Optimisation I Problems

1. Solve the following problem by the Two-Phase Simplex Method:

$$\text{maximize } z = 4x_1 + 8x_2 + 3x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\le 5 \\ 2x_1 + 3x_2 - x_3 &= 3 \\ x_1 - 5x_2 + 6x_3 &\ge 4 \end{aligned}$$

and x_1, x_2, $x_3 \ge 0$.

2. Maximize

$$z = 5x_1 + 3x_2$$

subject to

$$\begin{aligned} x_1 - x_2 &\le 2 \\ 2x_1 + x_2 &\le 4 \\ -3x_1 + 2x_2 &\le 6. \end{aligned}$$

3. Maximise $8x_1 + 9x_2 + 5x_3$ subject to

$$\begin{aligned} x_1 + x_2 + 2x_3 &\le 2 \\ 2x_1 + 3x_2 + 4x_3 &\le 3 \\ 6x_1 + 6x_2 + 2x_3 &\le 8 \end{aligned}$$

and x_1, x_2, $x_3 \ge 0$. [XBA]

4. Use Phase 1 of the Two-Simplex Method to show that the following LP is infeasible: maximise $2x_1 + x_2 + x_3$ subject to

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &\le 9 \\ 2x_1 - x_3 &\ge 4 \\ 2x_1 - x_2 + x_3 &= 4 \end{aligned}$$

and x_1, x_2, $x_3 \ge 0$. [XBB]

5. Use the Two-Phase Simplex Method to determine if the following LP has multiple optima. maximise $2x_1 + x_2 + x_3$ subject to

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &\le 9 \\ 2x_2 + 4x_3 &\ge 4 \\ x_1 + x_2 &= 6 \end{aligned}$$

and x_1, x_2, $x_3 \ge 0$. [XBC]

6. Show that the following LP produces a degenerate tableau after one pivot: maximise $2x_1 - x_2 + 8x_3$ subject to

$$\begin{aligned} 2x_3 &\le 1 \\ 2x_1 - 4x_2 + 6x_3 &\le 3 \\ -x_1 + 3x_2 + 4x_3 &\le 2 \end{aligned}$$

and $x_1, x_2, x_3 \ge 0$. [XBD]

7. Use the Simplex Algorithm to show that the following LP is unbounded: maximise $2x_1 + x_2$ subject to

$$\begin{aligned} x_1 - x_2 &\le 10 \\ 2x_1 &\le 40 \end{aligned}$$

and $x_1, x_2 \ge 0$. [XBE]

8. Maximise $3x_1 + x_2$ subject to

$$\begin{aligned} x_1 - x_2 &\le -1 \\ -x_1 - x_2 &\le -3 \\ 2x_1 + x_2 &\le 4 \end{aligned}$$

and $x_1, x_2 \ge 0$. [XCA]

9. Maximise $3x_1 + x_2$ subject to

$$\begin{aligned} x_1 - x_2 &\le -1 \\ -x_1 - x_2 &\le -3 \\ 2x_1 + x_2 &\le 2 \end{aligned}$$

and $x_1, x_2 \ge 0$. [XCB]

10. Maximise $3x_1 + x_2$ subject to

$$\begin{aligned} x_1 - x_2 &\le -1 \\ -x_1 - x_2 &\le -3 \\ 2x_1 - x_2 &\le 2 \end{aligned}$$

and $x_1, x_2 \ge 0$. [XCC]

11. Using the Revised Simplex Algorithm, maximise $8x_1 + 9x_2 + 5x_3$ subject to

$$\begin{aligned} x_1 + x_2 + 2x_3 &\le 2 \\ 2x_1 + 3x_2 + 4x_3 &\le 3 \\ 6x_1 + 6x_2 + 2x_3 &\le 8 \end{aligned}$$

and $x_1, x_2, x_3 \ge 0$. [XCD]

12. Apply the revised simplex method to the problem

$$\text{maximise } z = 5x_1 + 4x_2 + 3x_3$$

subject to

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &\le 5 \\ 4x_1 + x_2 + 2x_3 &\le 11 \\ 3x_1 + 4x_2 + 2x_3 &\le 8 \end{aligned}$$

and $x_1,\, x_2,,\, x_3 \ge 0.$ [C]

13. Solve the following by the dual simplex method:
(a) minimise $2x_1 + 3x_2$ subject to $2x_1 + 3x_2 \le 30$, $x_1 + 2x_2 \ge 10$, $x_1,\, x_2 \ge 0$.
(b) Minimise $5x_1 + 6x_2$ subject to $x_1 + x_2 \ge 2$, $4x_1 + x_2 \ge 4$, $x_1,\, x_2 \ge 0$. [E]

