Group Theory Problems

1. (a) Define *semigroup*. Explain what is meant by an identity in a semigroup and an inverse of an element in a semigroup with identity.

(b) Show that identities and inverses, if they exist at all, are unique.

(c) Let F be a semigroup. Suppose that there is an $e \in F$ such that $xe = x$ for every $x \in F$, and for every $x \in F$ there is a $y \in F$ such that $xy = e$. Show that

(i) for every $x \in F$ there is a $z \in F$ such that $ez = x$;

(ii) e is an identity of F;

(iii) F is a group.

(d) Let F be a non-empty semigroup. Suppose that for all x, $y \in F$ there are z, $z' \in F$ such that $xz = z'x = y$. Show that

(i) if x_0, $e \in F$ are such that $x_0 e = x_0$, then $xe = x$ for every $x \in F$;

(ii) F is a group

(e) Let F be a non-empty finite semigroup which satisfies both cancellation laws (i.e., $xz = yz \Rightarrow x = y$ and $zx = zy \Rightarrow x = y$). Show that F is a group.

(This question is quite hard. But each fragment is supposed to follow without an impossibly large jump from what has gone before. So even if you can't do (c-i), you ought to be able to use it to manage (c-ii).)

[AA]

2. For integers $n \geq 2$ write $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ and let $\times_n$ be multiplication mod n on $\mathbb{Z}_n$, so that $3 \times_7 5 = 1$.

(a) Show that $(\mathbb{Z}_n, \times_n)$ is a semigroup with identity, for any $n \geq 2$.

(b) Show that an element i of $\mathbb{Z}_n$ is invertible, for $\times_n$, iff i and n have no common factor other than 1. (You may assume the fact that there are integers r and s such that $ir + ns$ is a factor of both i and n.)

(c) Write $\mathbb{Z}_n^*$ for the group of invertible elements of $\mathbb{Z}_n$ (for $\times_n$). Which of the groups $\mathbb{Z}_n^*$, for $2 \leq n \leq 12$, are cyclic? Find generators of those which are. [AB]

3. (a) Define *subgroup*, *coset*.

(b) State and prove Lagrange's theorem.

(c) What is meant by 'the subgroup generated by' an element of a group? Describe the subgroup of the group of non-singular 2×2 real matrices generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

(d) Show that if G is a finite group of order n and $a \in G$ then $a^n = e$. [BA]

4. (a) Let G be the group of all bijective functions from $\mathbb{R} \setminus \{0, 1\}$ to itself. Define g, $h \in G$ by writing

$$g(x) = \frac{1}{x}, \quad h(x) = 1 - \frac{1}{x}$$

for $x \in \mathbb{R} \setminus \{0, 1\}$. Find g^2, h^2, h^3, $g^{-1}hg$. Show that the subgroup of G generated by g and h is finite. To which familiar group is it isomorphic?

(b) Show that a finite group H is isomorphic to a subgroup of $\mathbb{C} \setminus \{0\}$ (with multiplication) iff it is cyclic. Which finite groups are isomorphic to subgroups of $\mathbb{R} \setminus \{0\}$? [BB]

5. (a) Define *normal subgroup*.

(b) Explain how to form the quotient group of a given group by a normal subgroup. At which point does the construction fail if the subgroup is not normal?

(c) Let G be the group of non-singular 2×2 real matrices. Describe (with proofs)

(i) a normal subgroup H of G which is neither $\{e\}$ nor G;

(ii) a familiar group isomorphic to G/H;

(iii) a subgroup K of G which is not normal. [BC]

6. A_4 is the group of permutations e=identity, $a = (\overleftarrow{12})(\overleftarrow{34})$, $b = (\overleftarrow{13})(\overleftarrow{24})$ $c = (\overleftarrow{14})(\overleftarrow{23})$, $p = (\overleftarrow{123})$, $q = (\overleftarrow{243})$, $r = (\overleftarrow{142})$, $s = (\overleftarrow{134})$, $w = (\overleftarrow{132})$, $x = (\overleftarrow{143})$, $y = (\overleftarrow{234})$, $z = (\overleftarrow{124})$.

