}_1$ such that $\theta V_1 \subseteq E_2[\{x\}]$. But $V_1 \not\subseteq E_1[\{x\}]$, so there must be an open $U \subseteq X$ and a $V_1' \in \mathcal{V}_1$ such that $x \in U$, $V_1'' = V_1 \cap V_1' \neq \emptyset$ and $U \times \theta V_1'$ does not meet E_2; in which case $\theta V_1 \cap \theta V_1' = \emptyset$. And this is impossible, because $\theta V_1'' \subseteq \theta V_1 \cap \theta V_1'$. **X** This shows that $\{x : \operatorname{int} E_2[\{x\}] = \emptyset\}$ includes A and is comeager. As E_2 is arbitrary, Y_2 is uKU′.

(b) Now suppose that Y_2 is uKU′ and that $E_1 \subseteq X \times Y_1$ is closed and nowhere dense. Set

$$E_2 = (X \times Y_2) \setminus \bigcup\{U \times \theta V : V \in \mathcal{V}_1,\ (U \times V) \cap E_1 = \emptyset\}.$$

Then E_2 is nowhere dense in $X \times Y_2$. **P** Let $U \subseteq X$, $V_2 \subseteq Y_2$ be non-empty open sets. Let $V_1 \subseteq Y_1$ be a non-empty open set such that $\theta V_1 \subseteq V_2$. Then there are non-empty open $U' \subseteq U$, $V_1' \subseteq V_1$ such that $U' \times V_1'$ does not meet E_1; in which case $U' \times \theta V_1'$ is a non-empty open set included in $U \times V_2$ and not meeting E_2. **Q**

Because Y_2 is uKU′, $A = \{x : \operatorname{int} E_2[\{x\}] = \emptyset\}$ is comeager. **?** Suppose, if possible, that $x \in A$ and that $V_1 = \operatorname{int} E_1[\{x\}] \neq \emptyset$. Then $\theta V_1 \setminus E_2[\{x\}]$ is not empty, so there are non-empty open sets U, V_1' such that $U \times V_1'$ does not meet E_1 and $x \in U$ and $\theta V_1' \cap \theta V_1$ is non-empty; in which case $V_1'' = V_1' \cap V_1$ is not empty. But $V_1' \cap E_1[\{x\}] = \emptyset$. **X** Thus $\{x : \operatorname{int} E_1[\{x\}] = \emptyset\}$ includes A and is comeager. As X and E_1 are arbitrary, Y_1 is uKU′.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

4. Corollary (a) If Y_1, Y_2 have order-isomorphic π-bases and one is uKU$'$, so is the other.
(b) If Y is a topological space, it is uKU$'$ iff the Stone space of its regular open algebra is.
(c) If Y is any topological space and $Y_0 \subseteq Y$ is dense, then Y is uKU$'$ iff Y_0 is.

proof (a) Immediate from Lemma.

(b) Write $\mathfrak{T}$ for the original topology of Y and $\mathfrak{S}$ for the topology generated by the family $\mathcal{U}$ of $\mathfrak{T}$-regular-open sets. If $G \in \mathfrak{T}$, then $Y \setminus \overline{G}^{\mathfrak{T}} \in \mathcal{U}$, so $\overline{G}^{\mathfrak{S}} \subseteq \overline{G}^{\mathfrak{T}}$; but $\mathfrak{S}$ is coarser than $\mathfrak{T}$, so the two closures are the same. Applying this now to the complement of $\overline{G}^{\mathfrak{T}}$, we see that $\operatorname{int}_{\mathfrak{T}}\overline{G}^{\mathfrak{T}} = \operatorname{int}_{\mathfrak{S}}\overline{G}^{\mathfrak{S}}$. And this is true for every $G \in \mathfrak{T}$; *a fortiori*, it is true for every $G \in \mathfrak{S}$. Thus $\mathfrak{S}$ and $\mathfrak{T}$ have the same regular open algebras.

