Complex Variable I Questions

1. Plot the following complex numbers on Argand diagrams:
(a) (for $0 \le n \le 10$) (i) $z_n = \exp(in)$ (ii) $z_n = \sum_{k=0}^{n} \exp(ik)$
(b) (for $0 \le n \le 10$) (i) $z_n = (\frac{3}{4})^n \exp(in)$ (ii) $z_n = \sum_{k=0}^{n} (\frac{3}{4})^k \exp(ik)$
(c) (for $0 \le n \le 10$) (i) $z_n = \frac{(i\pi)^n}{n!}$ (ii) $z_n = \sum_{k=0}^{n} \frac{(i\pi)^k}{k!}$
(d) (for $0 \le n \le 10$) (i) $z_n = \frac{\exp(in)}{n+1}$ (ii) $z_n = \sum_{k=0}^{n} \frac{\exp(ik)}{k+1}$.

[A]

2. Sketch the following subsets of $\mathbb{C}$:
(a) $\{z : |z - i| \le 1\}$ (b) $\{z : \arg(z-1) = \frac{\pi}{4}\}$ (c) $\{z : |z - 2i| = |z - 2|\}$
(d) $\{z : |z + i| > 1\}$ (e) $\{z : \mathcal{R}\mathrm{e}(iz) > 1\}$.

[AA]

3. Calculate the following:
(a) $(2+i)^2$ (b) i^{19} (c) $(2i+1)^2$ (d) $(2i-1)^2$
(e) $(1-2i)^2$ (f) $\frac{3+4i}{2-i}$ (g) $\frac{2-i}{3+4i}$ (h) $\frac{2+i}{3-4i}$.

[AB]

4. Find the modulus and argument of
(a) $6+3i$ (b) -4 (c) $3-4i$ (d) $-3+4i$ (e) $-4-3i$.
(Give your answers exactly, with square roots and the arctan function if necessary.) [AC]

5. (a) If $|z| = 3$ what are $|\frac{1}{z}|$, $|z^2|$, $|\bar{z}|$?
(b) If $\arg z = \frac{\pi}{4}$ what are $\arg \frac{1}{z}$, $\arg z^2$, $\arg \bar{z}$?
(c) If $\arg z = -\frac{\pi}{4}$ what are $\arg \frac{1}{z}$, $\arg z^2$, $\arg \bar{z}$?
(d) If $\arg z = -\frac{3\pi}{4}$ what are $\arg \frac{1}{z}$, $\arg z^2$, $\arg \bar{z}$?
(e) If $\arg z = \pi$ what are $\arg \frac{1}{z}$, $\arg z^2$, $\arg \bar{z}$? [AD]

6. Sketch the following subsets of $\mathbb{C}$:
(a)(i) $\{z : |z| < 1\}$
(ii) $\{z : \frac{1}{2} < |z - 1 - i| \le 2\}$
(iii) $\{z : z^7 = i\}$
(b)(i) $\{z : |z + i| > |z - i|\}$
(ii) $\{z : \mathcal{R}\mathrm{e}(z(1+i)) > 1\}$
(c)(i) $\{2\exp(i\theta) : 0 \le \theta \le \pi\}$
(ii) $\{i + \exp(i\theta) : 0 \le \theta \le 2\pi\}$
(iii) $\{\exp(i\theta) : -\pi \le \theta \le 0\}$.

[B]

7. For each of the following complex functions f, express it in the form $f(x+iy) = u(x,y) + iv(x,y)$, where u and v are real functions of two real variables, and find the four partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$; say whether f is analytic.
(i) $f(z) = z^2$ for $z \in \mathbb{C}$ (ii) $f(z) = |z|$ for $z \in \mathbb{C}$
(iii) $f(z) = \mathcal{I}\mathrm{m}\, z$ for $z \in \mathbb{C}$ (iv) $f(z) = |z|^2$ for $z \in \mathbb{C}$
(v) $f(z) = \exp z$ for $z \in \mathbb{C}$ (vi) $f(z) = \frac{1}{z}$ for $z \ne 0$.

