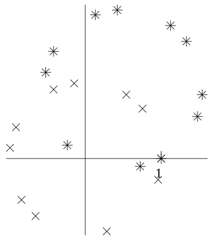
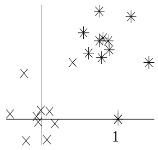
Complex Variable I Answers

1. In the diagrams below, the sequences of parts (i) are plotted with $\times$ and the sequences of parts (ii) are plotted with $*$.

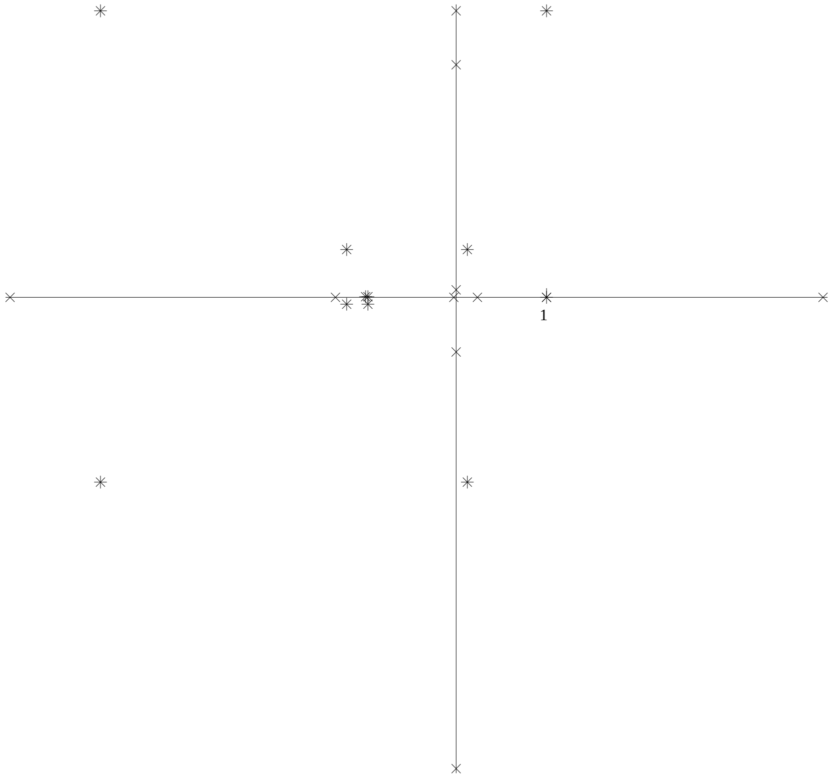
(a) $z_n = \exp(in)$



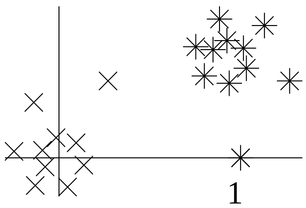
(b) $z_n = (\frac{3}{4})^n \exp(in)$



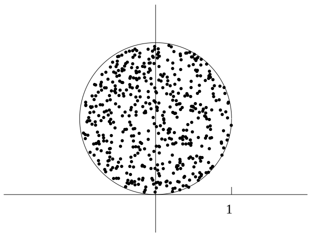
(c) $z_n = \frac{(i\pi)^n}{n!}$

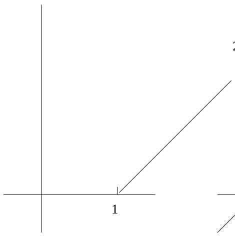


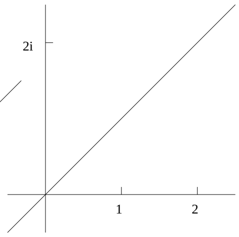
(d) $z_n = \frac{\exp(in)}{n+1}$



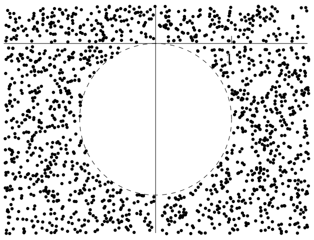
2. (a) $\{z : |z - i| \leq 1\}$ (b) $\{z : \arg(z - 1) = \frac{\pi}{4}\}$ (c) $\{z : |z - 2i| = |z - 2|\}$

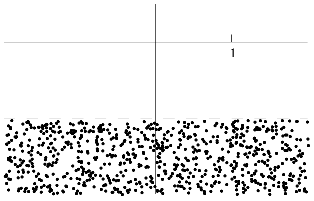






(d) $\{z : |z + i| > 1\}$ (e) $\{z : \mathcal{R}\mathrm{e}(iz) > 1\}$



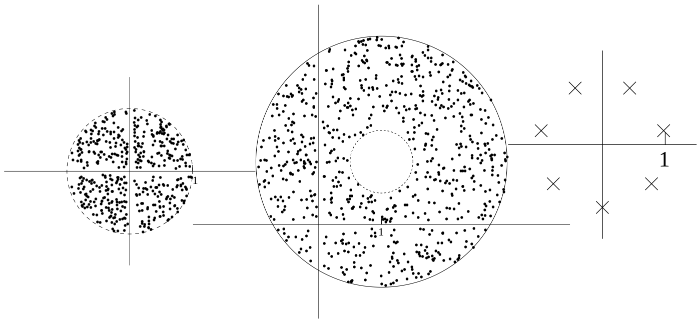


3.
(a) $3 + 4i$ (b) $-i$ (c) $-3 + 4i$ (d) $-3 - 4i$
(e) $-3 - 4i$ (f) $\frac{2}{5} + \frac{11}{5}i$ (g) $\frac{2}{25} - \frac{11}{25}i$
(h) $\frac{2}{25} + \frac{11}{25}i$.

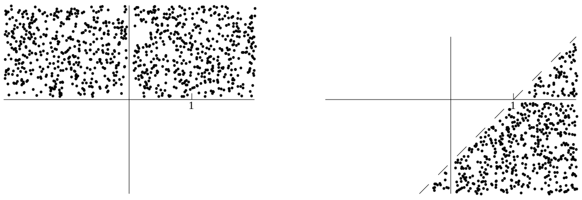
4. (a) $3\sqrt{5} \simeq 6.71$, $\arctan \frac{1}{2} \simeq 0.46$.
(b) 4, $\pi \simeq 3.14$.
(c) 5, $-\arctan \frac{4}{3} \simeq -0.93$.
(d) 5, $\pi - \arctan \frac{4}{3} \simeq 2.21$.
(e) 5, $\arctan \frac{3}{4} - \pi \simeq -2.50$.

5. (a) $\frac{1}{3}$, 9, 3.
(b) $-\frac{\pi}{4}$, $\frac{\pi}{2}$, $-\frac{\pi}{4}$.
(c) $\frac{\pi}{4}$, $-\frac{\pi}{2}$, $\frac{\pi}{4}$.
(d) $\frac{3\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$.
(e) π, 0, π.

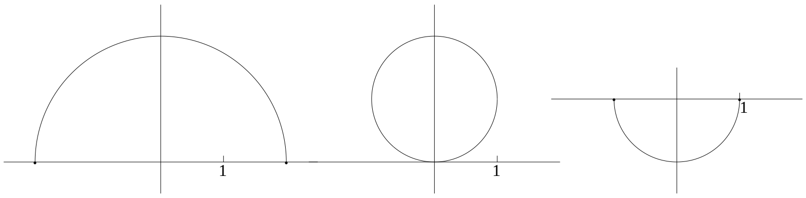
6. (a)(i) $\{z : |z| < 1\}$ (ii) $\{z : \frac{1}{2} < |z - 1 - i| \le 2\}$ (iii) $\{z : z^7 = i\}$



(b)(i) $\{z : |z + i| > |z - i|\}$ (ii) $\{z : \mathcal{R}\mathrm{e}(z(1 + i)) > 1\}$



(c)(i) $\{2\exp(i\theta) : 0 \le \theta \le \pi\}$ (ii) $\{i + \exp(i\theta) : 0 \le \theta \le 2\pi\}$ (iii) $\{\exp(i\theta) : -\pi \le \theta \le 0\}$.