Note next that, writing $\theta G = \operatorname{int}_{\mathfrak{T}}\overline{G}^{\mathfrak{T}}$ for $G \in \mathfrak{T} \setminus \{\emptyset\}$, $\theta : \mathfrak{T} \setminus \{\emptyset\} \to \mathcal{U} \setminus \{\emptyset\}$ satisfies the conditions of the Lemma, while $\mathcal{U}$ is a π-base for $\mathfrak{S}$. So $(Y, \mathfrak{T})$ is uKU$'$ iff $(Y, \mathfrak{S})$ is. Next, $\mathcal{U}$ is a base for $\mathfrak{S}$ and is isomorphic, as partially ordered set, to the family of open-and-closed sets in its Stone space Z, which is a base for the topology of Z; so Z is uKU$'$ iff $(Y, \mathfrak{S})$ is iff $(Y, \mathfrak{T})$ is, as claimed.

(c) Set $\theta V = V \cap Y_0$ for non-empty open sets $V \subseteq Y$; then θ satisfies the conditions of the Lemma.

5. Proposition (a) Any space of countable π-weight is uKU$'$.
(b) Every uKU$'$ space is productively ccc.

proof (a) If X is any topological space, Y has a countable π-base $\mathcal{V}$ not containing $\emptyset$, and $E \subseteq X \times Y$ is closed and nowhere dense, then for each $V \in \mathcal{V}$ the set $A_V = \{x : V \subseteq E[\{x\}]\}$ is closed and nowhere dense, because $\operatorname{int} A_V \times V \subseteq E$; but $\{x : \operatorname{int} E[\{x\}] \neq \emptyset\} = \bigcup_{V \in \mathcal{V}} A_V$, so is meager.

(b) If Y is not productively ccc, then there are a ccc topological space Z and families $\langle V_\xi \rangle_{\xi<\omega_1}$, $\langle W_\xi \rangle_{\xi<\omega_1}$ of non-empty open subsets of Y, Z respectively such that $\langle W_\xi \times V_\xi \rangle_{\xi<\omega_1}$ is disjoint. Taking $W'_\xi = \operatorname{int} \overline{W_\xi}$ for each ξ, $\langle W'_\xi \times V_\xi \rangle_{\xi<\omega_1}$ is still disjoint, while each W'_ξ is a regular open set. The regular open algebra of Z is ccc, so its Stone space $\tilde{Z}$ is also ccc; and if $\tilde{W}_\xi$ is the open-and-closed set in $\tilde{Z}$ corresponding to $W'_\xi \subseteq Z$, then $\langle \tilde{W}_\xi \times V_\xi \rangle_{\xi<\omega_1}$ is disjoint in $\tilde{Z} \times Y$.

Set $X = \{0,1\}^{\omega_1}$ and for $\xi < \omega_1$ set $X_\xi = \{x : x \in X,\ x(\eta) = 0 \text{ for } \eta \geq \xi\}$; then every X_ξ is nowhere dense, but $\bigcup_{\xi<\omega_1} X_\xi$ is not meager. Set

$$E = \overline{\bigcup_{\xi<\omega_1} X_\xi \times \tilde{W}_\xi \times V_\xi} \subseteq (X \times \tilde{Z}) \times Y.$$

Then E is nowhere dense in $X \times \tilde{Z} \times Y$. **P** If $U \subseteq X$ and $V \subseteq Y$ and $W \subseteq \tilde{Z}$ are non-empty open sets, then if $(W \times V) \cap (\tilde{W}_\xi \times V_\xi) = \emptyset$ for every ξ, we have $U \times W \times V$ disjoint from E; while if $(W \times V) \cap (\tilde{W}_\xi \times V_\xi) \neq \emptyset$, then $(U \setminus X_\xi) \times (W \cap \tilde{W}_\xi) \times (V \cap V_\xi)$ is a non-empty open set disjoint from E. **Q**

Now examine $\{(x,z) : \operatorname{int} E[\{(x,z)\}] \neq \emptyset\} \supseteq \bigcup_{\xi<\omega_1} X_\xi \times \tilde{W}_\xi$. This is not meager in $X \times \tilde{Z}$. **P** Let $\langle H_n \rangle_{n\in\mathbb{N}}$ be a sequence of dense open subsets of $X \times \tilde{Z}$. Because X is productively ccc, $X \times \tilde{Z}$ is ccc and for each $n \in \mathbb{N}$ we can find a dense open subset H'_n of H_n which is a countable union of open sets of the form $U \times W$, where $U \subseteq X$ is determined by finitely many coordinates. There is therefore, for each $n \in \mathbb{N}$, a countable set $J_n \subseteq \omega_1$ such that if $(x,z) \in H'_n$ and $x' \restriction J_n = x \restriction J_n$ then $(x',z) \in H'_n$. Let $\xi \in \omega_1$ be such that $J_n \subseteq \xi$ for every $n \in \mathbb{N}$.

