Optimisation I Answers

1. Introduce slack variables x_4, x_5 for the inequalities, so that we have

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 5 \\ 2x_1 + 3x_2 - x_3 &= 3 \\ x_1 - 5x_2 + 6x_3 - x_5 &= 4 \end{aligned}$$

and $x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geq 0$. Because the point $x_1 = x_2 = x_3 = 0$, $x_4 = 5$, $x_5 = -4$ is not feasible, we add two artificial variables x_6, x_7 to get

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 5 \\ 2x_1 + 3x_2 - x_3 + x_6 &= 3 \\ x_1 - 5x_2 + 6x_3 - x_5 + x_7 &= 4 \end{aligned}$$

and $x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ x_7 \geq 0$; now we have a feasible point $x_1 = x_2 = x_3 = 0$, $x_4 = 5$, $x_6 = 3$, $x_5 = 0$ and $x_7 = 4$. We now seek a feasible point with $x_6 = x_7 = 0$, that is, with $x_6 + x_7 = 0$; we do this by minimising $x_6 + x_7$, that is, by maximising $s = -x_6 - x_7$. Our tableau is therefore

	s	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	θ-ratio	
s	1	0	0	0	0	0	1	1	0		
x_4	0	1	2	1	1	0	0	0	5	$5/1 = 5$	
x_6	0	2	3	-1	0	0	1	0	3	< 0	
x_7	0	1	-5	$6 <$	0	-1	0	1	4	$4/6 \simeq 0.7 \leftarrow$	
s	1	-3	2	$-5\uparrow$	0	1	0	0	-7		$[s - x_6 - x_7]$
x_3	0	$\frac{1}{6}$	$-\frac{5}{6}$	1	0	$-\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}/\frac{1}{6} = 4$	$[x_7 \div 6]$
x_4	0	$\frac{5}{6}$	$\frac{17}{6}$	0	1	$\frac{1}{6}$	0	$-\frac{1}{6}$	$\frac{13}{3}$	$\frac{13}{3}/\frac{5}{6} = 5.2$	$[x_4 - x_3]$
x_6	0	$\frac{13}{6} <$	$\frac{13}{6}$	0	0	$-\frac{1}{6}$	1	$\frac{1}{6}$	$\frac{11}{3}$	$\frac{11}{3}/\frac{13}{6} \simeq 1.7 \leftarrow$	$[x_6 + x_3]$
s	1	$-\frac{13}{6}\uparrow$	$-\frac{13}{6}$	0	0	$\frac{1}{6}$	0	$\frac{5}{6}$	$-\frac{11}{3}$		$[s + 5x_3]$
x_1	0	1	1	0	0	$-\frac{1}{13}$	$\frac{6}{13}$	$\frac{1}{13}$	$\frac{22}{13}$		$[x_6 \div \frac{13}{6}]$
x_3	0	0	-1	1	0	$-\frac{2}{13}$	$-\frac{1}{13}$	$\frac{2}{13}$	$\frac{5}{13}$		$[x_3 - \frac{1}{6}x_1]$
x_4	0	0	2	0	1	$\frac{3}{13}$	$-\frac{5}{13}$	$-\frac{3}{13}$	$\frac{38}{13}$		$[x_4 - \frac{5}{6}x_1]$
s	1	0	0	0	0	0	1	1	0		$[s + \frac{13}{6}x_1]$

Thus we do indeed have a feasible point in which $s = 0$, with $x_2 = x_5 = x_6 = 0$ and $x_1 = \frac{22}{13}$, $x_3 = \frac{5}{13}$ and $x_4 = \frac{38}{13}$. We therefore enter the second phase with basic variables x_1, x_3 and x_4, and our tableau is

	z	x_1	x_2	x_3	x_4	x_5	RHS	θ-ratio	
z	1	-4	-8	-3	0	0	0		
x_1	0	1	1	0	0	$-\frac{1}{13}$	$\frac{22}{13}$	$\frac{22}{13}/1 \simeq 1.7$	
x_3	0	0	-1	1	0	$-\frac{2}{13}$	$\frac{5}{13}$	< 0	
x_4	0	0	$2 <$	0	1	$\frac{3}{13}$	$\frac{38}{13}$	$\frac{38}{13}/2 \simeq 1.5 \leftarrow$	
z	1	0	$-7\uparrow$	0	0	$-\frac{10}{13}$	$\frac{103}{13}$		$[z + 4x_1 + 3x_3]$
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{3}{26}$	$\frac{19}{13}$		$[x_4 \div 2]$
x_1	0	1	0	0	$-\frac{1}{2}$	$-\frac{5}{26}$	$\frac{3}{13}$		$[x_1 - x_2]$
x_3	0	0	0	1	$\frac{1}{2}$	$-\frac{1}{26}$	$\frac{24}{13}$		$[x_3 + x_2]$
z	1	0	0	0	$\frac{7}{2}$	$\frac{1}{26}$	$\frac{236}{13}$		$[z + 7x_2]$