7. (i) $u = x^2 - y^2$, $v = 2xy$; $\dfrac{\partial u}{\partial x} = 2x = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -2y$, $\dfrac{\partial v}{\partial x} = 2y$; f is analytic.

(ii) $u = \sqrt{x^2 + y^2}$, $v = 0$; $\dfrac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$ if x, y not both 0, $\dfrac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$ if x, y not both 0, $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} = 0$; f is not analytic.

(iii) $u = 0$, $v = y$; $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x} = 0$, $\dfrac{\partial v}{\partial y} = 1$; f is not analytic.

(iv) $u = x^2 + y^2$, $v = 0$; $\dfrac{\partial u}{\partial x} = 2x$, $\dfrac{\partial u}{\partial y} = 2y$, $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} = 0$; f is not analytic.

(v) $u = e^x \cos y$, $v = e^x \sin y$; $\dfrac{\partial u}{\partial x} = e^x \cos y = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -e^x \sin y$, $\dfrac{\partial v}{\partial x} = e^x \sin y$; f is analytic.

(vi) $u = \frac{x}{x^2+y^2}$ if x, y not both 0, $v = -\frac{y}{x^2+y^2}$ if x, y not both 0; $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$ if x, y not both 0, $\dfrac{\partial u}{\partial y} = -\frac{2xy}{(x^2+y^2)^2}$ if x, y not both 0, $\dfrac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$ if x, y not both 0; f is analytic.

8. (i) 1 (ii) ∞ (iii) 0 (iv) $\frac{1}{\sqrt{2}} \simeq 0.71$ (v) $\frac{1}{e} \simeq 0.37$ (vi) $\frac{27}{4}$

9. (i) 1 (ii) 1 (iii) $\sqrt[3]{3}$ (iv) $\frac{1}{3}$.
(v) If c is an integer ≤ 0, the series is undefined, as some denominator is 0.
If c is not an integer ≤ 0, but one or both of a, b are, then the coefficients are eventually 0, so the radius of convergence is ∞.
In other cases, $R = \lim_{n\to\infty} |c_n/c_{n+1}| = \lim_{n\to\infty} \frac{(n+1)(c+n)}{(a+n)(b+n)} = 1$.

10. (a)(i) iff $x > 0$ (ii) iff $y \geq 0$ and x, y are not both 0 (iii) iff $x \geq 0$ and x, y are not both 0
(b)(i) $z = -i$ (ii) $z = i$ (iii) $z = -1$

11.

$$\begin{aligned}
\mathcal{R}\mathrm{e}\left(\frac{w+z}{w-z}\right) &= \frac{1}{2}\left(\frac{w+z}{w-z} + \frac{\bar{w}+\bar{z}}{\bar{w}-\bar{z}}\right) \\
&= \frac{1}{2}\left(\frac{(w+z)(\bar{w}-\bar{z}) + (w-z)(\bar{w}+\bar{z})}{(w-z)(\bar{w}-\bar{z})}\right) \\
&= \frac{1}{2}\left(\frac{w\bar{w} - w\bar{z} + z\bar{w} - z\bar{z} + w\bar{w} + w\bar{z} - z\bar{w} - z\bar{z}}{(w-z)(\bar{w}-\bar{z})}\right) \\
&= \frac{w\bar{w} - z\bar{z}}{(w-z)(\bar{w}-\bar{z})} \\
&= \frac{|w|^2 - |z|^2}{|w-z|^2}.
\end{aligned}$$

12. If $w = z_2 - z_1$, then $z_3 - z_2 = we^{2\pi i/3}$ and $z_1 - z_3 = we^{-2\pi i/3}$. So $(z_1-z_2)^2 + (z_2-z_3)^2 + (z_3-z_1)^2 = w^2(1 + e^{4\pi i/3} + e^{-4\pi i/3}) = 0$. Rearranging the terms, we have the result.
The same formula is still valid if the points are taken in any other order.

13. (a)(i) does not exist (ii) does not exist
(b)(i) 0 (ii) $\frac{4}{4-3e^i} \doteq 0.79 + 0.84i$ (geometric series)
(c)(i) 0 (ii) $\exp(i\pi) = -1$ (exponential series)
(d)(i) 0 (ii) $-e^{-i}\mathrm{Ln}(1-e^i) \doteq 0.92 + 0.54i$ (series for $\mathrm{Ln}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots$ for $|z| \leq 1$, $z \neq -1$).

14. (i) summable to $1+i$
(ii) not summable (ratio test)
(iii) summable (comparison with $\frac{1}{n^2}$)
(iv) not summable (comparison with $\frac{1}{n}$).

15. (a) Singularities at $\pm i$, residues both $\frac{\mathrm{Ln}\, i}{2i} = \frac{\pi}{4}$.

(b) Taking the left-hand straight part of the curve to lie just above the axis, and the small almost-semicircle to lie in the upper half-plane, we get a contour lying entirely within the domain of $\frac{\mathrm{Ln} z}{1+z^2}$.

By the Residue Theorem, the value of the integral will be $2\pi i \,\mathrm{Res}_i \frac{\mathrm{Ln} z}{1+z^2} = \frac{\pi^2 i}{2}$. Now, writing Γ_R for the large almost-semicircle, the integral is approximately

$$\int_{\Gamma_R} \frac{\mathrm{Ln} z}{1+z^2} dz + \int_{-R}^{-\epsilon} \frac{\ln(-x) + i\pi}{1+x^2} dx + \int_{\Delta_\epsilon} \frac{\mathrm{Ln} z}{1+z^2} dz + \int_{\epsilon}^{R} \frac{\ln x}{1+x^2} dx.$$

The second integral, substituting $\xi = -x$, is

$$\int_{\epsilon}^{R} \frac{\ln \xi + i\pi}{1+\xi^2} d\xi,$$

so the sum of the second and fourth integrals is

$$\int_{\epsilon}^{R} \frac{2\ln x + i\pi}{1+z^2} dx = 2\int_{\epsilon}^{R} \frac{\ln x}{1+x^2} dx + i\pi \int_{\epsilon}^{R} \frac{1}{1+x^2} dx.$$

To estimate the first integral, observe that on Γ_R we have

$$\left|\frac{\mathrm{Ln} z}{1+z^2}\right| = \left|\frac{\ln R + i\arg z}{1+z^2}\right| \leq \frac{\ln R + \pi}{R^2 - 1},$$

while the length of Γ_R is (approximately) πR; so the integral is in modulus at most

$$\frac{\ln R + \pi}{R^2 - 1} \pi R \to 0 \text{ as } R \to \infty.$$

The next step is to estimate the integral around the small semicircle. Here we have

$$\left|\frac{\mathrm{Ln} z}{1+z^2}\right| = \left|\frac{\ln \epsilon + i\arg z}{1+z^2}\right| \leq \frac{|\ln \epsilon| + \pi}{1 - R^2},$$

while the length of Δ_ϵ is (approximately) $\pi\epsilon$; so the integral is in modulus at most

$$\frac{|\ln \epsilon| + \pi}{1 - \epsilon^2} \pi\epsilon \to 0 \text{ as } \epsilon \to 0.$$

Accordingly

$$2\int_0^\infty \frac{\ln x}{1+x^2} dx + i\pi \int_0^\infty \frac{1}{1+x^2} dx = \frac{\pi^2 i}{2}.$$

Finally, taking the real part,

$$\int_0^\infty \frac{\ln x}{1+x^2} = 0.$$

(Of course it is easy to show, using the substitution $\xi = \frac{1}{x}$, that

$$\int_0^1 \frac{\ln x}{1+x^2} dx = -\int_1^\infty \frac{\ln x}{1+x^2} dx.)$$

16. (a) Singularities at 0, residue 1; i, residue $-\frac{1}{2}e^{-\lambda}$; and $-i$, residue $-\frac{1}{2}e^{\lambda}$.

