Non-linear Optimisation Problems

1. For each of the following functions, find its gradient vector $\nabla f(\mathbf{x})$, its Hessian $\mathbf{H}f(\mathbf{x})$, and its stationary points. Determine whether the stationary points are local maxima, local minima or saddle points; give reasons for your answers.
   (a) $f_1(\mathbf{x}) = x_1^2 - 3x_1x_2 + 3x_2^2$,
   (b) $f_2(\mathbf{x}) = 2x_1^2 - 5x_1x_2 + 2x_2^2$,
   (c) $f_3(\mathbf{x}) = x_1x_2(1 - x_1 - x_2)$,
   (d) $f_4(\mathbf{x}) = \sin(x_1 + x_2)$
for $\mathbf{x} \in \mathbb{R}^2$, and
   (e) $f_5(\mathbf{x}) = 2x_1^2 + 3x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_1x_3$
   (f) $f_6(\mathbf{x}) = x_3 \exp(-x_1^2 - x_2^2)$
for $\mathbf{x} \in \mathbb{R}^3$. [AA]

2. Which of the following functions are coercive, and how do you know?
(a) $f_1(\mathbf{x}) = x_1^2 - 3x_1x_2 + 2x_2^2$ for $\mathbf{x} \in \mathbb{R}^2$,
(b) $f_2(\mathbf{x}) = x_1^2 - 2x_1x_2 + 3x_2^2$ for $\mathbf{x} \in \mathbb{R}^2$,
(c) $f_3(\mathbf{x}) = x_1^2 + x_2^3 + x_3^4$ for $\mathbf{x} \in \mathbb{R}^3$,
(d) $f_4(\mathbf{x}) = x_1^4 + x_2^4 + x_3^4 - 4x_1^2x_2x_3$ for $\mathbf{x} \in \mathbb{R}^3$,
(e) $f_5(\mathbf{x}) = (x_1 + \cos x_2x_3)^2 + (x_2 + \cos x_3x_1)^2 + (x_3 + \cos x_1x_2)^2$ for $\mathbf{x} \in \mathbb{R}^3$. [AB]

3. For which values of c is the function $f(x) = x^2 + cx^3 + x^4$ convex? [AC]

4. Which of the following functions are convex?
(i) $f(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_3x_1$ for $\mathbf{x} \in \mathbb{R}^3$.
(ii) $f(\mathbf{x}) = x_1^4 + x_1x_2^3 + x_2^4$ for $\mathbf{x} \in \mathbb{R}^2$.
(iii) $f(\mathbf{x}) = e^{x_1 - x_2}$ for $\mathbf{x} \in \mathbb{R}^2$. [AD]

5. As question 4, with
(i) $f(\mathbf{x}) = (e^{x_1+x_2} - 2)^2$ for $\mathbf{x} \in \mathbb{R}^2$.
(ii) $f(\mathbf{x}) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + x_3^2 + x_1x_3 + x_2x_3$ for $\mathbf{x} \in \mathbb{R}^3$. [AE]

6. (a) Let C_1, C_2 be convex subsets of $\mathbb{R}^n$.
(i) Is $C_1 \cap C_2$ necessarily convex?
(ii) Is $C_1 \cup C_2$ necessarily convex?
(b) Suppose that f and g are convex functions defined on a convex set $C \subseteq \mathbb{R}^n$, and $a \in \mathbb{R}$.
(i) Is $f + g : C \to \mathbb{R}$ necessarily convex?
(ii) Is $af : C \to \mathbb{R}$ necessarily convex?
(iii) Is $\max(f, g) : C \to \mathbb{R}$ necessarily convex?
(iv) Is $\min(f, g) : C \to \mathbb{R}$ necessarily convex?
(v) Is $f \times g : C \to \mathbb{R}$ necessarily convex?
Justify your answers. [AF]

7. Which of the following functions are continuous, and which are not? Justify your answers.
(a) $f_1 : \mathbb{R} \to \mathbb{R}$, where $f_1(x) = (x-1)^2$ if $x \geq 0$ and $f_1(x) = e^x$ if $x < 0$.
(b) $f_2 : \mathbb{R} \to \mathbb{R}$, where $f_2(x) = x^2$ if $x \geq 0$ and $f_2(x) = e^{x+1}$ if $x < 0$.
(c) $g_1 : \mathbb{R}^2 \to \mathbb{R}$, where $g_1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \sin x_1 \cos x_2$ if $x_1 \geq 0$ and $g_1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \cos x_1 \sin x_2$ if $x_1 < 0$.
(d) $g_2 : \mathbb{R}^2 \to \mathbb{R}$, where $g_2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \sin x_1 \cos x_2$ if $x_1 \geq x_2$ and $g_2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \cos x_1 \sin x_2$ if $x_1 < x_2$.
[AG]