so we have an optimum when $x_1^* = \frac{3}{13}$, $x_2^* = \frac{19}{13}$, $x_3^* = \frac{24}{13}$ and $z^* = \frac{236}{13} \simeq 18.15$.

alternative route

In the first tableau, there was a point at which we had two equal most-negative coefficients in the s-row; both were $-\frac{13}{6}$. If we choose the *second* to determine our leaving variable we get a different path, as follows:

	s	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	θ-ratio	
x_3	0	$\frac{1}{6}$	$-\frac{5}{6}$	1	0	$-\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}/\frac{1}{6} = 4$	
x_4	0	$\frac{5}{6}$	$\frac{17}{6} <$	0	1	$\frac{1}{6}$	0	$-\frac{1}{6}$	$\frac{13}{3}$	$\frac{13}{3}/\frac{17}{6} \simeq 1.5 \leftarrow$	
x_6	0	$\frac{13}{6} <$	$\frac{13}{6}$	0	0	$-\frac{1}{6}$	1	$\frac{1}{6}$	$\frac{11}{3}$	$\frac{11}{3}/\frac{13}{6} \simeq 1.7$	
s	1	$-\frac{13}{6}$	$-\frac{13}{6}\uparrow$	0	0	$\frac{1}{6}$	0	$\frac{5}{6}$	$-\frac{11}{3}$		
x_2	0	$\frac{5}{17}$	1	0	$\frac{6}{17}$	$\frac{1}{17}$	0	$-\frac{1}{17}$	$\frac{26}{17}$	$\frac{26}{17}/\frac{5}{17} = 5.2$	$[x_4 \div \frac{17}{6}]$
x_3	0	$\frac{7}{17}$	0	1	$\frac{5}{17}$	$-\frac{2}{17}$	0	$\frac{2}{17}$	$\frac{33}{17}$	$\frac{33}{17}/\frac{7}{17} \simeq 4.7$	$[x_3 + \frac{5}{6}x_2]$
x_6	0	$\frac{26}{17} <$	0	0	$-\frac{13}{17}$	$-\frac{5}{17}$	1	$\frac{5}{17}$	$\frac{6}{17}$	$\frac{6}{17}/\frac{26}{17} \simeq 0.2 \leftarrow$	$[x_6 - \frac{13}{6}x_2]$
s	1	$-\frac{26}{17}\uparrow$	0	0	$\frac{13}{17}$	$\frac{5}{17}$	0	$\frac{12}{17}$	$-\frac{6}{17}$		$[s + \frac{13}{6}x_2]$
x_1	0	1	0	0	$-\frac{1}{2}$	$-\frac{5}{26}$	$\frac{17}{26}$	$\frac{5}{26}$	$\frac{3}{13}$		$[x_6 \div \frac{26}{17}]$
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{3}{26}$	$-\frac{5}{26}$	$-\frac{3}{26}$	$\frac{19}{13}$		$[x_2 - \frac{5}{17}x_1]$
x_3	0	0	0	1	$\frac{1}{2}$	$-\frac{1}{26}$	$-\frac{7}{26}$	$\frac{1}{26}$	$\frac{24}{13}$		$[x_3 - \frac{7}{17}x_1]$
s	1	0	0	0	0	0	1	1	0		$[s + \frac{26}{17}x_1]$

Thus we have a feasible point $x_1 = \frac{3}{13}$, $x_2 = \frac{19}{13}$, $x_3 = \frac{24}{13}$ and $x_4 = x_5 = x_6 = x_7 = 0$. The tableau to maximise z, with basic variables x_1, x_2, x_3, becomes

	z	x_1	x_2	x_3	x_4	x_5	RHS	θ-ratio	
z	1	-4	-8	-3	0	0	0		
x_1	0	1	0	0	$-\frac{1}{2}$	$-\frac{5}{26}$	$\frac{3}{13}$		
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{3}{26}$	$\frac{19}{13}$		
x_3	0	0	0	1	$\frac{1}{2}$	$-\frac{1}{26}$	$\frac{24}{13}$		
z	1	0	0	0	$\frac{7}{2}$	$\frac{1}{26}$	$\frac{236}{13}$		$[z + 4x_1 + 8x_2 + 3x_3]$

which is already optimal.