(b) $\frac{1}{1+z^2} = 1 - z^2 + z^4 - \ldots,$

$$\frac{e^{i\lambda z}}{1+z^2} = (1 + i\lambda z - \frac{\lambda^2}{2!}z^2 - \ldots)(1 - z^2 + z^4 - \ldots) = 1 + i\lambda z - (1 + \frac{\lambda^2}{2})z^2 - \ldots.$$

17. By the Residue Theorem, the value of the integral will be $2\pi i \operatorname{Res}_{z=i} \frac{e^{iz}}{z(1+z^2)} = -\frac{\pi i}{e}$, if we take the small semicircle (let us call it Δ_ϵ) to lie above the axis. Now, writing Γ_R for the large semicircle, the integral is

$$\int_{\Gamma_R} \frac{e^{iz}}{z(1+z^2)}dz + \int_{-R}^{-\epsilon} \frac{e^{ix}}{x(1+x^2)}dx + \int_{\Delta_\epsilon} \frac{e^{iz}}{z(1+z^2)}dz + \int_{\epsilon}^{R} \frac{e^{ix}}{x(1+x^2)}dx.$$

The second integral, substituting $\xi = -x$, is

$$-\int_{\epsilon}^{R} \frac{e^{-i\xi}}{\xi(1+\xi^2)}d\xi,$$

so the sum of the second and fourth integrals is

$$\int_{\epsilon}^{R} \frac{e^{ix} - e^{-ix}}{x(1+x^2)}dx = 2i\int_{\epsilon}^{R} \frac{\sin x}{x(1+x^2)}dx.$$

Next, Jordan's Lemma tells us that the first integral tends to 0 as $R \to \infty$, because $|\frac{1}{z(1+z^2)}| \le \frac{1}{R(R^2-1)}$ for all z on Γ_R, and $\lim_{R\to\infty} \frac{1}{R(R^2-1)} = 0$. So we get

$$2i\int_{\epsilon}^{\infty} \frac{\sin x}{x(1+x^2)} + \int_{\Delta_\epsilon} \frac{e^{i\lambda z}}{z(1+z^2)}dz = -\frac{\pi i}{e}.$$

The next step is to estimate the integral around the small semicircle. We know from (b) of question 16 that $\frac{e^{iz}}{z(1+z^2)}$ has a Laurent series, valid near 0, of the form $\frac{1}{z} + i + \ldots = \frac{1}{z} + g(z)$ say; because g is continuous at 0, $\int_{\Delta_\epsilon} g(z)dz \to 0$ as $\epsilon \downarrow 0$, and we are left with

$$\int_{\Delta_\epsilon} \frac{1}{z}dz = \int_{-\pi}^{0} \frac{1}{\epsilon e^{-i\theta}}(-i\epsilon e^{-i\theta})d\theta = \int_{-\pi}^{0}(-i)d\theta = -i\pi.$$

Accordingly

$$\lim_{\epsilon\downarrow 0}\int_{\Delta_\epsilon} \frac{e^{iz}}{z(1+z^2)}dz = -i\pi.$$

and

$$2i\int_{0}^{\infty} \frac{\sin x}{x(1+x^2)}dx - i\pi = -\frac{\pi i}{e}.$$

Finally

$$\int_{0}^{\infty} \frac{\sin x}{x(1+x^2)}dx = \frac{\pi}{2}(1 - \frac{1}{e}).$$

18. (a) The singularities are at the zeroes of $4+z^4$, that is, at $\pm1\pm i$. At each singularity z_0 the residue is

$$\lim_{z\to z_0}(z-z_0)\frac{e^{i\lambda z}}{f}s4+z^4 = e^{i\lambda z_0}\lim_{z\to z_0}\frac{z-z_0}{4+z^4} = e^{i\lambda z_0}\lim_{z\to z_0}\frac{1}{4z^3}$$
$$= e^{i\lambda z_0}\lim_{z\to z_0}\frac{z}{4z^4} = e^{i\lambda z_0}\big(-\frac{1}{16}z_0\big).$$

So the residues are
at $1+i$,

$$-\tfrac{1+i}{16}e^{i\lambda(1+i)};$$

at $1-i$,

$$-\tfrac{1-i}{16}e^{i\lambda(1-i)};$$

at $-1+i$,

$$-\tfrac{-1+i}{16}e^{i\lambda(-1+i)};$$

at $-1-i$,

$$-\tfrac{-1-i}{16}e^{i\lambda(-1-i)}.$$

(b) If $R>\sqrt{2}$, then for z lying on Γ_R, $|\frac{1}{4+z^4}|\le\frac{1}{R^4-4}$, while the length of Γ_R is πR. So

$$|\textstyle\int_{\Gamma_R}\frac{1}{4+z^4}dz|\le\frac{\pi R}{R^4-4}\to 0$$

as $R\to\infty$.

(c) By the Residue Theorem, if $R>\sqrt{2}$,

$$\int_\Gamma\frac{1}{4+z^4}dz = 2\pi i\big(\operatorname*{Res}_{1+i}\frac{1}{4+z^4}+\operatorname*{Res}_{-1+i}\frac{1}{4+z^4}\big),$$

taking Γ to be Γ_R followed by the straight-line segment from $-R$ to R. Now the sum of the residues is (taking $\lambda=0$ in part (a))

$$-\tfrac{1}{16}((1+i)+(-1+i)) = -\tfrac{i}{8}.$$

So the integral is $\frac{\pi}{4}$. Next, parametrising the straight-line path by $z=x$, $-R\le x\le R$, $dz=dx$ we have

$$\textstyle\int_{-R}^{R}\frac{1}{4+x^4}dx+\int_{\Gamma_R}\frac{1}{4+z^4}dz=\frac{\pi}{4},$$

so using (b) we get

$$\textstyle\int_{-\infty}^{\infty}\frac{1}{4+x^4}dx=\lim_{R\to\infty}\int_{-R}^{R}\frac{1}{4+x^4}dx=\frac{\pi}{4}.$$

(d) Start with $\lambda > 0$, and consider the integral of $\frac{e^{i\lambda z}}{4+z^4}dz$ around the same contour Γ as in part (c). By the Residue Theorem

$$\int_\Gamma \frac{e^{i\lambda z}}{4+z^4}dz = 2\pi i\Big(\mathrm{Res}_{1+i}\,\frac{e^{i\lambda z}}{4+z^4} + \mathrm{Res}_{-1+i}\,\frac{e^{i\lambda z}}{4+z^4}\Big).$$

Now the sum of the residues is

$$\begin{aligned}
&-\frac{1}{16}\big((1+i)e^{i\lambda(1+i)} + (-1+i)e^{i\lambda(-1+i)}\big)\\
&\quad= -\frac{1}{16}\big((1+i)e^{i\lambda}e^{-\lambda} + (-1+i)e^{-i\lambda}e^{-\lambda}\big)\\
&\quad= -\frac{e^{-\lambda}}{16}\big((1+i)(\cos\lambda + i\sin\lambda) + (-1+i)(\cos\lambda - i\sin\lambda)\big)\\
&\quad= -\frac{e^{\lambda}}{16}\big(\cos\lambda + i\sin\lambda + i\cos\lambda - \sin\lambda - \cos\lambda + i\sin\lambda + i\cos\lambda + \sin\lambda\big)\\
&\quad= -\frac{ie^{-\lambda}}{8}(\cos\lambda + \sin\lambda).
\end{aligned}$$

So the integral is $\frac{\pi e^{-\lambda}}{4}(\cos\lambda + \sin\lambda)$.