8. Find the first and second partial derivatives of the following functions:
(a) $f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 e^{-x_2}$;
(b) $g\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 x_2^2 x_3^3 + x_2 x_3^2 x_1^3 + x_3 x_1^2 x_2^3$.
[AH]

9. Find the determinants of the matrices $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. [BA]

10. For each of the following matrices, say whether it is positive definite/positive semi-definite/negative semi-definite/negative definite, or none of these.
(a) $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$.
Give reasons for your answers. [BB]

11. Set $f(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 + x_2x_3$ for $\mathbf{x} \in \mathbb{R}^3$.
(a) Show that f is convex.
(b) Show that f has a unique global minimum. Where is it?
(c) Show that f is coercive.
(d) Show that $f(\mathbf{x})$ has a least value subject to the condition $x_1x_2x_3 \le 1$. Does it have a greatest value subject to this condition? Justify your answer. [BC]

12. Consider the problem

$$\text{maximise } x_1x_2x_3 \text{ subject to } x_1^2 + x_2^2 + x_3^2 = 6,\ x_1 + x_2 + x_3 = 4.$$

(a) Show that this problem has at least one solution, and that all solutions can be found by the Lagrange multiplier method.
(b) Use Lagrange multipliers to solve the problem. (*Hint*: at any stationary point of the Lagrangean, at least two of x_1, x_2, x_3 are equal.) [CA]

13. Use the method of Lagrange multipliers to minimise $f(\mathbf{x}) = x_2 + x_2^2 - 2x_1x_2$ subject to the constraints $x_1^2 \le x_2$ and $x_1^2 \le 1$. (You should examine separately the unconstrained problem, the two problems with a single equality constraint and the problem with two equality constraints. Take care to check which of the stationary points you find are feasible for the original problem.) [CB]

14. Use the bisection method to approximate a solution of the equation $f(x) = x + 4\cos x = 0$, starting from $x_0 = 0$, $x_1 = 3$. (Do all calculations to at least four decimal places. Continue until you find an x such that $|f(x)| \le 0{\cdot}01$.) [DA]

15. Use the method of false position to approximate a solution of the equation $f(x) = x + 4\cos x = 0$, starting from $x_0 = 0$, $x_1 = 3$. (Do all calculations to at least four decimal places. Continue until you find an x such that $|f(x)| \le 0{\cdot}01$.) [DB]

16. Use Newton's method to approximate a solution of the equation $f(x) = x + 4\cos x = 0$, starting from $x_0 = 0$. (Do all calculations to at least four decimal places. Continue until you find an x such that $|f(x)| \le 0{\cdot}01$.)
Compared with the false-position method, does this use more, or fewer, calculations involving the function f? [DC]

17. What happens if you attempt to find a solution of $f(x) = \arctan x = 0$ by the method of false position, starting from $x_0 = -1$ and $x_1 = 5$? [DD]

18. (a) Show that the function $f(x) = \cosh x - \sin x$, for $x \in \mathbb{R}$, is convex. Is it coercive? Why?
(b) Use the three-point method to locate a minimum of f within an interval of length at most 0·1, starting from the three points $a_0 = 0$, $b_0 = 1$ and $c_0 = 2$. Give the upper and lower ends of your final interval. (Do all your calculations to at least five decimal places.) [DE]

19. Consider the function $f(x) = x^4 - 9x^2$ for $x \in \mathbb{R}$. Use Newton's method to search for stationary points of f, starting from (i) $x_0 = 0{\cdot}94$ (ii) $x_0 = 0{\cdot}95$ (iii) $x_0 = 0{\cdot}96$. (In each case continue the iteration until you have $|x_n - x_{n-1}| < 0{\cdot}1$, and find $f(x_n)$.) [DF]

20. Consider the function $f(\mathbf{x}) = x_1^4 - x_1^2 x_2 + x_2^2$ for $\mathbf{x} \in \mathbb{R}^2$. Starting from $\mathbf{x}^{(0)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, find the next two points $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ when searching for a stationary point of f by Newton's method. [DG]

21. (a) Show that the function f of question 20 is coercive. Is it convex? Why?
(b) Starting from $\mathbf{x}^{(0)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, find the next two points $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ when searching for a minimum of f by the method of steepest descent. (Use the three-point method, with final interval length at most 0·1, for the line search problem.) [DH]

22. Use the method of Lagrange multipliers to find the stationary values of the function $f(\mathbf{x}) = x_1^3 - x_2^4$ subject to the constraint $x_1^2 + 2x_2^2 = 4$. Which, if any, of these are global maxima or minima? [IB]

23. Use the method of Lagrange multipliers to find the minima of $f(\mathbf{x}) = x_1 x_2 + x_3$ subject to the constraint $x_1^2 + x_2^2 + x_3^2 = c$, where $c > 0$. Give the value f^* of the function at these minima, as a function of c, and calculate $\frac{df^*}{dc}$. [ID]

24. Consider the problem

Find the minima of $f(\mathbf{x}) = x_1 + x_2 + x_3$ subject to the constraints $x_1^2 + x_2^2 = \gamma$, $x_2^2 + x_3^2 = \delta$, where γ, $\delta > 0$.