[BA]

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8. Give the radii of convergence of the following power series:
(i) $\sum_{n=0}^{\infty} \frac{n^2+2n-1}{n^3-n^2-3} z^n$
(ii) $\sum_{n=0}^{\infty} \frac{\sqrt{n}}{n!} z^n$
(iii) $\sum_{n=0}^{\infty} n^n z^n$
(iv) $\sum_{n=0}^{\infty} 2^n z^{2n}$
(v) $\sum_{n=0}^{\infty} \frac{n^n}{n!} z^n$
(vi) $\sum_{n=0}^{\infty} \frac{n!(2n)!}{(3n)!} z^n$ [BB]

9. Give the radii of convergence of the following power series:
(i) $\sum_{n=0}^{\infty} z^n$
(ii) $\sum_{n=0}^{\infty} \frac{n^3-n^2-3}{n^2+2n-1} z^n$
(iii) $\sum_{n=0}^{\infty} \frac{1}{3^n} z^{3n}$
(iv) $\sum_{n=2}^{\infty} \frac{2.5.8.11\ldots(3n-4)}{n!} z^n$.
(v) $\sum_{n=0}^{\infty} \frac{a(a+1)\ldots(a+n-1)b(b+1)\ldots(b+n-1)}{n!c(c+1)\ldots(c+n-1)} z^n$, where a, b, c are complex numbers. (The answer depends on the values of a, b and c; take care to consider all cases.) [BC]

10. (a) For precisely which real numbers x, y is each of the following formulae true?
(i) $\arg(x+iy) = \arctan \frac{y}{x}$
(ii) $\arg(x+iy) = \arccos \frac{x}{\sqrt{x^2+y^2}}$
(iii) $\arg(x+iy) = \arcsin \frac{y}{\sqrt{x^2+y^2}}$.
(b) For each of the following formulae find a complex number z for which it is true:
(i) $\arg(z^2) > 2\arg z$
(ii) $\arg(z^3) < \arg(z^2) + \arg(z)$
(iii) $\arg \frac{1}{z} + \arg z = 2\pi$.
[C]

11. Show that if w and z are complex numbers and $|z| < |w|$, then $\mathcal{R}\mathrm{e}(\frac{w+z}{w-z}) = \frac{|w|^2-|z|^2}{|w-z|^2}$. [CA]

12. Show that if z_1, z_2, z_3 are the vertices of an equilateral triangle in $\mathbb{C}$, taken anticlockwise around the triangle, then $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$. What happens if they are the vertices of an equilateral triangle taken *clockwise*? [CB]

13. For each of the eight sequences of which you plotted the first few terms in question 1, state whether the sequence is convergent, and if so what its limit is (if you can - (d-ii) is hard).
[D]

14. Which of the following series are summable? For those which are, give the sum if you can. Briefly justify your answers.
(i) $\sum_{n=0}^{\infty} (\frac{1+i}{2})^n$
(ii) $\sum_{n=0}^{\infty} \frac{2^n}{(1+i)^n(n^4+1)}$
(iii) $\sum_{n=1}^{\infty} \frac{\exp in^2}{n^2}$
(iv) $\sum_{n=0}^{\infty} \frac{n}{n^2+1}$.
[E]

15. (a) Find the isolated singularities of $\frac{\mathrm{Ln}(z)}{1+z^2}$ and the residue at each.