14. Consider the undirected graph G with nodes $1, \ldots, 6$ and arcs $(1,4)$, $(1,6)$, $(2,4)$, $(2,5)$, $(3,4)$, $(3,5)$, $(3,6)$.
(a) Sketch G; try to find a diagram with the smallest possible number of crossing arcs.
(b) Give the adjacency matrix for G.
(c) Find three different chains from node 1 to node 3.
(d) Find two cycles of different lengths starting and finishing at node 4. [GA]

15. Consider the undirected graph G with nodes $1, \ldots, 8$ and arcs $(1,5)$, $(1,6)$, $(1,7)$, $(2,8)$, $(3,6)$, $(3,8)$, $(4,6)$, $(4,8)$, $(5,7)$.
(a) Listing each step, look for a spanning tree using a breadth-first search, and sketch the tree.
(b) Listing each step, look for a spanning tree using a depth-first search, and sketch the tree.
(In both searches, start from node 1, take arcs from each node under consideration in strict order (e.g., when examining arcs from node 6, take them in the order $(6,1)$, $(6,3)$, $(6,4)$), and fully list rejected arcs as well as accepted arcs in the order in which they are examined.) [GB]

16. What happens, in question 15, if the arcs $(3,8)$ and $(4,8)$ are deleted? Sketch the trees which will be formed in this case (a) by a breadth-first search (b) by a depth-first search (both starting from node 1, and taking arcs in strict order, as in q. 15).
[GC]

17. Consider the directed graph with nodes $1, \ldots, 8$ and arcs $(1,2)$, $(1,3)$, $(1,4)$, $(2,3)$, $(2,5)$, $(3,7)$, $(4,3)$, $(5,3)$, $(5,8)$, $(6,4)$, $(6,5)$, $(6,7)$, $(6,8)$, $(7,3)$, $(7,8)$.
(a) Give the adjacency matrix of the graph.
(b) Which nodes (if any) are sources? Which are sinks?
(c) Find two paths from node 1 to node 8 which are *independent*, i.e., do not share any other nodes.
(d) Find two arcs such that if you delete them there will be no paths from node 1 to node 8 in the resulting graph.
Hint: you will probably find it much easier to answer part (c) if you first sketch the graph.
[GD]

18. Consider the network with nodes $1, \ldots, 5$ and capacities $c_{12} = 3$, $c_{13} = 5$, $c_{14} = 4$, $c_{23} = 1$, $c_{25} = 6$, $c_{35} = 2$, $c_{43} = 2$, $c_{45} = 7$. Find a cut stopping all flows from node 1 to node 5 and a flow from node 1 to node 5 with the same value. Are these unique? [GE]

19. Consider the network with source 1, sink 6, transshipment nodes $2, \ldots, 5$ and capacities

$$c_{12} = 5,\ c_{13} = 8,\ c_{24} = 4,\ c_{25} = 2,\ c_{34} = 3,\ c_{35} = 4,\ c_{45} = 4,\ c_{46} = 8,\ c_{56} = 7.$$

Use the Ford-Fulkerson algorithm to find a maximal flow and minimal cut for this network; give their value. (Show your working.) [HA]

20. In question 19, suppose we start from a flow of 4 along $1 - 2 - 4 - 5 - 6$. Show how the Ford-Fulkerson algorithm transforms this into a maximal flow. [HB]

21. Consider the network with nodes $1, \dots, 4$, source 1, sink 4, and arc capacities

$$c_{12} = 2,\ c_{13} = 5,\ c_{23} = c_{32} = 2 \text{ (bi-directional arc)},\ c_{24} = 4,\ c_{34} = 3;$$

suppose moreover that the transshipment nodes 2 and 3 have capacities 3 and 4 respectively. Use the Ford-Fulkerson algorithm to find a maximal flow and a minimal cut. [HC]

22. For each of the following functions, find its gradient vector $\nabla f(\mathbf{x})$, its Hessian $\mathbf{H}f(\mathbf{x})$, and its stationary points. Determine whether the stationary points are local maxima, local minima or saddle points; show your working.
(a) $f_1(\mathbf{x}) = x_1^2 - 3x_1x_2 + 3x_2^2$,
(b) $f_2(\mathbf{x}) = 2x_1^2 - 5x_1x_2 + 2x_2^2$,
(c) $f_3(\mathbf{x}) = x_1x_2(1 - x_1 - x_2)$,
(d) $f_4(\mathbf{x}) = \sin(x_1 + x_2)$
for $\mathbf{x} \in \mathbb{R}^2$, and
(e) $f_5(\mathbf{x}) = 2x_1^2 + 3x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_1x_3$
(f) $f_6(\mathbf{x}) = x_3 \exp(-x_1^2 - x_2^2)$
for $\mathbf{x} \in \mathbb{R}^3$. [IA]