Next, by Jordan's Lemma, because $\lambda > 0$ and $|\frac{1}{4+z^4}| \le \frac{1}{R^4-4}$ for z on Γ_R and $\lim_{R\to\infty}\frac{1}{R^4-4} = 0$,

$$\lim_{R\to\infty}\int_{\Gamma_R}\frac{e^{i\lambda z}}{4+z^4}dz = 0.$$

So, just as in (c) above, we get

$$\int_{-\infty}^{\infty}\frac{e^{i\lambda x}}{4+x^4}dx = \frac{\pi e^{-\lambda}}{4}(\cos\lambda + \sin\lambda),$$

so that

$$\int_0^\infty \frac{\cos\lambda x}{4+x^4}dx = \frac{\pi}{8}e^{-\lambda}(\cos\lambda + \sin\lambda).$$

This deals with $\lambda > 0$. The case $\lambda = 0$ has already been done in (c), and the same formula is of course valid. But for negative λ we need to observe that

$$\begin{aligned}
\int_0^\infty \frac{\cos\lambda x}{4+x^4}dx &= \int_0^\infty \frac{\cos(-\lambda x)}{4+x^4}dx\\
&= \frac{\pi}{8}e^{-(-\lambda)}(\cos(-\lambda) + \sin(-\lambda))\\
&= \frac{\pi}{8}e^{\lambda}(\cos\lambda - \sin\lambda).
\end{aligned}$$

To get a formula valid for all λ we could write

$$\int_0^\infty \frac{\cos\lambda x}{4+x^4}dx = \frac{\pi}{8}e^{-|\lambda|}(\cos\lambda + \sin|\lambda|).$$

The argument given for $\lambda > 0$ fails for $\lambda < 0$ because if $\lambda < 0$ then $\lim_{R\to\infty}\int_{\Gamma_R}\frac{e^{i\lambda z}}{4+z^4}dz$ is not zero.

19. As the integrand is even, we have

$$\int_0^\infty \frac{\cos \lambda x}{x^2+1} dx = \frac{1}{2}\int_\infty^\infty \frac{\cos \lambda x}{x^2+1} dx = \frac{1}{2}\,\mathcal{R}\mathrm{e}\int_\infty^\infty \frac{e^{i\lambda x}}{x^2+1} dx.$$

Consider the function $f(z) = \frac{e^{i\lambda z}}{z^2+1}$. This is an analytic function defined except when $z^2+1=0$, that is, except when $z = \pm i$. The residue at i is $\frac{e^{i\lambda i}}{2i} = \frac{e^{-\lambda}}{2i}$.

Consider the integral $\int_\Gamma \frac{e^{i\lambda z}}{z^2+1} dz$ where Γ is a semi-circle in the upper half-plane with diameter $[-R, R]$, where $R > 1$. The integral is

$$\int_{-R}^{R} \frac{e^{i\lambda x}}{x^2+1} dx + \int_{\Gamma_R} \frac{e^{i\lambda z}}{z^2+1} dz$$

(where Γ_R is the curved section from R to $-R$)

$$\begin{aligned} &= \int_\Gamma \frac{e^{i\lambda z}}{z^2+1} dz \\ &= 2\pi i \operatorname*{Res}_{z=i} \frac{e^{i\lambda z}}{z^2+1} \\ &= \frac{2\pi i e^{-\lambda}}{2i} = \pi e^{-\lambda}. \end{aligned}$$

case 1 If $\lambda > 0$, then

$$\lim_{R\to\infty} \max_{z \text{ lies on } \Gamma_R} \left|\frac{1}{z^2+1}\right| \leq \lim_{R\to\infty} \frac{1}{R^2-1} = 0,$$

so Jordan's Lemma tells us that $\lim_{R\to\infty} \int_{\Gamma_R} \frac{e^{i\lambda z}}{z^2+1} dz = 0$, and

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x^2+1} dx &= \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{i\lambda x}}{x^2+1} dx \\ &= \lim_{R\to\infty} \pi e^{-\lambda} - \int_{\Gamma_R} \frac{e^{i\lambda z}}{z^2+1} dz = \pi e^{-\lambda}. \end{aligned}$$

Taking real parts and halving,

$$\int_0^\infty \frac{\cos \lambda x}{x^2+1} dx = \frac{\pi}{2} e^{-\lambda}.$$

case 2 If $\lambda = 0$, then

$$|\int_{\Gamma_R} \frac{1}{z^2+1} dz| \leq (\text{length of curve}) \times (\text{maximum modulus}) \leq \pi R \cdot \frac{1}{R^2-1} \to 0$$

as $R \to \infty$, so

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx &= \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^2+1} dx \\ &= \lim_{R\to\infty} \pi - \int_{\Gamma_R} \frac{1}{z^2+1} dz = \pi.\end{aligned}$$

Taking real parts and halving,

$$\int_0^{\infty} \frac{1}{x^2+1} dx = \frac{\pi}{2}.$$

case 3 If $\lambda < 0$, then $-\lambda > 0$, so

$$\int_0^{\infty} \frac{\cos \lambda x}{x^2+1} dx = \int_0^{\infty} \frac{\cos(-\lambda)x}{x^2+1} dx = \frac{\pi}{2} e^{-(-\lambda)} = \frac{\pi}{2} e^{\lambda}.$$

[In all cases,

$$\int_0^{\infty} \frac{\cos \lambda x}{x^2+1} dx = \frac{\pi}{2} e^{-|\lambda|}.]$$

20. Consider the function $f(z) = \frac{e^{i\lambda z}}{z^2-2z+1}$. This is an analytic function defined except when $z^2-2z+1 = 0$, that is, except when $z = 1 \pm i$. The residue at $1+i$ is

$$\frac{e^{i\lambda(1+i)}}{2(1+i)-2} = \frac{e^{-\lambda}}{2i}(\cos\lambda + i\sin\lambda).$$

Consider the integral $\int_\Gamma \frac{e^{i\lambda z}}{z^2-2z+1}dz$ where Γ is a semi-circle in the upper half-plane with diameter $[-R, R]$, where $R > 1$. The integral is

$$\int_{-R}^{R} \frac{e^{i\lambda x}}{x^2-2x+1}dx + \int_{\Gamma_R} \frac{e^{i\lambda z}}{z^2-2z+1}dz$$

(where Γ_R is the curved section from R to $-R$)

$$\begin{aligned} &= \int_\Gamma \frac{e^{i\lambda z}}{z^2-2z+1}dz \\ &= 2\pi i \operatorname*{Res}_{z=1+i} \frac{e^{i\lambda z}}{z^2-2z+1} \\ &= \frac{2\pi i e^{-\lambda}}{2i}(\cos\lambda + i\sin\lambda) \\ &= \pi e^{-\lambda}(\cos\lambda + i\sin\lambda). \end{aligned}$$

case 1 If $\lambda > 0$, then

$$\lim_{R\to\infty} \max_{z \text{ lies on } \Gamma_R} \left|\frac{1}{z^2-2z+1}\right| \le \lim_{R\to\infty} \frac{1}{R^2-2R-1} = 0,$$

so Jordan's Lemma tells us that $\lim_{R\to\infty} \int_{\Gamma_R} \frac{e^{i\lambda z}}{z^2-2z+1}dz = 0$, and

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x^2-2x+1}dx &= \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{i\lambda x}}{x^2-2x+1}dx \\ &= \lim_{R\to\infty} \pi e^{-\lambda}(\cos\lambda + i\sin\lambda) - \int_{\Gamma_R} \frac{e^{i\lambda z}}{z^2-2z+1}dz \\ &= \pi e^{-\lambda}(\cos\lambda + i\sin\lambda). \end{aligned}$$

Taking real and imaginary parts,

$$\int_{-\infty}^{\infty} \frac{\cos\lambda x}{x^2-2x+1}dx = \pi e^{-\lambda}\cos\lambda,$$

$$\int_{-\infty}^{\infty} \frac{\sin\lambda x}{x^2-2x+1}dx = \pi e^{-\lambda}\sin\lambda.$$

case 2 If $\lambda = 0$, then

$$|\int_{\Gamma_R} \frac{1}{z^2-2z+1} dz| \leq (\text{length of curve}) \times (\text{maximum modulus}) \leq \pi R \cdot \frac{1}{R^2-2R-1} \to 0$$

as $R \to \infty$, so

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{x^2-2x+1} dx &= \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^2-2x+1} dx \\ &= \lim_{R\to\infty} \pi - \int_{\Gamma_R} \frac{1}{z^2-2z+1} dz = \pi.\end{aligned}$$