Write down the equations provided by the method of Lagrange multipliers, and show that $x_1 x_3 = x_2(x_1 + x_3)$ at any minimum. Complete the solution for the case $\gamma = \delta$. Writing $f^* = f^*(\gamma, \delta)$ for the minimum value of f, regarded as a function of γ and δ, find $\frac{\partial f^*}{\partial \gamma}$ when $\gamma = \delta = 10$. [IE]

25. Set $f(\mathbf{x}) = x_1^2 + 5x_2^2 + 8x_3^2$, $g(\mathbf{x}) = x_1 x_2 + x_2 x_3 + x_3 x_1 + 1$ for $\mathbf{x} \in \mathbb{R}^3$.
(a) Show that there is a minimal value of $f(\mathbf{x})$ subject to the condition $g(\mathbf{x}) = 0$. Is there a maximal value? Justify your answer.
(b) Suppose that $\mathbf{x}^*$ minimises f subject to the condition $g = 0$.
(i) Use the method of Lagrange multipliers to show that, for some λ,

$$2x_1^* = \lambda(x_2^* + x_3^*), \quad 10x_2^* = \lambda(x_1^* + x_3^*), \quad 16x_3^* = \lambda(x_1^* + x_2^*).$$

(ii) Explain why the determinant of the matrix $\begin{pmatrix} 2 & -\lambda & -\lambda \\ -\lambda & 10 & -\lambda \\ -\lambda & -\lambda & 16 \end{pmatrix}$ must be zero. Show that λ is either -4 or $-5 \pm \sqrt{65}$.
(iii) Show that at least one of x_1^*, x_2^*, x_3^* is less than 0 and at least one is greater than 0. Show that $x_3^{*2} \le x_2^{*2} \le x_1^{*2}$, and therefore that $x_1^* \ne 0$ and x_2^* and x_3^* are of the same sign (or zero). Hence show that $\lambda \le 0$.
(iv) Show that $(16 + \lambda)x_3^* = (10 + \lambda)x_2^*$, so that neither x_2^* nor x_3^* can be zero and that $\lambda \notin [-16, -10]$.
(c) Find λ and $f(\mathbf{x}^*)$. How many possible values are there for $\mathbf{x}^*$? [IF]

26. Consider the function $f(x) = x + 3\cos x$ for $x \in \mathbb{R}$.
(a) Use the dichotomous search method, with indistinguishability constant 0·1, to locate the minimum of f in the interval $[1, 4]$ within an interval of length at most 0·2.
(b) Use the bisection search method to locate the minimum of f in the interval $[1, 4]$ within an interval of length at most 0·2.
(c) Use the method of false position, starting from $x_0 = 1$ and $x_1 = 4$, to find an x such that $|f'(x)| \leq 0{\cdot}01$.
[JA]

27. Consider the function $f(\mathbf{x}) = x_1^2 - x_1x_3 + x_2^2 + x_3^2$ for $\mathbf{x} \in \mathbb{R}^3$. Starting from $\mathbf{x}^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, find the next two points $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ when searching for a minimum of f by the method of steepest descent. (For each line search problem, express $f(\mathbf{x} - t\nabla f\mathbf{x})$ as a quadratic in t and find its exact minimum.)
[JB]

28. Consider the problem of minimising the function $f(x) = e^{-x} + x^2$.
(a) Starting from the initial interval $[-1, 1]$, use the bisection search method to reduce the interval of uncertainty to one of length at most 0·2.
(b) Starting from $x_0 = -1$, $x_1 = 1$, use the method of false position to find a point x such that $|f'(x)| \leq 0{\cdot}01$.
(c) Starting from the initial point $x_0 = 0$, use Newton's method to find a point x such that $|f'(x)| \leq 0{\cdot}01$.
[JE]

29. (a) What is a *convex program*? For a given convex program, what is its *dual*?
(b) Consider the problem
(P) minimise $x_1^2 + 2x_2^2 + 3x_3^2$
for $\mathbf{x} \in \mathbb{R}^3$
subject to the constraints $x_1 + x_2 \geq 2$, $x_2 + x_3 \geq 1$.
Show that (P), suitably reformulated, is a convex program.
(c) State the dual problem (D) to the problem (P) of part (b), and solve (D) and (P). [LA]