2. With slack variables for each inequality the tableau is

	z	x_1	x_2	x_3	x_4	x_5	RHS	θ-ratio	
x_3	0	$1 <$	-1	1	0	0	2	$2/1 = 2 \leftarrow$	
x_4	0	2	1	0	1	0	4	$4/2 = 2$	
x_5	0	-3	2	0	0	1	6	< 0	
z	1	$-5 \uparrow$	-3	0	0	0	0		
x_1	0	1	-1	1	0	0	2	< 0	$[x_3]$
x_4	0	0	$3 <$	-2	1	0	$0 \to \epsilon$	$\frac{\epsilon}{3} \leftarrow$	$[x_4 - 2x_1]$
x_5	0	0	-1	3	0	1	12	< 0	$[x_5 + 3x_1]$
z	1	0	-8	5	0	0	10		$[z + 5x_1]$
x_2	0	0	1	$-\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{\epsilon}{3}$	< 0	$[x_4 \div 3]$
x_1	0	1	0	$\frac{1}{3}$	$\frac{1}{3}$	0	$2 + \frac{\epsilon}{3}$	$6 + \epsilon$	$[x_1 + x_2]$
x_5	0	0	0	$\frac{7}{3} <$	$\frac{1}{3}$	1	$12 + \frac{\epsilon}{3}$	$\frac{36}{7} + \frac{\epsilon}{7} \simeq 5.1 \leftarrow$	$[x_5 + x_2]$
z	1	0	0	$-\frac{1}{3} \uparrow$	$\frac{8}{3}$	0	$10 + \frac{8\epsilon}{3}$		$[z + 8x_2]$
x_3	0	0	0	1	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{36}{7} + \frac{\epsilon}{7}$		$[x_5 \div \frac{7}{3}]$
x_2	0	0	1	0	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{24}{7} + \frac{3\epsilon}{7}$		$[x_2 - \frac{2}{3}x_3]$
x_1	0	1	0	0	$\frac{2}{7}$	$-\frac{1}{7}$	$\frac{2}{7} - \frac{2\epsilon}{7}$		$[x_1 - \frac{1}{3}x_3]$
z	1	0	0	0	$\frac{19}{7}$	$\frac{1}{7}$	$\frac{82}{7} + \frac{19\epsilon}{7}$		$[z + \frac{1}{3}x_3]$

and we have an optimal solution. Letting $\epsilon \downarrow 0$ we get $x_1^* = \frac{2}{7}$, $x_2^* = \frac{24}{7}$ and $z^* = \frac{82}{7}$.