Taking real and imaginary parts,

$$\int_{-\infty}^{\infty} \frac{\cos \lambda x}{x^2-2x+1} dx = \pi,$$

$$\int_{-\infty}^{\infty} \frac{\sin \lambda x}{x^2-2x+1} dx = 0.$$

case 3 If $\lambda < 0$, then $-\lambda > 0$, so

$$\int_{-\infty}^{\infty} \frac{\cos \lambda x}{x^2-2x+1} dx = \int_{-\infty}^{\infty} \frac{\cos(-\lambda)x}{x^2-2x+1} dx = \pi e^{-(-\lambda)} \cos(-\lambda) = \pi e^{\lambda} \cos \lambda,$$

$$\int_{-\infty}^{\infty} \frac{\sin \lambda x}{x^2-2x+1} dx = -\int_{-\infty}^{\infty} \frac{\sin(-\lambda)x}{x^2-2x+1} dx = -\pi e^{\lambda} \sin(-\lambda) = \pi e^{\lambda} \sin \lambda.$$

[In all cases,

$$\int_{-\infty}^{\infty} \frac{\cos \lambda x}{x^2-2x+1} dx = \pi e^{-|\lambda|} \cos \lambda,$$

$$\int_{-\infty}^{\infty} \frac{\sin \lambda x}{x^2-2x+1} dx = \pi e^{-|\lambda|} \sin \lambda.]$$

21. Consider the function $f(z) = \frac{e^{i\lambda z}}{z^2+2z+1}$. This is an analytic function defined except when $z^2+2z+1 = 0$, that is, except when $z = -1 \pm i$. The residue at $-1+i$ is

$$\frac{e^{i\lambda(-1+i)}}{2(-1+i)+2} = \frac{e^{-\lambda}}{2i}(\cos\lambda - i\sin\lambda).$$

Consider the integral $\int_\Gamma \frac{e^{i\lambda z}}{z^2+2z+1}dz$ where Γ is a semi-circle in the upper half-plane with diameter $[-R, R]$, where $R > 1$. The integral is

$$\int_{-R}^{R} \frac{e^{i\lambda x}}{x^2+2x+1}dx + \int_{\Gamma_R} \frac{e^{i\lambda z}}{z^2+2z+1}dz$$

(where Γ_R is the curved section from R to $-R$)

$$\begin{aligned}
&= \int_\Gamma \frac{e^{i\lambda z}}{z^2+2z+1}dz \\
&= 2\pi i \operatorname*{Res}_{z=-1+i} \frac{e^{i\lambda z}}{z^2+2z+1} \\
&= \frac{2\pi i e^{-\lambda}}{2i}(\cos\lambda - i\sin\lambda) \\
&= \pi e^{-\lambda}(\cos\lambda - i\sin\lambda).
\end{aligned}$$

case 1 If $\lambda > 0$, then

$$\lim_{R\to\infty} \max_{z \text{ lies on } \Gamma_R} |\frac{1}{z^2+2z+1}| \leq \lim_{R\to\infty} \frac{1}{R^2-2R-1} = 0,$$

so Jordan's Lemma tells us that $\lim_{R\to\infty} \int_{\Gamma_R} \frac{e^{i\lambda z}}{z^2+2z+1}dz = 0$, and

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x^2+2x+1}dx &= \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{i\lambda x}}{x^2+2x+1}dx \\
&= \lim_{R\to\infty} \pi e^{-\lambda}(\cos\lambda + i\sin\lambda) - \int_{\Gamma_R} \frac{e^{i\lambda z}}{z^2+2z+1}dz \\
&= \pi e^{-\lambda}(\cos\lambda - i\sin\lambda).
\end{aligned}$$

Taking real and imaginary parts,

$$\int_{-\infty}^{\infty} \frac{\cos\lambda x}{x^2+2x+1}dx = \pi e^{-\lambda}\cos\lambda,$$

$$\int_{-\infty}^{\infty} \frac{\sin\lambda x}{x^2+2x+1}dx = -\pi e^{-\lambda}\sin\lambda.$$

case 2 If $\lambda = 0$, then

$$|\int_{\Gamma_R} \frac{1}{z^2+2z+1} dz| \leq \text{(length of curve)} \times \text{(maximum modulus)} \leq \pi R \cdot \frac{1}{R^2-2R-1} \to 0$$

as $R \to \infty$, so

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{x^2+2x+1} dx &= \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^2+2x+1} dx \\ &= \lim_{R\to\infty} \pi - \int_{\Gamma_R} \frac{1}{z^2+2z+1} dz = \pi.\end{aligned}$$

Taking real and imaginary parts,

$$\int_{-\infty}^{\infty} \frac{\cos \lambda x}{x^2+2x+1} dx = \pi,$$

$$\int_{-\infty}^{\infty} \frac{\sin \lambda x}{x^2+2x+1} dx = 0.$$

case 3 If $\lambda < 0$, then $-\lambda > 0$, so

$$\int_{-\infty}^{\infty} \frac{\cos \lambda x}{x^2+2x+1} dx = \int_{-\infty}^{\infty} \frac{\cos(-\lambda) x}{x^2+2x+1} dx = \pi e^{-(-\lambda)} \cos(-\lambda) = \pi e^{\lambda} \cos \lambda,$$

$$\int_{-\infty}^{\infty} \frac{\sin \lambda x}{x^2+2x+1} dx = -\int_{-\infty}^{\infty} \frac{\sin(-\lambda) x}{x^2+2x+1} dx = \pi e^{\lambda} \sin(-\lambda) = -\pi e^{\lambda} \sin \lambda.$$

[In all cases,

$$\int_{-\infty}^{\infty} \frac{\cos \lambda x}{x^2+2x+1} dx = \pi e^{-|\lambda|} \cos \lambda,$$

$$\int_{-\infty}^{\infty} \frac{\sin \lambda x}{x^2+2x+1} dx = -\pi e^{-|\lambda|} \sin \lambda.]$$

22. As the integrand is even, we have

$$\int_0^\infty \frac{x\sin\lambda x}{x^2+1}dx = \frac{1}{2}\int_\infty^\infty \frac{x\sin\lambda x}{x^2+1}dx = \frac{1}{2}\,\mathcal{I}\mathrm{m}\int_\infty^\infty \frac{xe^{i\lambda x}}{x^2+1}dx.$$

Consider the function $f(z) = \frac{ze^{i\lambda z}}{z^2+1}$. This is an analytic function defined except when $z^2+1=0$, that is, except when $z=\pm i$. The residue at i is $\frac{ie^{i\lambda i}}{2i} = \frac{e^{-\lambda}}{2}$.