30. Consider the problem
(P) minimise $(x_1 + 1)^2 + x_2^2$
for $\mathbf{x} \in C$
subject to the constraint $x_1 + 2x_2 \geq 2$,
where $C = \{\mathbf{x} : x_1 \geq 0,\ x_2 \geq 0\}$.
(a) Show that (P), suitably reformulated, is a convex program.
(b) State the dual problem (D).
(c) Solve (D).
(d) Use Lagrange multipliers to solve the problem
minimise $(x_1 + 1)^2 + x_2^2$
for $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{u} \in \mathbb{R}^3$,
subject to $x_1 - u_1^2 = x_2 - u_2^2 = x_1 + 2x_2 - 2 - u_3^2 = 0$,
and hence solve (P).
(e) Explain why, given the solution to (D), you can be sure that it is also the solution to (P). [LB]

31. Consider the problem
(P) minimise $f(\mathbf{x}) = x_2$
subject to $\mathbf{x} \in \mathbb{R}^2$
and $x_1 \geq 1$, $x_1^2 + x_2^2 \leq 1$.
(a) Check that (P), suitably reformulated, is a convex program.
(b) Show that (P) has an optimum $\mathbf{x}^*$ with $f(\mathbf{x}^*) = 0$.
(c) State the dual problem (D).
(d) Show that the objective function $h(\boldsymbol{\lambda})$ of (D) has no maximum, but that $\sup_{\boldsymbol{\lambda} \leq \mathbf{0}} h(\boldsymbol{\lambda})$ is 0. [LC]

32. Consider the problem
(P) minimise $e^x(2+\sin x)$
subject to $e^x \le 0$.
(a) Show that (P) is a convex program.
(b) State the dual problem (D).
(c) Show that $L(x,\lambda)$ has no minimum for $x \in \mathbb{R}$, for any $\lambda \le 0$, but that $h(\lambda) = \inf_{x\in\mathbb{R}} L(x,\lambda)$ is always 0.
(d) Is 0 the optimum value for the primal problem (P)? Explain your answer. [LD]

33. Consider the problem
(P) minimise $f(\mathbf{x}) = x_1$
subject to $\mathbf{x} \in C$, $x_2 \le 0$
where $C = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^2, \text{ either } \mathbf{x} = \mathbf{0} \text{ or } x_2 > 0\}$.
(a) Check that C is convex.
(b) State the dual problem (D).
(c) Solve (P). Show that (D) has $h(\lambda) = -\infty$ for every $\lambda \le 0$. [LE]

34. Consider the problem
(P) minimise $f(\mathbf{x}) = x_1$
subject to $\mathbf{x} \in C$, $\frac{{x_2}^2}{x_1} \le 0$
where C is the set

$$\{\mathbf{x} : \mathbf{x} \in \mathbb{R}^2, \text{ either } x_1 > 0 \text{ and } x_2 > 0 \text{ or } x_1 \ge 1,\, x_2 = 0\}.$$

(a) Show that $g(\mathbf{x}) = \frac{x_2^2}{x_1}$ is a convex function on C, so that (P) is a convex program. Show that the minimum value of f is attained and is 1.
(b) State the dual problem (D). Show that $h(\lambda) = \inf_{\mathbf{x}\in C} L(\mathbf{x},\lambda) = 0$ for every $\lambda \le 0$, so that the duality gap for this problem is 1. [LF]

35. (a) Describe the set

$$C_0 = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^3,\, x_1^2 + x_2^2 \le x_3^2,\, x_3 \ge 0\}$$

and show that it is convex.
(b) Show that $x_2 + x_3 \ge 0$ for every $\mathbf{x} \in C_0$.
(c) Show that $\begin{pmatrix} -1 \\ -\alpha \\ \sqrt{\alpha^2+1} \end{pmatrix} \in C_0$ for every $\alpha \in \mathbb{R}$.
(d) Show that $\lim_{\alpha\to\infty} \sqrt{\alpha^2+1} - \alpha = 0$. [LGA]

36. Consider the problem
(P) minimise $f(\mathbf{x}) = x_1$
subject to $\mathbf{x} \in C$, $x_2 + x_3 \le 0$,
where C is the set

$$\{\mathbf{x} : \mathbf{x} \in \mathbb{R}^3,\, x_1^2 + x_2^2 \le x_3^2,\, x_3 \ge 0,\, x_1 \ge -1\}.$$

(a) Show that (P) is a convex program.
(b) Show that the optimal value of $f(\mathbf{x})$ is 0.
(c) State the dual problem (D).
(d) Show that $h(\lambda) = \inf_{\mathbf{x}\in C} L(\mathbf{x},\lambda) = -1$ for every $\lambda \le 0$.
(You may find the facts listed in question 35 helpful.) [LGB]