(b) By integrating $\frac{\mathrm{Ln}(z)}{1+z^2}$ around a curve approximating to that made up of a semicircle in the upper half-plane from R to $-R$, a straight-line segment from $-R$ to $-\epsilon$, another semicircle from $-\epsilon$ to ϵ, and a straight-line segment back to R, where $0 < \epsilon < 1 < R$, find $\int_0^\infty \frac{\ln x}{1+x^2} dx$. [EA]

16. (a) Find the isolated singularities of $\frac{e^{i\lambda z}}{z(1+z^2)}$, for $\lambda \in \mathbb{R}$, and the residue at each.
(b) Give the first three non-zero terms of the Taylor series centered at 0 for the functions $\frac{1}{1+z^2}$, $\frac{e^{i\lambda z}}{1+z^2}$. [EB]

17. By integrating $\frac{e^{iz}}{z(1+z^2)}$ around a curve made up of a semicircle in the upper half-plane from R to $-R$, a straight-line segment from $-R$ to $-\epsilon$, another semicircle from $-\epsilon$ to ϵ, and a straight-line segment back to R, where $0 < \epsilon < 1 < R$, find $\int_{-\infty}^{\infty} \frac{\sin x}{x(1+x^2)} dx$. [EC]

18. (a) Find the isolated singularities of $\frac{e^{i\lambda z}}{4+z^4}$, where $\lambda \in \mathbb{R}$, and the residue at each.
(b) Writing Γ_R for the semicircle from R to $-R$ in the upper half-plane, where $R > \sqrt{2}$, show that

$$\lim_{R\to\infty} \left| \int_{\Gamma_R} \frac{1}{4+z^4} dz \right| = 0.$$

(c) By considering the integral of $\frac{1}{4+z^4}$ around Γ_R and its diameter, find $\int_{-\infty}^{\infty} \frac{1}{4+x^4} dx$.
(d) Find $\int_0^{\infty} \frac{\cos \lambda x}{4+x^4} dx$ for $\lambda \in \mathbb{R}$. Explain why you need different arguments for the cases $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$. [ED]

19. Use contour integration to find $\int_0^{\infty} \frac{\cos \lambda x}{x^2+1} dx$ for $\lambda \in \mathbb{R}$. [EE]

20. Use contour integration to find $\int_{-\infty}^{\infty} \frac{\cos \lambda x}{x^2-2x+2} dx$ and $\int_{-\infty}^{\infty} \frac{\sin \lambda x}{x^2-2x+2} dx$ for $\lambda \in \mathbb{R}$. [EF]

21. Use contour integration to find $\int_{-\infty}^{\infty} \frac{\cos \lambda x}{x^2+2x+2} dx$ and $\int_{-\infty}^{\infty} \frac{\sin \lambda x}{x^2+2x+2} dx$ for $\lambda \in \mathbb{R}$. [EG]

22. Use contour integration to find $\int_0^{\infty} \frac{x \sin \lambda x}{x^2+1} dx$ for $\lambda \in \mathbb{R}$. [EH]

23. Give the radii of convergence of the following power series:
(i) $\sum_{n=0}^{\infty} z^n$
(ii) $\sum_{n=0}^{\infty} \frac{n^2+2n-1}{n^3-n^2-3} z^n$
(iii) $\sum_{n=0}^{\infty} \frac{n^3-n^2-3}{n^2+2n-1} z^n$
(iv) $\sum_{n=0}^{\infty} \frac{\sqrt{n}}{n!} z^n$
(v) $\sum_{n=0}^{\infty} n^n z^n$
(vi) $\sum_{n=0}^{\infty} 2^n z^{2n}$
(vii) $\sum_{n=0}^{\infty} \frac{1}{3^n} z^{3n}$
(viii) $\sum_{n=0}^{\infty} \frac{n^n}{n!} z^n$
(ix) $\sum_{n=0}^{\infty} \frac{n!(2n)!}{(3n)!} z^n$
(x) $\sum_{n=2}^{\infty} \frac{2.5.8.11\ldots(3n-4)}{n!} z^n$.
(xi) $\sum_{n=0}^{\infty} \frac{a(a+1)\ldots(a+n-1)b(b+1)\ldots(b+n-1)}{n!c(c+1)\ldots(c+n-1)} z^n$, where a, b, c are complex numbers. (The answer depends on the values of a, b and c; take care to consider all cases.) [F]