Consider the integral $\int_\Gamma \frac{ze^{i\lambda z}}{z^2+1}dz$ where Γ is a semi-circle in the upper half-plane with diameter $[-R,R]$, where $R>1$. The integral is

$$\int_{-R}^{R} \frac{xe^{i\lambda x}}{x^2+1}dx + \int_{\Gamma_R} \frac{ze^{i\lambda z}}{z^2+1}dz$$

(where Γ_R is the curved section from R to $-R$)

$$\begin{aligned} &= \int_\Gamma \frac{ze^{i\lambda z}}{z^2+1}dz \\ &= 2\pi i \operatorname*{Res}_{z=i} \frac{ze^{i\lambda z}}{z^2+1} \\ &= \frac{2\pi i e^{-\lambda}}{2} = \pi i e^{-\lambda}. \end{aligned}$$

case 1 If $\lambda>0$, then

$$\lim_{R\to\infty}\max_{z \text{ lies on } \Gamma_R} \left|\frac{z}{z^2+1}\right| \le \lim_{R\to\infty}\frac{R}{R^2-1} = 0,$$

so Jordan's Lemma tells us that $\lim_{R\to\infty}\int_{\Gamma_R}\frac{ze^{i\lambda z}}{z^2+1}dz = 0$, and

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x^2+1}dx &= \lim_{R\to\infty}\int_{-R}^{R}\frac{e^{i\lambda x}}{x^2+1}dx \\ &= \lim_{R\to\infty} \pi i e^{-\lambda} - \int_{\Gamma_R}\frac{e^{i\lambda z}}{z^2+1}dz = \pi i e^{-\lambda}. \end{aligned}$$

Taking imaginary parts and halving,

$$\int_0^\infty \frac{x\sin\lambda x}{x^2+1}dx = \frac{\pi}{2}e^{-\lambda}.$$

case 2 If $\lambda=0$, then $\int_0^\infty \frac{x\sin\lambda x}{x^2+1}dx = 0$.
case 3 If $\lambda<0$, then $-\lambda>0$, so

$$\int_0^\infty \frac{x\sin\lambda x}{x^2+1}dx = -\int_0^\infty \frac{x\sin(-\lambda)x}{x^2+1}dx = -\frac{\pi}{2}e^{-(-\lambda)} = -\frac{\pi}{2}e^{\lambda}.$$

23. (i) 1 (ii) 1 (iii) 1 (iv) ∞ (v) 0 (vi) $\frac{1}{\sqrt{2}} \simeq 0.71$ (vii) $\sqrt[3]{3}$ (viii) $\frac{1}{e} \simeq 0.37$ (ix) $\frac{27}{4}$ (x) $\frac{1}{3}$.
(xi) If c is an integer ≤ 0, the series is undefined, as some denominator is 0.
If c is not an integer ≤ 0, but one or both of a, b are, then the coefficients are eventually 0, so the radius of convergence is ∞.
In other cases, $R = \lim_{n\to\infty}|c_n/c_{n+1}| = \lim_{n\to\infty}\frac{(n+1)(c+n)}{(a+n)(b+n)} = 1.$

24. (a)(i) $1+2z+3z^2+\ldots=\sum_{n=0}^{\infty}(n+1)z^n$
(ii) $1+3z+6z^2+10z^3+\ldots=\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}z^n$
(iii) $1-2z+3z^2-\ldots=\sum_{n=0}^{\infty}(-1)^n(n+1)z^n$.

(b)(i) $\sum_{n=0}^{\infty}(n+1)z^n-\sum_{n=0}^{\infty}z^n=\frac{1}{(1-z)^2}-\frac{1}{1-z}=\frac{z}{(1-z)^2}$.
(ii)

$$\begin{aligned}\sum_{n=0}^{\infty}(n+1)(n+2)z^n-3\sum_{n=0}^{\infty}nz^n-2\sum_{n=0}^{\infty}z^n&\\ &=\frac{2}{(1-z)^3}-\frac{3z}{(1-z)^2}-\frac{2}{1-z}\\ &=\frac{z(1+z)}{(1-z)^3}.\end{aligned}$$

All these expressions are valid iff $|z|<1$.

25. (a) $\{z:\mathcal{I}\mathrm{m}(z)\neq 0\}$. (b) $f'(z)=0$ wherever it is defined. (c) $f(i)=i\pi$, $f(-i)=-i\pi$. (d) $\{i\pi,-i\pi\}$. (e) No.

26. (i) $c_n=\frac{1}{2^n n!}$, $R=\infty$.
(ii) $\frac{1}{2-3z}=\frac{1}{2}\cdot\frac{1}{1-\frac{3}{2}z}=\frac{1}{2}\sum_{n=0}^{\infty}(\frac{3}{2}z)^n$, $c_n=\frac{3^n}{2^{n+1}}$, $R=\frac{2}{3}$.
(iii) $\frac{\sin z}{z}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n}$; $c_n=\frac{(-1)^{n/2}}{(n+1)!}$ if n is even, 0 if n is odd; $R=\infty$.
(iv) $\frac{1+z}{1-z}=(1+z)\sum_{n=0}^{\infty}z^n=\sum_{n=0}^{\infty}z^n+\sum_{n=0}^{\infty}z^{n+1}=\sum_{n=0}^{\infty}z^n+\sum_{n=1}^{\infty}z^n$; $c_0=1$, $c_n=2$ if $n\geq 1$; $R=1$.

27. (i) $\exp(3z)$ for all $z\in\mathbb{C}$.
(ii) $\frac{z}{1-(1+i)z^2}$ when $|z|<1/\sqrt[4]{2}$.
(iii) $-\frac{1}{z^2}(z+\mathrm{Ln}(1-z))$ for $|z|<1$ (in fact, for $|z|\leq 1$ and $z\neq 1$; but this refinement is outside the scope of the course).
(iv) $\cos z^{1/2}$ when $z^{1/2}$ is defined, that is, z is not real and negative.
(v) Recall that $c_n=\frac{1}{\sqrt{5}}(\alpha^n-\beta^n)$ where $\alpha=\frac{1}{2}(1+\sqrt{5})$, $\beta=\frac{1}{2}(1-\sqrt{5})$. (Proof by induction on n.) So

$$\sum_{n=0}^{\infty}c_nz^n=\frac{1}{\sqrt{5}}\Big(\frac{1}{1-\alpha z}-\frac{1}{1-\beta z}\Big)=\frac{1}{1-z+z^2}$$

if $|z|<\frac{1}{\alpha}$.

28. (a) 1, unless a is an integer ≥ 0, in which case it is ∞.
(b) $f'(z)=a+a(a-1)z+\frac{a(a-1)(a-2)}{2!}z^2+\ldots=\sum_{n=0}^{\infty}\frac{a(a-1)\ldots(a-n)}{n!}z^n$ if $|z|<1$.
(c) 0.
(d) $\frac{d}{dz}f(z)\exp(-a\mathrm{Ln}(1+z))=f'(z)\exp(-a\mathrm{Ln}(1+z))-\frac{a}{1+z}f(z)\exp(-a\mathrm{Ln}(1+z))=\frac{(1+z)f'(z)-af(z)}{1+z}\exp(-a\mathrm{Ln}(1+z))=0$.
(e) By (d), $f(z)\exp(-a\mathrm{Ln}(1+z))$ must be constant on the connected open set $\{z:|z|<1\}$; setting $z=0$, the constant must be $f(0)\exp(-a\mathrm{Ln}1)=1$; so $f(z)=1/\exp(-a\mathrm{Ln}(1+z))=\exp(a\mathrm{Ln}(1+z))$ for $|z|<1$.
(This is the *binomial series*.)

29. If $z \in D$, $\mathrm{Ln}z = \ln(|z|) + i \arg z$, so $f(z) = \exp(\frac{1}{3}\ln(|z|))\exp(\frac{1}{3}\arg z)$ and $|f(z)| = \exp(\frac{1}{3}\ln(|z|)) = |z|^{\frac{1}{3}}$, $\arg f(z) = \frac{1}{3}\arg z$. So $w = f(z)$ iff $|z| = |w|^3$ and $\arg z = 3\arg w$, that is to say, iff $|\arg w| < \frac{\pi}{3}$ and $z = w^3$. In particular, there can be at most one solution of the equation $f(z) = w$, that is, f is injective; the set of values of f is just $\{w : -\frac{\pi}{3} < \arg w < \frac{\pi}{3}\}$; and $f^{-1}(w) = w^3$ for these w.

30. (a) f is defined on $D = \{z : z \in \mathbb{C},\ z \notin [0,1]\}$, g is defined on $D' = \{z : z \in \mathbb{C},\ z \notin\,]-\infty, 0],\ z \notin [1, \infty[\}$.

(b) $f(z)^2 = z(z-1)$ for $z \in D$, $g(z)^2 = z(z-1)$ for $z \in D'$.