(a) Write out the group table of A_4, listing the elements in the order given both in rows and in columns.

(b) Find a normal subgroup H with four elements; list its cosets, and give the group table of A_4/H.

(c) How many subgroups with three elements are there? Are any of them normal?

(d) Is there any subgroup with six elements? [BD]

7. (a) Define *homomorphism*, *isomorphism*, *kernel* (of a homomorphism).

(b) Show that the kernel of a homomorphism is a normal subgroup and that the set of values of a homomorphism is a subgroup.

(c) State and prove the First Isomorphism Theorem.

(d) Let G be a group, H a normal subgroup of G and K a subgroup of G. Show that

(i) HK is a subgroup of G;

(ii) $H \cap K$ is a normal subgroup of K;

(iii) HK/H is isomorphic to $K/H \cap K$. [BE]

8. Let G be a group and H, K subgroups of G.

(a) Show that $H \cap K$ is a subgroup of H.

(b) Show that if $HK \subseteq KH$ then KH is a subgroup of G and $HK = KH$.

(c) Show that if H and K are finite then $\#(HK)\#(H \cap K) = \#(H)\#(K)$. (Hint: show that for h_1, $h_2 \in H$, $h_1K = h_2K$ iff $h_1(H \cap K) = h_2(H \cap K)$ iff $h_1^{-1}h_2 \in K$.)

(d) Give an example in which HK is not a subgroup of G. (Hint: take $G = A_4$, $\#(H) = 3$, $\#(K) = 2$.) [BF]

9. (a) For $k = 1, 2, \dots$, how many elements of A_4 have k orbits? (Hint: question 6.)

(b) For $k = 1, 2, \dots$ how many elements of S_4 have k orbits? (You should be able to work out how many of each kind of permutation there are without listing them, but if in doubt do the exhaustive check.)

(c) Suppose that $\rho \in S_n$, where $n \geq 3$, has m orbits. (i) How many orbits does $(\overleftarrow{123})\rho$ have if 1, 2, 3 all belong to different orbits of ρ? (ii) How many orbits does $(\overleftarrow{123})\rho$ have if 1, 2 belong to the same orbit of ρ and 3 belongs to a different one? (iii) Can we determine how many orbits $(\overleftarrow{123})\rho$ has if we are told that 1, 2, 3 all belong to the same orbit of ρ? Justify your answers. [BG]

10. (a) In the group S_6, set $\rho = (\overleftarrow{12})(\overleftarrow{3456})$. What is the order of ρ? List the members of the subgroup generated by ρ.

(b) In the group S_7, set $\rho = (\overleftarrow{123})(\overleftarrow{4567})$. What is the order of ρ?

(c) In the group S_8, what is the largest order which any element can have? Why? [BH]

11. The '16-puzzle' is a square tray with 15 numbered square pieces, one square being left empty, thus:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Each piece is free to move up, down or sideways; so that the pieces can be rearranged by shifting them successively into the empty space, which wanders accordingly around the tray. (For instance, the first three moves could be 15 right, 11 down, 7 down.) Show that at most $\frac{1}{2} \cdot 16!$ positions can be achieved from the starting position. (*Hint*: if you think of the empty space as '16', find a relation between the signature of a permutation in S_{16} and the position of '16' in the tray.) [CA]

12. Two elements x, y of a group G are **conjugate** in G if there is an $a \in G$ such that $x = aya^{-1}$. Let σ be the permutation $(\overleftarrow{123})$.

(a) How many permutations are conjugate to σ in the symmetric group S_4?

(b) Which permutations are conjugate to σ in the alternating group A_4? (You may find your group table for A_4 useful.)

(c) How many permutations are conjugate to σ in the symmetric group S_5?

(d) How many permutations are conjugate to σ in the alternating group A_5? [CB]

13. (a) Let G be a group, K a normal subgroup of G and H a subgroup of G such that $H \cap K = \{e\}$. Show that $x \mapsto xK : H \to G/K$ is an injective homomorphism.