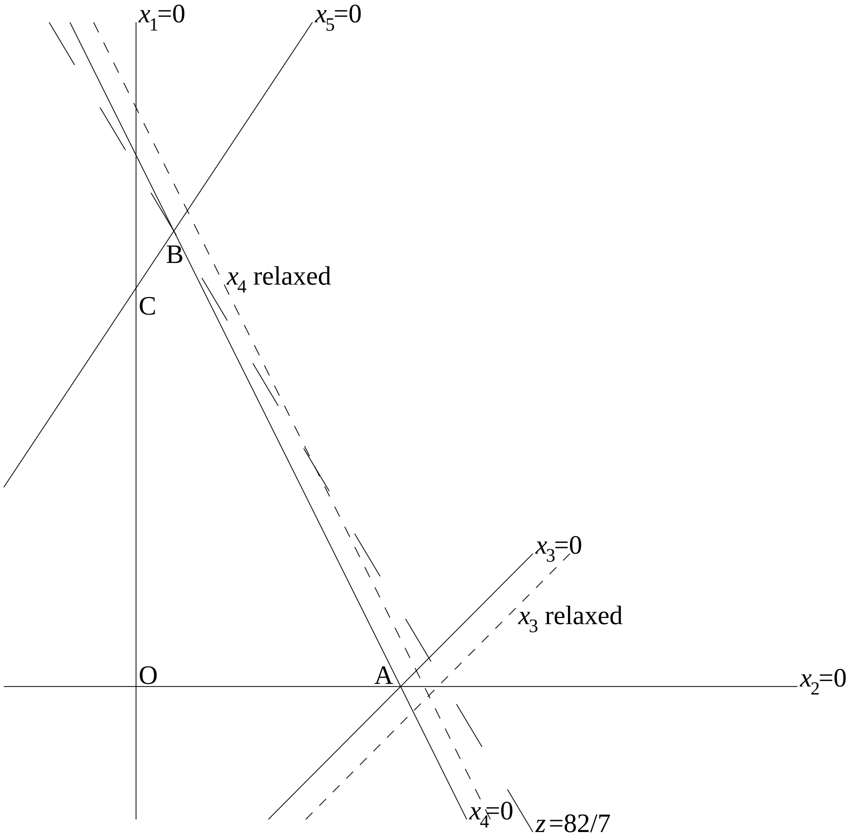
Alternative solution

If we take the other row in the first tableau with minimal θ-ratio, so that x_4 is the leaving variable, we get

	z	x_1	x_2	x_3	x_4	x_5	RHS	θ-ratio	
x_3	0	1	-1	1	0	0	2	$2/1 = 2$	
x_4	0	$2 <$	1	0	1	0	4	$4/2 = 2 \leftarrow$	
x_5	0	-3	2	0	0	1	6	< 0	
z	1	$-5 \uparrow$	-3	0	0	0	0		
x_1	0	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	2	4	$[\frac{1}{2}x_4]$
x_3	0	0	$-\frac{3}{2}$	1	$-\frac{1}{2}$	0	$0 \to \epsilon$	< 0	$[x_3 - x_1]$
x_5	0	0	$\frac{7}{2} <$	0	$\frac{3}{2}$	1	12	$\frac{24}{7} \leftarrow$	$[x_5 + 3x_1]$
z	1	0	$-\frac{1}{2} \uparrow$	0	$\frac{5}{2}$	0	10		$[z + 5x_1]$
x_2	0	0	1	0	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{24}{7}$		$[\frac{2}{7}x_5]$
x_1	0	1	0	0	$\frac{2}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$		$[x_1 - \frac{1}{2}x_2]$
x_3	0	0	0	1	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{16}{7} + \epsilon$		$[x_3 + \frac{3}{2}x_2]$
z	1	0	0	0	$\frac{19}{7}$	$\frac{1}{7}$	$\frac{82}{7}$		$[z + \frac{1}{2}x_2]$

and we are at the same solution as before.

Graphically, we can express the situation as follows:



Here I have marked each constraint line by the coordinate which is zero on that row, so that the x_1-axis becomes the line '$x_2 = 0$'. The feasible region is the quadrilateral OABC, and the lines on which z is constant are parallel to the dashed line $z = \frac{82}{7}$.

In the tableaux above, we start at the feasible point O ($x_1 = x_2 = 0$, $x_3 = 2$, $x_4 = 4$, $x_5 = 6$) and move towards the point A (being the direction in which z increases fastest). We get stopped simultaneously by the two constraints $x_3 \geq 0$ and $x_4 \geq 0$, the two lines in the first tableau in which the θ-ratio is minimal. 'Degeneracy' corresponds to the point A being the intersection of 'too many' constraint hyperplanes; since this is a two-dimensional problem, 'hyperplanes' are just lines, and 'too many' means three or more (the lines $x_2 = 0$, $x_3 = 0$ and $x_4 = 0$). At the point A we have $x_2 = x_3 = x_4 = 0$, $x_1 = 2$ and $x_5 = 12$.

What happens next depends on what we think has happened. In the 'first solution' we thought that x_3 was the leaving variable, that is, we thought we were being stopped by the constraint $x_3 \geq 0$. In the next tableau we therefore had an x_4 row with RHS equal to 0; if you like, it was a surprise to find that $x_4 = 0$. In order to cope with this, we *relaxed* the x_4 constraint by ϵ. Graphically, this amounts to moving the $x_4 = 0$ line to a parallel line just outside the feasible region. (We mustn't move it the other way at this stage, since we must make sure that our point A remains feasible.) This shift gave us the option of moving north-east to the new constraint $x_4 = 0$ to the point $x_1 = 2 + \frac{\epsilon}{3}$, $x_2 = \frac{\epsilon}{3}$, $x_3 = x_4 = 0$ and $x_5 = 12 + \frac{\epsilon}{3}$; from which it becomes obvious that x_3 must become the leaving variable and we move north-west to the point $x_1 = \frac{2}{7} - \frac{2\epsilon}{7}$, $x_2 = \frac{24}{7} + \frac{3\epsilon}{7}$, which approximates the true optimal point $\mathbf{x}^* = (\frac{2}{7}, \frac{24}{7})$.