(c) $f(i) = i\exp(\frac{1}{2}\mathrm{Ln}(1+i))$ has argument $\frac{5\pi}{8}$,
$f(-i) = -i\exp(\frac{1}{2}\mathrm{Ln}(1-i))$ has argument $-\frac{5\pi}{8}$
$g(i) = -\exp(\frac{1}{2}\mathrm{Ln}(-i-1))$ has argument $\frac{5\pi}{8}$
$g(-i) = \exp(\frac{1}{2}\mathrm{Ln}(i-1))$ has argument $\frac{3\pi}{8}$.

(d) The set on which both f and g are defined is $D \cap D' = \{z : \mathcal{I}\mathrm{m}(z) \neq 0\}$. We know that on this domain $f(z) = \pm g(z)$ and neither is zero. It follows that on each half of the domain, either $f(z) = g(z)$ or $f(z) = -g(z)$. Looking at the results of (c), we see that we must have $f(i) = g(i)$ and $f(-i) = -g(-i)$; consequently $f(z) = g(z)$ iff $\mathcal{I}\mathrm{m}(z) > 0$ and $f(z) = -g(z)$ iff $\mathcal{I}\mathrm{m}(z) < 0$.

(e) If $z \doteq \frac{1}{2}$ then $g(z) \doteq \frac{i}{2}$, while
if $\mathcal{I}\mathrm{m}(z) > 0$ then $f(z) = g(z) \doteq \frac{i}{2}$,
if $\mathcal{I}\mathrm{m}(z) < 0$ then $f(z) = -g(z) \doteq -\frac{i}{2}$.

If $z \doteq 2$ then $f(z) \doteq \sqrt{2}$ while
if $\mathcal{I}\mathrm{m}(z) > 0$ then $g(z) = f(z) \doteq \sqrt{2}$,
if $\mathcal{I}\mathrm{m}(z) < 0$ then $g(z) = -f(z) \doteq -\sqrt{2}$.

31. (a)(i) $2e^{it}$ for $0 \le t \le 2\pi$.
(ii) $2e^{i(\pi - t)}$ for $0 \le t \le \pi$ *or* $2e^{-it}$ for $-\pi \le t \le 0$.
(iii) t for $-2 \le t \le 2$.

	(i)	(ii)	(iii)	(iv)	(v)
(b) $\mathcal{R}\mathrm{e}(z)$	$4\pi i$	$-2\pi i$	0	$2\pi i$	$8i$
(c) z^n	0	$\psi(n)$	$\psi(n)$	0	0
(d) z^n	0	$\psi(n)$	—	—	0
(e) $\frac{1}{z}$	$2\pi i$	$-i\pi$	—	—	$2\pi i$
(f) $z^2\exp(z^3)$	(0)	$\frac{2}{3}\sinh 8$	$\frac{2}{3}\sinh 8$	0	(0)
(g) $\exp(z^2 - z)$	0	?	?	0	0

writing $\psi(n) = \frac{2^{n+2}}{n+1}$ if n is even, 0 if n is odd (that is, $\psi(n) = \frac{1}{n+1}(2^{n+1} - (-2)^{n+1})$).

Notes(b)(i) $\int_0^{2\pi} 2\cos t.2ie^{it}dt = 2i\int_0^{2\pi} 1 - \cos 2t + i\sin 2t\, dt = 4\pi i$.
(ii) $\int_0^{\pi}(-2\cos t)(-2ie^{i(\pi-t)})dt = 2i\int_0^{\pi}\cos 2t - 1 + i\sin 2t\, dt = -2\pi i$.
(iii) $\int_{-2}^{2} t dt = 0$.
(iv) $0 - (-2\pi i) = 2\pi i$.
(v) $\int_{-2}^{2} 1.idy + \int_1^{-1} xdx + \int_2^{-2}(-1)idy + \int_{-1}^{1} xdx = 4i + 0 + 4i + 0 = 8i$.
(c)(d) $z^n = \frac{1}{n+1}\frac{d}{dz}z^{n+1}$.
(e)(i) $\int_0^{2\pi}\frac{1}{2e^{it}}2ie^{it}dt = 2\pi i$.
(ii) $\int_0^{\pi}\frac{1}{2e^{i(\pi-t)}}(-2ie^{i(\pi-t)})dt = -i\pi$.
(v) (If you know about 'deformation of contours') deform Γ_5 into Γ_1 without passing over 0; (if you know the Residue Theorem) $\int_{\Gamma_5}\frac{1}{z}dz = 2\pi i\,\mathrm{Res}_{z=0}\,\frac{1}{z}$.
(f) $z^2\exp(z^3) = \frac{1}{3}\frac{d}{dz}\exp(z^3)$.
(g) Use Cauchy's theorem.

32. (a) ∞.
(b) $\mathbb{C}\setminus\{0\}$.
(c) $z^2 f'(z) = \sum_{n=1}^{\infty} \frac{1}{(n+1)!} z^{n+1} = \sum_{n=2}^{\infty} \frac{1}{n!} z^n = \exp(z) - 1 - z$.
(d) $g'(z) = \exp(\frac{1}{z}) - \frac{1}{z}$.
(e) $\int_\Gamma \exp(\frac{1}{z})dz - \int_\Gamma \frac{1}{z}dz = \int_\Gamma g'(z)dz = 0$.
(f) Both are $2\pi i$.

33. (a) For $|\mathcal{I}\mathrm{m}(z)| < \pi$.
(b) For purely imaginary z (other than i).
(c) $\{z : \mathcal{R}\mathrm{e}(z) \neq 0 \ or\ |z| > 1\}$; that is, all complex z except for the straight line segment from $-i$ to i.
(d) $f'(z) = \frac{1}{1+z^2}$ for $z \in \mathrm{dom}(f)$.
(e) $f(2) = \frac{\pi}{2} - \frac{1}{2}\arctan\frac{4}{3} \simeq 1.11$, $f(-2) = \frac{\pi}{2} + \frac{1}{2}\arctan\frac{4}{3} \simeq 2.03$.
(f)(Γ_1) $f(2) - f(2) = 0$.
(Γ_2) $f(2) - f(-2) = -\arctan\frac{4}{3} \simeq -0.93$.
(Γ_3) $\int_{-2}^{2} \frac{1}{1+x^2}dx = \arctan 2 - \arctan(-2) = 2\arctan 2 \simeq 2.21$.
(Γ_4) $2\arctan 2 + \arctan\frac{4}{3} = \pi \simeq 3.14$. (To see that the sum is π exactly, consider the angles $A\hat{O}B$, $B\hat{O}C$, $C\hat{O}D$, where $A = 1$, $B = 1 + 2i$, $C = -1 + 2i$, $D = -1$ and $O = 0$ in $\mathbb{C}$. The first and last are just $\arctan 2$, while the middle one is $\arg(\frac{-1+2i}{1+2i}) = \arctan\frac{4}{3}$.)
(Γ_5) $f(1) - f(1) = 0$.
(g) For all $z \in \mathrm{dom}(f)$.
(h) For those z such that $0 < \mathcal{R}\mathrm{e}(z) < \pi$ and $z \neq \frac{\pi}{2}$.
(i) Yes (use part (g)).