23. Use the method of Lagrange multipliers to find the stationary values of the function $f(\mathbf{x}) = x_1^3 - x_2^4$ subject to the constraint $x_1^2 + 2x_2^2 = 4$. Which, if any, of these are global maxima or minima? [IB]

24. Which of the following functions are coercive, and how do you know?
(a) $f_1(\mathbf{x}) = x_1^2 - 3x_1x_2 + 2x_2^2$ for $\mathbf{x} \in \mathbb{R}^2$,
(b) $f_2(\mathbf{x}) = x_1^2 - 2x_1x_2 + 3x_2^2$ for $\mathbf{x} \in \mathbb{R}^2$,
(c) $f_3(\mathbf{x}) = x_1^2 + x_2^3 + x_3^4$ for $\mathbf{x} \in \mathbb{R}^3$,
(d) $f_4(\mathbf{x}) = x_1^4 + x_2^4 + x_3^4 - 4x_1^2x_2x_3$ for $\mathbf{x} \in \mathbb{R}^3$,
(e) $f_5(\mathbf{x}) = (x_1 + \cos x_2x_3)^2 + (x_2 + \cos x_3x_1)^2 + (x_3 + \cos x_1x_2)^2$ for $\mathbf{x} \in \mathbb{R}^3$. [IC]

25. Use the method of Lagrange multipliers to find the minima of $f(\mathbf{x}) = x_1x_2 + x_3$ subject to the constraint $x_1^2 + x_2^2 + x_3^2 = c$, where $c > 0$. Give the value f^* of the function at these minima, as a function of c, and calculate $\frac{df^*}{dc}$. [ID]

26. Consider the problem
Find the minima of $f(\mathbf{x}) = x_1 + x_2 + x_3$ subject to the constraints $x_1^2 + x_2^2 = \gamma$, $x_2^2 + x_3^2 = \delta$, where $\gamma, \delta > 0$.
Write down the equations provided by the method of Lagrange multipliers, and show that $x_1x_3 = x_2(x_1+x_3)$ at any minimum. Complete the solution for the case $\gamma = \delta$. Writing $f^* = f^*(\gamma, \delta)$ for the minimum value of f, regarded as a function of γ and δ, find $\frac{\partial f^*}{\partial \gamma}$ when $\gamma = \delta = 10$. [IE]

27. Set $f(\mathbf{x}) = x_1^2 + 5x_2^2 + 8x_3^2$, $g(\mathbf{x}) = x_1x_2 + x_2x_3 + x_3x_1 + 1$ for $\mathbf{x} \in \mathbb{R}^3$.
(a) Show that there is a minimal value of $f(\mathbf{x})$ subject to the condition $g(\mathbf{x}) = 0$. Is there a maximal value? Justify your answer.
(b) Suppose that $\mathbf{x}^*$ minimises f subject to the condition $g = 0$.
(i) Use the method of Lagrange multipliers to show that, for some λ,

$$2x_1^* = \lambda(x_2^* + x_3^*), \quad 10x_2^* = \lambda(x_1^* + x_3^*), \quad 16x_3^* = \lambda(x_1^* + x_2^*).$$

(ii) Explain why the determinant of the matrix $\begin{pmatrix} 2 & -\lambda & -\lambda \\ -\lambda & 10 & -\lambda \\ -\lambda & -\lambda & 16 \end{pmatrix}$ must be zero. Show that λ is either -4 or $-5 \pm \sqrt{65}$.
(iii) Show that at least one of x_1^*, x_2^*, x_3^* is less than 0 and at least one is greater than 0. Show that $x_3^{*2} \le x_2^{*2} \le x_1^{*2}$, and therefore that $x_1^* \neq 0$ and x_2^* and x_3^* are of the same sign (or zero). Hence show that $\lambda \le 0$.
(iv) Show that $(16 + \lambda)x_3^* = (10 + \lambda)x_2^*$, so that neither x_2^* nor x_3^* can be zero and that $\lambda \notin [-16, -10]$.
(c) Find λ and $f(\mathbf{x}^*)$. How many possible values are there for $\mathbf{x}^*$? [IF]