24. By differentiating the series $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots$, find (a) series for (i) $(1-z)^{-2}$ (ii) $(1-z)^{-3}$ (iii) $(1+z)^{-2}$ (b) the sums (i) $\sum_{n=0}^{\infty} nz^n$ (ii) $\sum_{n=0}^{\infty} n^2 z^n$. For which values of z are your expressions valid? [G]

25. Consider the formula $f(z) = \mathrm{Ln}(z) - \mathrm{Ln}(-z)$. (a) What is the domain on which this formula may be used to define an analytic function f? (b) Find its derivative f'. (c) Find $f(i)$, $f(-i)$. (d) What is the set of values of f? (e) Is f constant? [H]

26. Express the following functions as sums $\sum_{n=0}^{\infty} c_n z^n$ of power series; in each case find the coefficient c_n of z^n, and give the radius of convergence of the power series:

(i) $\exp(\frac{z}{2})$ (ii) $\frac{1}{2-3z}$ (iii) $\frac{\sin z}{z}$ (iv) $\frac{1+z}{1-z}$.

[HA]

27. Find expressions for the sums

(i) $\sum_{n=0}^{\infty} \frac{3^n z^n}{n!}$ (ii) $\sum_{n=0}^{\infty} (1+i)^n z^{2n+1}$ (iii) $\sum_{n=0}^{\infty} \frac{z^n}{n+2}$ (iv) $\sum_{n=0}^{\infty} \frac{(-z)^n}{(2n)!}$

(v) $\sum_{n=0}^{\infty} c_n z^n$ where $\langle c_n \rangle_{n\in\mathbb{N}}$ is the Fibonacci sequence ($c_0 = 0$, $c_1 = 1$, $c_n = c_{n-1} + c_{n-2}$ for $n \geq 2$).

In each case, say for which values of z your expression is valid.

(Hint for (v): show that $c_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$ for suitable α, β.) [HB]

28. Let $a \in \mathbb{C}$. Consider the series

$$f(z) = 1 + az + \frac{a(a-1)}{2!} z^2 + \ldots + \frac{a(a-1)\ldots(a-n+1)}{n!} z^n + \ldots.$$

(a) What is the radius of convergence of this series?

(b) Find a series for $f'(z)$.

(c) Find $(1+z)f'(z) - af(z)$. (Hint: compute the coefficient of z^n.)

(d) Using (c), show that $\frac{d}{dz} f(z) \exp(-a\mathrm{Ln}(1+z)) = 0$.

(e) Hence show that $f(z) = \exp(a\mathrm{Ln}(1+z))$ for $|z| < 1$. [I]

29. Set $D = \mathrm{dom}(\mathrm{Ln}) = \{z : z \text{ is not real and negative}\}$, $f(z) = \exp(\frac{1}{3}\mathrm{Ln}z)$ for $z \in D$. Show that f is injective. Find its set of values and its inverse. [J]

30. Consider the formulae $f(z) = z\exp(\frac{1}{2}\mathrm{Ln}(1 - \frac{1}{z}))$, $g(z) = iz\exp(\frac{1}{2}\mathrm{Ln}(\frac{1}{z} - 1))$.

(a) What are the domains on which these formulae can be used to define analytic functions f, g?

(b) Find $f(z)^2$, $g(z)^2$.

(c) Find the arguments of $f(i)$, $g(i)$, $f(-i)$, $g(-i)$.

(d) Find $\{z : f(z) = g(z)\}$, $\{z : f(z) = -g(z)\}$.

(e) Describe the behaviour of $f(z)$, $g(z)$ for z near $\frac{1}{2}$, 2. [K]

31. In this question, Γ_1 is the circle centre 0 radius 2 taken anticlockwise; Γ_2 is the semicircle from -2 to 2 in the upper half-plane; Γ_3 is the straight line segment from -2 to 2; Γ_4 is the semicircle from 2 to -2 in the upper half-plane, followed by the straight line segment from -2 to 2; Γ_5 is the rectangle with vertices at $\pm 1 \pm 2i$ taken anticlockwise.