Because X and $\tilde{Z}$ are both compact Hausdorff spaces, $X \times \tilde{Z}$ is a Baire space and $\bigcap_{n\in\mathbb{N}} H'_n$ is dense. It therefore meets the non-empty open set $X \times \tilde{W}_\xi$; take $x \in X$, $z \in \tilde{W}_\xi$ such that $(x,z) \in \bigcap_{n\in\mathbb{N}} H'_n$. Define $x' \in X$ by setting $x'(\eta) = x(\eta)$ if $\eta < \xi$, 0 otherwise; then $(x',z) \in (X_\xi \times \tilde{W}_\xi) \cap \bigcap_{n\in\mathbb{N}} H_n$. As $\langle H_n \rangle_{n\in\mathbb{N}}$ is arbitrary, $\bigcup_{\xi<\omega_1} X_\xi \times \tilde{W}_\xi$ is non-meager. **Q**

Thus $E \subseteq (X \times \tilde{Z}) \times Y$ witnesses that Y is not uKU$'$.

6. Examples (a) There is a separable compact Hausdorff space which is not uKU$'$.
(b) There is a countable regular Hausdorff space which is not uKU$'$.
(c) There is a compact Hausdorff uKU$'$ space with a Hausdorff continuous image which is not uKU$'$.

proof (a) Give $\mathbb{N}^{\mathbb{N}}$ the topology $\mathfrak{T}_d$ of Fremlin Natkaniec & Recław 00, Example 1. This is not uKU$'$ (Fremlin Natkaniec & Recław 00, Corollary 5). So the Stone space Z of its regular open algebra $\mathfrak{G}$ is not uKU$'$, by 4b. But $\mathfrak{G}$ is σ-centered so Z is separable.

(b) If $Z_0 \subseteq Z$ is a countable dense set, then it is not uKU$'$, by 4c.

(c) Enumerate Z_0 as $\langle z_n \rangle_{n \in \mathbb{N}}$. Let $\langle w_n \rangle_{n \in \mathbb{N}}$ be a sequence of distinct points not in Z, and set $W = Z \cup \{w_n : n \in \mathbb{N}\}$; define $f : W \to Z$ by setting $f(w_n) = z_n$ for every n, $f(z) = z$ for every $z \in Z$. Give W the topology generated by

$$\{\{w_n\} : n \in \mathbb{N}\} \cup \{W \setminus \{w_n\} : n \in \mathbb{N}\} \cup \{f^{-1}[H] : H \subseteq Z \text{ is open}\}.$$

It is easy to check that W is compact and Hausdorff, and of course $f : W \to Z$ is a continuous surjection. But $\{\{w_n\} : n \in \mathbb{N}\}$ is a countable π-base for the topology of W, so W is uKU$'$, while Z is not.

Remark By Fremlin Natkaniec & Recław 00, Theorem 1, any regular space which is a continuous image of a dense subset of $\{0,1\}^\kappa$, for any κ, is a uK-U space. So if it is Baire, it is a uKU$'$ space; but otherwise it needn't be. Note that the space W of (c) above is a uKU$'$ space which is not a continuous image of a dense subset of any $\{0,1\}^\kappa$.