28. Consider the function $f(x) = x + 3\cos x$ for $x \in \mathbb{R}$.
(a) Use the dichotomous search method, with indistinguishability constant 0·1, to locate the minimum of f in the interval $[1, 4]$ within an interval of length at most 0·2.
(b) Use the bisection search method to locate the minimum of f in the interval $[1, 4]$ within an interval of length at most 0·2.
(c) Use the method of false position, starting from $x_0 = 1$ and $x_1 = 4$, to find an x such that $|f'(x)| \leq 0{\cdot}01$. [JA]

29. Consider the function $f(\mathbf{x}) = x_1^2 - x_1x_3 + x_2^2 + x_3^2$ for $\mathbf{x} \in \mathbb{R}^3$. Starting from $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, find the next two points $\mathbf{x}_1$, $\mathbf{x}_2$ when searching for a minimum of f by the method of steepest descent. [JB]

30. Consider the function $f(x) = x^4 - 9x^2$ for $x \in \mathbb{R}$. Use Newton's method to search for stationary points of f, starting from (i) $x_0 = 0{\cdot}94$ (ii) $x_0 = 0{\cdot}95$ (iii) $x_0 = 0{\cdot}96$. (In each case continue the iteration until you have $|x_n - x_{n-1}| < 0{\cdot}1$, and find $f(x_n)$.) [JC]

31. Consider the function $f(\mathbf{x}) = x_1^4 - x_1^2x_2 + x_2^2$ for $\mathbf{x} \in \mathbb{R}^2$. Starting from $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, find the next two points $\mathbf{x}_1$, $\mathbf{x}_2$ when searching for a stationary point of f by Newton's method. [JD]

32. Consider the problem of minimising the function $f(x) = e^{-x} + x^2$.
(a) Starting from the initial interval $[-1, 1]$, use the bisection search method to reduce the interval of uncertainty to one of length at most 0·2.
(b) Starting from $x_0 = -1$, $x_1 = 1$, use the method of false position to find a point x such that $|f'(x)| \leq 0{\cdot}01$.
(c) Starting from the initial point $x_0 = 0$, use Newton's method to find a point x such that $|f'(x)| \leq 0{\cdot}01$. [JE]

33. For the function f and starting point $\mathbf{x}_0$ of question 31, find the next point $\mathbf{x}_1$ when searching for a minimum of f by the method of steepest descent. (Use the dichotomous search method, with final interval length at most 0·1, for the line search problem.) [JF]

34. For which values of c is the function $f(x) = x^2 + cx^3 + x^4$ convex? [KA]

35. Which of the following functions are convex?
(i) $f(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_3x_1$ for $\mathbf{x} \in \mathbb{R}^3$.
(ii) $f(\mathbf{x}) = x_1^4 + x_1x_2^3 + x_2^4$ for $\mathbf{x} \in \mathbb{R}^2$.
(iii) $f(\mathbf{x}) = e^{x_1 - x_2}$ for $\mathbf{x} \in \mathbb{R}^2$. [KBA]

36. As question 35, with
(i) $f(\mathbf{x} = (e^{x_1+x_2} - 2)^2$ for $\mathbf{x} \in \mathbb{R}^2$.
(ii) $f(\mathbf{x}) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + x_3^2 + x_1x_3 + x_2x_3$ for $\mathbf{x} \in \mathbb{R}^3$. [KBB]

37. (a) Let C_1, C_2 be convex subsets of $\mathbb{R}^n$.
(i) Is $C_1 \cap C_2$ necessarily convex?
(ii) Is $C_1 \cup C_2$ necessarily convex?
(b) Suppose that f and g are convex functions defined on a convex set $C \subseteq \mathbb{R}^n$, and $a \in \mathbb{R}$.
(i) Is $f + g : C \to \mathbb{R}$ necessarily convex?
(ii) Is $af : C \to \mathbb{R}$ necessarily convex?
(iii) Is $\max(f, g) : C \to \mathbb{R}$ necessarily convex?
(iv) Is $\min(f, g) : C \to \mathbb{R}$ necessarily convex?
(v) Is $f \times g : C \to \mathbb{R}$ necessarily convex?
Justify your answers. [KC]