(a) Find parametrizations for the curves Γ_1, Γ_2, Γ_3.

(b) Find $\int_{\Gamma_j} \mathcal{R}\mathrm{e}(z)dz$ for $j = 1, \ldots, 5$.

(c) Find $\int_{\Gamma_j} z^n dz$ for $j = 1, \ldots, 5$ and integer $n \geq 0$.

(d) Find $\int_{\Gamma_j} z^n dz$ for $j = 1, 2, 5$ and integer $n \leq -2$.

(e) Find $\int_{\Gamma_j} \frac{1}{z} dz$ for $j = 1, 2, 5$.

(f) Find $\int_{\Gamma_j} z^2 \exp(z^3) dz$ for $j = 2, 3, 4$.

(g) Find $\int_{\Gamma_j} \exp(z^2 - z) dz$ for $j = 1, 4, 5$.

(Please draw up a table to show your answers to parts (b)-(g).) [L]

32. Set $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{(n+1)!n}$, $g(z) = z - f(\frac{1}{z})$.

(a) What is the radius of convergence of the series for f?

(b) What is the domain of g?

(c) Show that $z^2 f'(z) = \exp(z) - 1 - z$.

(d) Find $g'(z)$.

(e) Show that $\int_\Gamma \exp(\frac{1}{z})dz = \int_\Gamma \frac{1}{z}dz$ for any closed curve Γ of finite length not passing through 0.

(f) Find $\int_{\Gamma_1} \exp(\frac{1}{z})dz$, $\int_{\Gamma_5} \exp(\frac{1}{z})dz$ where Γ_1 is the circle centre 0 radius 2, taken anticlockwise, and Γ_5 is the rectangle with vertices at $\pm 1 \pm 2i$, taken anticlockwise. [M]

33. (a) For which $z \in \mathbb{C}$ is $\mathrm{Ln}(\exp(z)) = z$?
(b) For which $z \in \mathbb{C}$ is $\frac{z+i}{z-i}$ real?
(c) Set $f(z) = \frac{\pi}{2} + \frac{i}{2}\mathrm{Ln}(\frac{z+i}{z-i})$. What is the domain of f?
(d) Find f'.
(e) Find $f(2)$, $f(-2)$.
(f) Find $\int_{\Gamma_j} \frac{1}{1+z^2} dz$ for $j = 1, 2, 3, 5$, and show that $\int_{\Gamma_4} \frac{1}{1+z^2} dz = \pi$.
(Γ_1 is the circle centre 0 radius 2 taken anticlockwise; Γ_2 is the semicircle from -2 to 2 in the upper half-plane; Γ_3 is the straight line segment from -2 to 2; Γ_4 is the semicircle from 2 to -2 in the upper half-plane, followed by the straight line segment from -2 to 2; Γ_5 is the rectangle with vertices at $\pm 1 \pm 2i$ taken anticlockwise.)
(g) For which $z \in \mathbb{C}$ is $\tan f(z) = z$?
(h) For which $z \in \mathbb{C}$ is $f(\tan z) = z$?
(i) Is f injective? [N]

34. Let $n \geq 1$. Set $f(z) = \exp(\frac{1}{z}) - \sum_{k=0}^{n} \frac{1}{k! z^k}$ for $z \neq 0$.
(a) Show that $\sum_{k=n+1}^{\infty} \frac{1}{k!} \leq \frac{n+2}{(n+1)!(n+1)}$.
(b) Show that $|f(z)| \leq \frac{n+2}{(n+1)!(n+1)}$ if $|z| \geq 1$.
(c) Show that $|\int_{\Gamma_1} f(z)dz| \leq \frac{4\pi(n+2)}{(n+1)!(n+1)}$, where Γ_1 is the circle centre 0 radius 2 taken anticlockwise.
(d) Hence show that $|\int_{\Gamma_1} \exp \frac{1}{z} dz - 2\pi i| \leq \frac{4\pi(n+2)}{(n+1)!(n+1)}$.
(e) Find $\int_{\Gamma_1} \exp \frac{1}{z} dz$. [O]