(b) Now let K be the normal subgroup of S_4 with 4 elements. How many elements does the quotient group S_4/K have? Show that there is a subgroup H of S_4, isomorphic to S_3, such that $H \cap K = \{e\}$. Show that $H \cong S_4/K$. [CC]

14. Let G be a group. Set

$$A = \{xyx^{-1}y^{-1} : x,\, y \in G\},$$

and

$$H = \{a_1a_2 \dots a_n : a_1, \dots , a_n \in A\}.$$

(a) Show that (i) $a^{-1} \in A$ for every $a \in A$ (ii) $zaz^{-1} \in A$ for every $a \in A$, $z \in G$ (iii) H is a normal subgroup of G.

(b) Show that the quotient group G/H is abelian. (*Hint*: $xH = yH$ iff $y^{-1}x \in H$.)

(c) Show that if K is any abelian group and $\phi : G \to K$ is a homomorphism, then the kernel of K includes H.

(d) Find H when G is

(i) $\mathbb{Z}_6$ (ii) S_3 (iii) A_4.

(e) Find A when G is

(i) S_3 (ii) A_4.

(H is the **commutator subgroup** of G.) [CD]

15. In the dihedral group D_n of order $2n$, set $\rho = (\overleftarrow{12 \dots n})$ and $\sigma = (\overleftarrow{1n})(\overleftarrow{2\ n-1})\dots$.

(a) Show that any subgroup of $< \rho >= \{e, \rho, \rho^2, \dots , \rho^{n-1}\}$ is a normal subgroup of D_n. (*Hint*: show that $(\sigma\rho^j)\rho^i(\sigma\rho^j)^{-1} = \rho^{-i}$.)

(b) Find $\rho\sigma\rho^{-1}$.

(c) Show that if n is *odd*, then the only normal subgroup of D_n containing σ is D_n itself. (*Hint*: $\rho = \rho^{n+1}$ is a power of ρ^2.)

(d) Show that if n is *even*, then

$$H^* = \{\sigma^i\rho^{2j} : 0 \le i < 2,\, 0 \le j < \tfrac{n}{2}\}$$

is a subgroup of D_n. Is it a normal subgroup? Why?

(e) Show that if n is odd, then the only normal subgroups of D_n are D_n itself and the subgroups of $< \rho >$.

(f) Show that if n is even, then the only normal subgroups of D_n are D_n itself, subgroups of $< \rho >$, and H^*.

(g) To which familiar group is the group H^* of (d) above isomorphic?

(h) How many elements of D_n are conjugate to ρ in D_n? How many elements of D_n are conjugate to σ in D_n? (Make sure your answer considers both odd and even n.) [DA]

16. (a) Sketch a regular tetrahedron with the edges labelled 1 to 6. Write out, in disjoint-cycle form, permutations of the edges corresponding (i) to a rotation of the tetrahedron through an angle of 120° about an axis through a vertex and the centre of the opposite face (ii) to a rotation of the tetrahedron through an angle of 180° about an axis through the midpoints of two opposite edges (iii) to a reflection of the tetrahedron across a plane through an edge and the midpoint of the opposite edge.

(b) Show that A_6 has a subgroup H, isomorphic to S_4, which is *transitive*, that is, for every $i, j \le 6$ there is a $\rho \in H$ such that $\rho(i) = j$.

(c) Find a non-transitive subgroup of A_6 which is isomorphic to S_4. [DB]

17. (a) Show that the group of rotations of a cube is isomorphic to S_4.

(b) Give geometric descriptions of those rotations which correspond (i) to members of K, the four-element normal subgroup of S_4 (ii) to members of A_4, the alternating group.

(c) Give geometric descriptions of (i) the two types of rotation of order 2 (ii) the rotations of order 3 (iii) the rotations of order 4. How many of each of these are there? [DC]

18. (a) Show that the group of rotations of a regular dodecahedron is isomorphic to A_5.

(b) Show that the group of isometries of a regular dodecahedron is not isomorphic to S_5.