7. Proposition The product of any family of uKU$'$ spaces is uKU$'$.

proof (a) The basis of the proof is the following fact: if $\langle Y_i \rangle_{i \in I}$ is a family of compact Hausdorff uKU$'$ spaces with product Y, X is a topological space, $E \subseteq X \times Y$ is nowhere dense, and $n \in \mathbb{N}$, then

$$\begin{aligned} A = \{x : x \in X, &\text{ there is a non-empty open set } V \subseteq E[\{x\}] \\ &\text{determined by } n \text{ or fewer coordinates}\} \end{aligned}$$

is meager. **P** Induce on n. If $n = 0$ this is trivial because the only non-empty open set determined by 0 coordinates is Y itself, so that $A \times Y \subseteq E$ and A must be nowhere dense. For the inductive step to $n \geq 1$, **?** suppose, if possible, that A is not meager. Then there is a non-empty open set $U \subseteq X$ such that $A \cap U'$ is non-meager for any non-empty open set $U' \subseteq U$. But as E is nowhere dense, there are a non-empty open set $U' \subseteq U$ and a non-empty open set $V \subseteq Y$, determined by a coordinates in a finite set $J \subseteq I$, such that $(U' \times V) \cap E = \emptyset$.

For each $x \in A \cap U'$, we have a non-empty open set $V_x \subseteq E[\{x\}]$, determined by coordinates in a set $J_x \in [I]^{\leq n}$. Since $V_x \cap V'$ is always empty, $J_x \cap J$ is never empty. Since J is finite, we have a $j \in J$ such that $A' = \{x : x \in A \cap U', j \in J_x\}$ is non-meager.

Now turn to regarding E as a nowhere dense set in $(X \times Y_j) \times Y'$, where $Y' = \prod_{i \neq j} Y_i$. Set $B = \{(x, u) : x \in A', u \in \pi_j[V_x]\}$, where $\pi_j : Y \to Y_j$ is the coordinate map. Since Y_j is uKU$'$ and Baire, and $B[\{x\}]$ includes a non-empty open set for every x in the non-meager set A', B is non-meager. For $(x, u) \in B$, $E[\{x, u\}]$, interpreted as a subset of Y', includes $V_x[\{u\}]$, which is a non-empty open set determined by coordinates in $J_x \setminus \{j\}$, which has at most $n - 1$ members. But this contradicts the inductive hypothesis applied to $X \times Y_j$ and $\prod_{i \in J} Y_i$. **X**

Thus A is meager and the induction proceeds. **Q**

(b) Accordingly any product of compact Hausdorff uKU$'$ spaces is uKU$'$. **P** If $\langle Y_i \rangle_{i \in I}$ is a family of compact Hausdorff uKU$'$ spaces with product Y, X is any topological space, and $E \subseteq X \times Y$ is closed and nowhere dense, set

$$\begin{aligned} A_n = \{x : x \in X, &\text{ there is a non-empty open set } V \subseteq E[\{x\}] \\ &\text{determined by } n \text{ or fewer coordinates}\} \end{aligned}$$

Then every A_n is meager, by (a), so $\bigcup_{n \in \mathbb{N}} A_n$ is meager; but this is just $\{x : \operatorname{int} E[\{x\}] \neq \emptyset\}$. As X and E are arbitrary, Y is uKU$'$. **Q**

(c) Finally, any product of uKU$'$ spaces is uKU$'$. **P** Let $\langle (Y_i, \mathfrak{T}_i) \rangle_{i \in I}$ be a family of uKU$'$ spaces, with product $(Y, \mathfrak{T})$. For each i, let $\mathfrak{S}_i$ be the topology on Y_i generated by the family $\mathcal{V}_i$ of $\mathfrak{T}_i$-regular-open sets, as in the proof of 4b, and Z_i the Stone space of the regular open algebra of Y_i, so that $\mathcal{V}_i$ is order-isomorphic to a base for the topology of Z_i. By 4b, every Z_i is uKU$'$, so $Z = \prod_{i \in I} Z_i$ is uKU$'$, by (b). Now set

$$\mathcal{V} = \{\prod_{i \in I} V_i : V_i \in \mathfrak{T}_i \text{ for every } i \in I, \{i : V_i \neq Y_i\} \text{ is finite}\}.$$

$$\mathcal{W} = \{\prod_{i\in I} V_i : V_i \in \mathcal{V}_i \text{ for every } i \in I,\ \{i : V_i \neq Y_i\} \text{ is finite}\}.$$