34. (a)

$$\begin{aligned}\sum_{k=n+1}^{\infty} \frac{1}{k!} &= \frac{1}{(n+1)!}(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \dots) \\ &\leq \frac{1}{(n+1)!}(1 + \frac{1}{n+2} + \frac{1}{(n+2)^2} + \frac{1}{(n+2)^3} + \dots) \\ &= \frac{1}{(n+1)!}\frac{1}{1-\frac{1}{n+2}} = \frac{1}{(n+1)!}\frac{n+2}{n+1}.\end{aligned}$$

(b) $|f(z)| = |\sum_{k=n+1}^{\infty} \frac{1}{k!z^k}| \leq \sum_{k=n+1}^{\infty} |\frac{1}{k!z^k}| \leq \sum_{k=n+1}^{\infty} \frac{1}{k!} \leq \frac{n+2}{(n+1)!(n+1)}$.
(c) $|\int_{\Gamma_1} f(z)dz|$ is at most the length of Γ_1 multiplied by the greatest value of $|f(z)|$ on Γ, so is less than or equal to $4\pi\frac{n+2}{(n+1)!(n+1)}$.
(d) $\int_{\Gamma_1} \sum_{k=0}^{n} \frac{1}{k!z^k}dz = \int_{\Gamma_1} \frac{1}{z}dz = 2\pi i$, so

$$|\int_{\Gamma_1} \exp\tfrac{1}{z}dz - 2\pi i| = |\int_{\Gamma_1} \exp\tfrac{1}{z} - \sum_{k=0}^{n} \tfrac{1}{k!z^k}dz| = |\int_{\Gamma_1} f(z)dz| \leq \tfrac{4\pi(n+2)}{(n+1)!(n+1)}.$$

(e) Letting $n \to \infty$, $\int_{\Gamma_1} \exp\frac{1}{z}dz - 2\pi i = 0$, so $\int_{\Gamma_1} \exp\frac{1}{z}dz = 2\pi i$.

35. (a)(i) Pole, order 3, residue 5 at 0;
pole, order 2, residue i at $-\frac{i}{2}$;
pole, order 4, residue 0 at $\frac{1}{3}$.
(ii) Poles, order 1, residue $\mp\frac{i}{2}$ at $\pm i$.
(iii) Poles, order 2, residue 0 at $\pm i$.
(iv) Pole, order 4, residue $\frac{1}{6}$ at 0.
(v) Removable singularity at 0 (residue 0 if you like);
pole, order 1, residue $\frac{(-1)^n n}{\pi}$ at n for every $n \in \mathbb{Z} \setminus \{0\}$.
(vi) Poles, order 1, at 0, $-\frac{i}{3}$, $-3i$; residues given by $\frac{z^2-1}{9z^2+20iz-3} = \frac{1}{3}, -\frac{5}{12}, \frac{5}{12}$.

(b)(i) $2\pi i(\mathrm{Res}_0 + \mathrm{Res}_{-i/2} + \mathrm{Res}_{1/3}) = 2\pi i(5 + i + 0) = 10\pi i - 2\pi$.
(ii) $2\pi i \,\mathrm{Res}_i = 0$; or, $\int_{\Gamma_2}\big(\frac{d}{dz}(-\frac{1}{2(z^2+1)})\big)dz = 0$.
(iii) $2\pi i(\mathrm{Res}_0 + \mathrm{Res}_{-i/3}) = 2\pi i(\frac{1}{3} - \frac{5}{12}) = -\frac{\pi i}{6}$.
(iv) $2\pi i \,\mathrm{Res}_0 = -2\pi i$.
(v)

$$\begin{aligned}
2\pi i(\operatorname*{Res}_{-3} + \operatorname*{Res}_{-2} + \operatorname*{Res}_{-1} + \operatorname*{Res}_{0} + \operatorname*{Res}_{1} + \operatorname*{Res}_{2} + \operatorname*{Res}_{3})&\\
&= 2\pi i(-\frac{1}{\pi} + \frac{1}{\pi} - \frac{1}{\pi} + \frac{1}{\pi} - \frac{1}{\pi} + \frac{1}{\pi} - \frac{1}{\pi}\\
&= -2i.
\end{aligned}$$

36. (i) Pole, order 1, residue $\frac{1}{3}$ at -1; poles, order 1, residues $\frac{1}{6}(-1 \mp \frac{i}{3})$ as $\frac{1}{2}(1 \pm i\sqrt{3})$.

(ii) Poles, order 1, residue $\frac{1}{2}$ at $\pm i$.

(iii) Poles, order 2, residues $\mp\frac{i}{4}$ at $\pm i$.

(iv) Removable singularity (residue 0) at 0; poles, order 1, residues $(-1)^n n\pi$ at $n\pi$ for $n \in \mathbb{Z} \setminus \{0\}$.
(v) No isolated singularities.
(vi) Pole, order 1, residue 1 at $2\pi ni$ for $n \in \mathbb{Z}$.
(vii) Removable singularity (residue 0) at 0; pole, order 1, residue $2\pi ni$ at $2\pi ni$ for $n \in \mathbb{Z} \setminus \{0\}$.
(viii) Removable singularity (resideue 0) at 0; pole, order 1, residue 1 at $2\pi ni$ for $n \in \mathbb{Z} \setminus \{0\}$.
(ix) Pole, order 2, residue -1 at $2\pi ni$ for $n \in \mathbb{Z}$.
(x) Pole, order 1, residue 1 at 0; poles, order 2, residue 1 at $2\pi ni$ for $n \in \mathbb{Z} \setminus \{0\}$.

(xi) Pole, order 1, residue 1 at $n\pi$ for $n \in \mathbb{Z}$; removable singularities (residue 0) at $n\pi + \frac{\pi}{2}$ for $n \in \mathbb{Z}$.

(xii) Removable singularities (residue 0) at 0 and $\pm 2\pi i$; poles, order 1, residues $8\pi^3 ni(1-n^2)$ at $2\pi ni$ for $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$.
(xiii) Essential singularities, residue 0, at $2n\pi$ for $n \in \mathbb{Z}$.
(You need to know that $\cos z = 1$ iff $e^{iz} = 1$ iff z is an integer multiple of 2π, so that these are the isolated singularities. To see that they are essential singularities, remember that $1 - \cos x \leq \frac{x^2}{2}$ for all real x. So $\exp(\frac{1}{1-\cos x}) \geq \exp(\frac{2}{x^2})$ for $0 < |x| < 2\pi$, and

$$\lim_{x\to 0} x^{2n} \exp(\frac{1}{1-\cos x)} \geq \lim_{y\downarrow 0} y^n \exp(\frac{2}{y}) = \lim_{y\to\infty} \frac{\exp(2y)}{y^n} = \infty$$

for every n, hence 0 is an essential singularity. Because $\exp(\frac{1}{1-\cos z})$ is periodic with period 2π, all the singularities must be of the same type.
In fact this is an easy function to find the residues of; because it is even, it must have residue 0 at 0, and therefore at every singularity.)

37. (i) $2\pi i \operatorname{Res}_{\frac{1}{2}} = 2\pi i(-\frac{1}{2}\mathrm{Ln}\frac{1}{2}) = \pi i \ln 2 \simeq 2{\cdot}178i$.
(ii) $2\pi i \operatorname{Res}_0 = 0$ (the function is even).

38.

$$\int_0^{2\pi} \frac{\sin t}{5+3\sin t} dt = \int_\Gamma \frac{\frac{1}{2i}(z - \frac{1}{z})}{5 + \frac{3}{2i}(z - \frac{1}{z})} \frac{dz}{iz}$$
$$= \frac{1}{i} \int_\Gamma \frac{z^2 - 1}{z(3z^2 + 10iz - 3)} dz$$
$$= \frac{1}{i} \cdot (-\frac{\pi i}{6})$$

(see question 35(b-iii))

$$= -\frac{\pi}{6}.$$

Note. Of course $\int_0^{2\pi} \frac{\sin t}{5+3\sin t} dt$ must be real. To see that it is negative, observe that it is

$$\int_0^\pi \frac{\sin t}{5+3\sin t} dt + \int_\pi^{2\pi} \frac{\sin t}{5+3\sin t} dt = \int_0^\pi \frac{\sin t}{5+3\sin t} dt + \int_0^\pi \frac{\sin(t+\pi)}{5+3\sin(t+\pi)} dt$$
$$= \int_0^\pi \frac{\sin t}{5+3\sin t} + \frac{-\sin t}{5-3\sin t} dt$$
$$= \int_0^\pi (\frac{1}{5+3\sin t} - \frac{1}{5-3\sin t}) \sin t \, dt$$
$$= -6 \int_0^\pi \frac{\sin^2 t}{25-9\sin^2 t} dt$$
$$< 0$$

because the integrand $\frac{\sin^2 t}{25-9\sin^2 t}$ is sometimes positive and never negative.