(c) Give geometric descriptions of (i) the rotations of order 3 (ii) the rotations of order 5 (iii) the rotations of order 2. How many of each are there? [DD]

19. Consider a double-tetrahedron X obtained by glueing two regular tetrahedra together at a common face. Let G be the group of isometries of X, and H the group of rotations.

(a) What are $\#(G)$ and $\#(H)$? Show that $H \lhd G$.

(b) How many elements of order 3 does G have? How many of these are in H?

(c) How many elements of order 2 does G have? How many of these are in H?

(d) Find an element of order 2 in the centre C of G. ($C = \{a : ab = ba \text{ for every } b \in G\}$.) What is C?

(e) How many normal subgroups does G have? Describe each normal subgroup of G in geometric terms. [DE]

20. (a) Show that every element of A_5 is conjugate to its inverse in A_5.

(b) Show that every element of every S_n is conjugate to its inverse in S_n.

(c) Show that for $n = 3$, 4, 7 or 8 there is an element of A_n which is not conjugate in A_n to its inverse. What happens in A_6 and A_9? [EA]

21. Let G be a group.

(a) Explain what is meant by the *centre* C of G and show that C is a normal subgroup of G.

(b) Show that G/C cannot be cyclic unless $C = G$.

(c) Show that if G is finite and $\#(G)$ is a prime power greater than 1, then $C \neq \{e\}$.

(d) Show that if $\#(G) = p^2$ where p is prime, then G is abelian.

(e) Give an example of a non-abelian group of order p^3, where p is a prime, and describe its centre. [EB]

22. (a) What is an *automorphism*? What is an *inner automorphism*? Explain why an inner automorphism, on the definition you give, is an automorphism. Show that the set $\operatorname{Aut} G$ of automorphisms of a group is again a group.

(b) Let G be a group, C its centre, J the set of inner automorphisms of G. Show that

(i) J is a subgroup of $\operatorname{Aut} G$ isomorphic to G/C;

(ii) $J \lhd \operatorname{Aut} G$.

(c) For each of the following groups G describe C, $\operatorname{Aut} G$ and J: (i) $G = K$, the four-element normal subgroup of S_4; (ii) $G = S_3$; (iii) $G = \mathbb{Z}_6$; (iv) $G = \mathbb{Z}_5$; (v) $G = D_4$, the dihedral group of order 8. [EC]

23. Let G and H be groups and $\phi : H \to \operatorname{Aut} G$ a homomorphism. Define an operation $*$ on $G \times H$ by writing

$$(g_1, h_1) * (g_2, h_2) = (g_1 \cdot \phi(h_1)(g_2), h_1 h_2) \text{ for all } g_1,\, g_2 \in G,\, h_1,\, h_2 \in H.$$

(a) Show that $(G \times H, *)$ is a group.

(b) Show that $\{(g, e_H) : g \in G\}$ is a normal subgroup of $G \times H$ isomorphic to G.

(c) Show that $(e_G, h) : h \in H\}$ is a subgroup of $G \times H$ isomorphic to H.

(d) Show that if Q is any group, $G \lhd Q$, $H \leq Q$, $G \cap H = \{e\}$ and $GH = Q$, then there is a homomorphism $\phi : H \to \operatorname{Aut} G$ such that $Q \cong (G \times H, *)$ if we define $*$ from ϕ as above.

($(G \times H, *)$ is called a *wreath product.*) [EE]

24. Let G be a finite group and H a subgroup of G.

(a) Taking X to be the set of left cosets of H in G, show that there is an action of G on X defined by setting $a \bullet D = aD$ for $a \in G$, $D \in X$.

(b) Let K be the kernel of this action. Show that $K \subseteq H$. Show that $\frac{\#(G)}{\#(K)}$ is a factor of $m!$, where $m = \frac{\#(G)}{\#(H)}$.

(c) Show that if $K = H$ then $H \lhd G$.

(d) Show that if m is a prime and is the smallest prime factor of $\#(G)$, then $H \lhd G$.

(e) Show that if H is a proper subgroup of A_5, then $\#(H) \leq 12$.

(f) For just which values of n does A_5 have a subgroup of order n? Justify your answer. [EF]