Then $\mathcal{W}$ is a base for the topology $\mathfrak{S}$ on Y which is the product of the topologies $\mathfrak{S}_i$, and is order-isomorphic to a base for the topology of Z; so $(Y,\mathfrak{S})$ is uKU$'$. Now, just as in 4b, we have a function $\theta : \mathcal{V}\setminus\{\emptyset\} \to \mathcal{W}\setminus\{\emptyset\}$, defined by setting $\theta V = \operatorname{int}_{\mathfrak{T}} \overline{V}^{\mathfrak{T}}$ for $V \in \mathcal{V}\setminus\{\emptyset\}$, which satisfies the conditions of Lemma 3; so $(Y,\mathfrak{T})$ is uKU$'$, as claimed. **Q**

8. Proposition (a) An open subset of a uKU$'$ space is uKU$'$.
(b) If Y is the union of countably many open uKU$'$ subspaces it is uKU$'$.
(c) If Y is uKU$'$ and $f : Y \to Z$ is a continuous open surjection, then Z is uKU$'$.

9. Proposition If X is a Baire space, Y is a Baire uKU$'$ space then $X \times Y$ is Baire.

10. More examples (a) Recall from Fremlin Natkaniec & Recław 00 that the density topology on $\mathbb{R}$ is not uKU$'$. Consequently the Stone space of the Lebesgue measure algebra is not uKU$'$. The argument shows that the Stone space of any atomless probability measure is not uKU$'$.

(b) Mathias forcing Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$ and give $Y = [\mathbb{N}]^\omega$ the topology generated by the usual topology and sets of the form $\{a : a \subseteq b\}$ where $b \in \mathcal{F}$. This is Baire (in fact, α-favourable). It is not uKU$'$; the witness is the set

$$E = \{(a,b) : b \subseteq a \text{ or } b \cap a = \emptyset\} \subseteq Y \times Y.$$

(c) Let $Y \subseteq \mathcal{P}(\mathbb{N}\times\mathbb{N})$ be the family of those sets $a \subseteq \mathbb{N}\times\mathbb{N}$ such that $\#(a[n]) \le n$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{1}{n}\#(a[n]) = 0$. (Here $a[n] = \{j : \exists\, i < n,\ (i,j) \in a\}$.) Give Y the topology generated by the usual topology and sets of the form $\{a : a \supseteq b\}$ where $b \in Y$. Then Y is not uKU$'$. The witness is the pair (X,E), where $X = Y$ and the topology of X is that generated by the topology of Y and sets of the form $\{a : \#(a[n]) \le \epsilon n \text{ for } n \ge n_0\}$; this is α-favourable, therefore Baire; and E is the set $\{(a,b) : a \subseteq b\} \subseteq X \times Y$.

Problems (a) Given a topological space Y – even a compact Hausdorff space – it is not clear how to approach the question of whether it is uKU$'$. In particular, it is not clear which test spaces X to look at first. If you look at the methods used here to prove that spaces are *not* uKU$'$, you see that we have a quick test to sort out spaces which are not ccc (5b), the space $\{0,1\}^{\omega_1}$. (If Y is not ccc, then we can apply the argument for 6b with $Z = \{0\}$.) In 6a, however, we seem to need to look at the product $Y \times Y$. The same attack serves when Y is the real line with the density topology (Fremlin Natkaniec & Recław 00, Example 2). But I do not know, for instance, whether

if Y is a compact Hausdorff space which is not uKU$'$, are there necessarily a compact Hausdorff space X and a nowhere dense closed $E \subseteq X \times Y$ such that $\{x : \operatorname{int} E[\{x\}] \neq \emptyset\}$ is not meager?

(b) Can we strengthen 5b? For instance, does every uKU$'$ space satisfy Knaster's condition, or have precaliber ω_1?

(c) Can we strengthen Lemma 3? For instance, if Y_1, Y_2 have π-bases (not containing $\emptyset$) which are Tukey equivalent downwards (see Fremlin 91), and one is uKU$'$, must the other also be?

Reference

Fremlin D.H. [91] 'The partially ordered sets of measure theory and Tukey's ordering,' Note di Matematica 11 (1991) 177-214.

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