35. (a) For each of the following functions (regarded as defined on the natural domain associated with the formula given), find its isolated singularities, classify them (i.e., state whether they are removable singularities, poles, or essential singularities, and if they are poles give their orders), and give the residues.
(i) $\frac{5}{z} + \frac{4i}{z^3} + \frac{1}{(2z+i)^2} + \frac{2i}{2z+i} - \frac{3}{(3z-1)^4}$.
(ii) $\frac{1}{z^2+1}$.
(iii) $\frac{z}{(z^2+1)^2}$.
(iv) $\frac{\exp(z)}{z^4}$.
(v) $\frac{z}{\sin \pi z}$. (*Hint*: $\sin z = 0$ iff z is an integer multiple of π.)
(vi) $\frac{z^2-1}{z(3z^2+10iz-3)}$. (*Hint*: factorize $3z^2 + 10iz - 3$, either by guessing at a factor or by solving the quadratic $3z^2 + 10iz - 3 = 0$ with the formula $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$.)
(b) Find the following line integrals:
(i) $\int_{\Gamma_1} \frac{5}{z} + \frac{4i}{z^3} + \frac{1}{(2z+i)^2} + \frac{2i}{2z+i} - \frac{3}{(3z-1)^4} dz$, where Γ_1 is the unit circle taken anticlockwise once.
(ii) $\int_{\Gamma_2} \frac{z}{(z^2+1)^2} dz$, where Γ_2 is the straight-line segment from $-217{\cdot}5$ to $217{\cdot}5$ followed by the semicircle from $217{\cdot}5$ to $-217{\cdot}5$ in the upper half-plane.
(iii) $\int_{\Gamma_1} \frac{z^2-1}{z(3z^2+10iz-3)} dz$.
(iv) $\int_{\Gamma_1} \frac{1}{(\exp(z)-1)^2} dz$.
(v) $\int_{\Gamma_3} \frac{1}{\sin \pi z} dz$, where Γ_3 is the circle with centre 0 and radius $3\frac{1}{2}$, taken anticlockwise. [P]

36. As question 35(a), with

(i) $\frac{1}{z^3+1}$ (ii) $\frac{z}{z^2+1}$ (iii) $\frac{1}{(z^2+1)^2}$

(iv) $\frac{z}{\sin z}$ (v) $\mathrm{Ln}(1+\frac{1}{z})$ (vi) $\frac{1}{\exp(z)-1}$

(vii) $\frac{z}{\exp(z)-1}$ (viii) $\frac{1}{z}-\frac{1}{\exp(z)-1}$ (ix) $\frac{1}{(\exp(z)-1)^2}$

(x) $\frac{1}{z^2}-\frac{1}{(\exp(z)-1)^2}$ (xi) $\frac{1}{\tan z}$ (xii) $\frac{z(z^2+4\pi^2)}{\exp(z)-1}$

(xiii) $\exp(\frac{1}{1-\cos z})$.

(The last one is hard.) [Q]

37. As question 35(b), with

(i) $\int_{\Gamma_4}\frac{\mathrm{Ln} z}{1-2z}dz$, where Γ_4 is the curve made up of a semicircle from $-2i$ to $2i$, a vertical line segment from $2i$ to $\frac{i}{3}$, a semicircle from $\frac{i}{3}$ to $-\frac{i}{3}$, and a line segment from $-\frac{i}{3}$ to $-2i$, both semicircles being taken in the right-hand half-plane.

(ii) $\int_{\Gamma_1}\frac{1}{z^2}-\frac{1}{\tan^2 z}dz$. [R]

38. Find $\int_0^{2\pi}\frac{\sin t}{5+3\sin t}dt$. (*Hint*: parametrize Γ_1 with $z=e^{it}$ for $0\le t\le 2\pi$, and use 35(b-iii).) [S]