In the 'second solution' we interpreted the point A as the point $x_2 = x_4 = 0$. In this case, the surprise was to find that x_3 was zero. But if we relax the x_3 constraint, the feasible region doesn't change – if you replace the solid line '$x_3 = 0$' with the dashed line 'x_3 relaxed' the feasible region is still OABC. In the algorithm, the θ-ratio in the x_3 row of the second tableau in the second solution came out negative when we allowed an ϵ relaxation in x_3, showing that we could ignore x_3 at this point. So of course we headed straight for the optimal point B.

3. Adding slack variables we have: maximise $8x_1 + 9x_2 + 5x_3$ subject to

$$\begin{aligned} x_1 + x_2 + 2x_3 + s_1 &= 2 \\ 2x_1 + 3x_2 + 4x_3 + s_2 &= 3 \\ 6x_1 + 6x_2 + 2x_3 + s_3 &= 8 \end{aligned}$$

$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$.

The tableaux are:

	z	x_1	x_2	x_3	s_1	s_2	s_3	RHS	ratio	
z	1	-8	$-9\downarrow$	-5	0	0	0	0		
s_1	0	1	1	2	1	0	0	2	$\frac{2}{1} = 2$	
s_2	0	2	$3 <$	4	0	1	0	3	$\frac{3}{3} = 1 \leftarrow$	
s_3	0	6	6	2	0	0	1	8	$\frac{8}{6} \simeq 1{\cdot}3$	
z	1	$-2\downarrow$	0	7	0	3	0	9		$[\rho_0 + 3\rho_2]$
s_1	0	$\frac{1}{3}$	0	$\frac{2}{3}$	1	$-\frac{1}{3}$	0	1	$1/\frac{1}{3} = 3$	$[\rho_1 - \frac{1}{3}\rho_2]$
x_2	0	$\frac{2}{3}$	1	$\frac{4}{3}$	0	$\frac{1}{3}$	0	1	$1/\frac{2}{3} = 1{\cdot}5$	$[\frac{1}{3}\rho_2]$
s_3	0	$2 <$	0	-6	0	-2	1	2	$\frac{2}{2} = 1 \leftarrow$	$[\rho_3 - 2\rho_2]$
z	1	0	0	1	0	1	1	11		$[\rho_0 + \rho_3]$
s_1	0	0	0	$\frac{5}{3}$	1	0	$-\frac{1}{6}$	$\frac{2}{3}$		$[\rho_1 - \frac{1}{6}\rho_3]$
x_2	0	0	1	$\frac{10}{3}$	0	1	$-\frac{1}{3}$	$\frac{1}{3}$		$[\rho_2 - \frac{1}{3}\rho_3]$
x_1	0	1	0	-3	0	-1	$\frac{1}{2}$	1		$[\frac{1}{2}\rho_3]$

Optimum is $x_3^* = s_2^* = s_3^* = 0$, $x_1^* = 1$, $x_2^* = \frac{1}{3}$, $s_1^* = \frac{2}{3}$, $z^* = 11$.

4. Adding slack variables for the inequalities we get constraints

$$\begin{aligned} x_1 + 2x_2 - x_3 + s_1 &= 9 \\ 2x_1 - x_3 - s_2 &= 4 \\ 2x_1 - x_2 + x_3 &= 4 \end{aligned}$$

and x_1, x_2, x_3, s_1, $s_2 \geq 0$. Because $x_1 = x_2 = x_3 = 0$ is not a feasible point we need to add two artificial variables to get

$$\begin{aligned} x_1 + 2x_2 - x_3 + s_1 &= 9 \\ 2x_1 - x_3 - s_2 + a_1 &= 4 \\ 2x_1 - x_2 + x_3 + a_2 &= 4 \end{aligned}$$

and the question is: can we solve these equations with x_1, x_2, x_3, s_1, $s_2 \geq 0$ and $a_1 = a_2 = 0$? We know that $x_1 = x_2 = x_3 = s_2 = 0$, $s_1 = 9$, $a_1 = 4$, $a_2 = 4$ is a feasible point. If there is a solution with $a_1 = a_2 = 0$, this is a solution of

$$\begin{aligned} x_1 + 2x_2 - x_3 + s_1 &= 9 \\ 2x_1 - x_3 - s_2 + a_1 &= 4 \\ 2x_1 - x_2 + x_3 + a_2 &= 4 \end{aligned}$$

x_1, x_2, x_3, s_1, s_2, a_1, $a_2 \geq 0$ maximising $w = -a_1 - a_2$. So we use the Simplex Method on this problem. (The first step is to put it into canonical form, expressing w as a function of the non-basic variables x_1, x_2, x_3, s_2.)