38. (a) What is a *convex program*? For a given convex program, what is its *dual*?
(b) Consider the problem
(P) minimise $x_1^2 + 2x_2^2 + 3x_3^2$
for $\mathbf{x} \in \mathbb{R}^3$
subject to the constraints $x_1 + x_2 \geq 2$, $x_2 + x_3 \geq 1$.
Show that (P), suitably reformulated, is a convex program.
(c) State the dual problem (D) to the problem (P) of part (b), and solve (D) and (P). [LA]

39. Consider the problem
(P) minimise $(x_1 + 1)^2 + x_2^2$
for $\mathbf{x} \in C$
subject to the constraint $x_1 + 2x_2 \geq 2$,
where $C = \{\mathbf{x} : x_1 \geq 0,\, x_2 \geq 0\}$.
(a) Show that (P), suitably reformulated, is a convex program.
(b) State the dual problem (D).
(c) Solve (D).
(d) Use Lagrange multipliers to solve the problem
minimise $(x_1 + 1)^2 + x_2^2$
for $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{u} \in \mathbb{R}^3$,
subject to $x_1 - u_1^2 = x_2 - u_2^2 = x_1 + 2x_2 - 2 - u_3^2$,
and hence solve (P).
(e) Explain why, given the solution to (D), you can be sure that it is also the solution to (P). [LB]

40. Consider the problem
(P) minimise $f(\mathbf{x}) = x_2$
subject to $\mathbf{x} \in \mathbb{R}^2$
and $x_1 \geq 1$, $x_2^2 + x_2^2 \leq 1$.
(a) Check that (P), suitably reformulated, is a convex program.
(b) Show that (P) has an optimum $\mathbf{x}^*$ with $f(\mathbf{x}^*) = 0$.
(c) State the dual problem (D).
(d) Show that objective function $h(\boldsymbol{\lambda})$ of (D) has no maximum, but that $\sup_{\boldsymbol{\lambda} \leq \mathbf{0}} h(\boldsymbol{\lambda})$ is 0. [LC]

41. Consider the problem
(P) minimise $e^x(2 + \sin x)$
subject to $e^x \leq 0$.
(a) Show that (P) is a convex program.
(b) State the dual problem (D).
(c) Show that $L(x, \lambda)$ has no minimum for $x \in \mathbb{R}$, for any $\lambda \leq 0$, but that $h(\lambda) = \inf_{x \in \mathbb{R}} L(x, \lambda)$ is always 0.
(d) Is 0 the optimum value for the primal problem (P)? Explain your answer. [LD]

42. Consider the problem
(P) minimise $f(\mathbf{x}) = x_1$
subject to $\mathbf{x} \in C$, $x_2 \leq 0$
where $C = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^2, \text{ either } \mathbf{x} = \mathbf{0} \text{ or } x_2 > 0\}$.
(a) Check that C is convex.
(b) State the dual problem (D).
(c) Solve (P). Show that (D) has $h(\lambda) = -\infty$ for every $\lambda \leq 0$. [LE]

43. Consider the problem
(P) minimise $f(\mathbf{x}) = x_1$
subject to $\mathbf{x} \in C$, $\frac{{x_2}^2}{x_1} \leq 0$
where C is the set

$$\{\mathbf{x} : \mathbf{x} \in \mathbb{R}^2, \text{ either } x_1 > 0 \text{ and } x_2 > 0 \text{ or } x_1 \geq 1,\, x_2 = 0\}.$$

(a) Show that $g(\mathbf{x}) = \frac{x_2^2}{x_1}$ is a convex function on C, so that (P) is a convex program. Show that the minimum value of f is attained and is 1.

(b) State the dual problem (D). Show that $h(\lambda) = \inf_{\mathbf{x} \in C} L(\mathbf{x}, \lambda) = 0$ for every $\lambda \leq 0$, so that the duality gap for this problem is 1. [LF]

44. Consider the problem
(P) minimise $f(\mathbf{x}) = x_1$
subject to $\mathbf{x} \in C$, $x_2 + x_3 \leq 0$,
where C is the set

$$\{\mathbf{x} : \mathbf{x} \in \mathbb{R}^3,\, x_1^2 + x_2^2 \leq x_3^2,\, x_3 \geq 0,\, x_1 \geq -1\}.$$

(a) Describe the set

$$C_0 = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^3,\, x_1^2 + x_2^2 \leq x_3^2,\, x_3 \geq 0\}$$

and show that (P) is a convex program.

(b) Show that the optimal value of $f(\mathbf{x})$ is 0.

(c) State the dual problem (D).

(d) Show that $h(\lambda) = \inf_{\mathbf{x} \in C} L(\mathbf{x}, \lambda) = -1$ for every $\lambda \leq 0$. [LG]