The tableaux are

	w	x_1	x_2	x_3	s_1	s_2	a_1	a_2	RHS	ratio	
w	1	0	0	0	0	0	1	1	0		
s_1	0	1	2	-1	1	0	0	0	1		
a_1	0	2	0	-1	0	-1	1	0	4		
a_2	0	2	-1	1	0	0	0	1	4		

Put into canonical form:

	w	x_1	x_2	x_3	s_1	s_2	a_1	a_2	RHS	ratio	
w	1	$-4\downarrow$	1	0	0	1	0	0	-8		$[\rho_0-\rho_2-\rho_3]$
s_1	0	$1<$	2	-1	1	0	0	0	1	$\frac{1}{1}=1\leftarrow$	
a_1	0	2	0	-1	0	-1	1	0	4	$\frac{4}{2}=2$	
a_2	0	2	-1	1	0	0	0	1	4	$\frac{4}{2}=2$	
w	1	0	9	$-4\downarrow$	4	1	0	0	-4		$[\rho_0+4\rho_1]$
x_1	0	1	2	-1	1	0	0	0	1	<0	$[\rho_1]$
a_1	0	0	-4	1	-2	-1	1	0	2	$\frac{2}{1}=2$	$[\rho_2-2\rho_1]$
a_2	0	0	-5	$3<$	-2	0	0	1	2	$\frac{2}{3}\bumpeq 1\leftarrow$	$[\rho_3-2\rho_1]$
w	1	0	$\frac{7}{3}$	0	$\frac{4}{3}$	1	0	$\frac{4}{3}$	$-\frac{4}{3}$		$[\rho_0+\frac{4}{3}\rho_3]$
x_1	0	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{5}{3}$		$[\rho_1+\frac{1}{3}\rho_3]$
a_1	0	0	$-\frac{7}{3}$	0	$-\frac{4}{3}$	-1	1	$-\frac{1}{3}$	$\frac{4}{3}$		$[\rho_2-\frac{1}{3}\rho_3]$
x_3	0	0	$-\frac{5}{3}$	1	$-\frac{2}{3}$	0	0	$\frac{1}{3}$	$\frac{2}{3}$		$[\frac{1}{3}\rho_3]$

Thus the optimum value of w is $-\frac{4}{3}<0$ and there is no way of getting a_2+a_3 down to 0 while all the variables are positive; that is, there is no way of getting $a_2=a_3=0$ while the other variables are positive, and the original problem is infeasible.

5. With slack variables we get
maximise $2x_1+x_2+x_3$ subject to

$$\begin{aligned} 2x_1+3x_2-x_3+s_1&=9\\ 2x_2+4x_3-s_2&=4\\ x_1+x_2&=6 \end{aligned}$$

and x_1, x_2, x_3, $s_1\geq 0$. We need artificial variables for the second and third constraints

$$\begin{aligned} 2x_1+3x_2-x_3+s_1&=9\\ 2x_2+4x_3-s_2+a_1&=4\\ x_1+x_2+a_2&=6 \end{aligned}$$

and x_1, x_2, x_3, s_1, a_1, $a_2\geq 0$; need a solution with $a_1+a_2=0$ so seek to maximise $w=-a_1-a_2$ under these constraints. Tableaux are

	w	x_1	x_2	x_3	s_1	s_2	a_1	a_2	RHS	ratio	
w	1	0	0	0	0	0	1	1	0		
s_1	0	2	3	-1	1	0	0	0	9		
a_1	0	0	2	1	0	-1	1	0	4		
a_2	0	1	0	1	0	0	0	1	6		
Put into canonical form											
w	1	-1	$-2\downarrow$	-2	0	1	0	0	-10		$[\rho_0-\rho_2-\rho_3]$
s_1	0	2	3	-1	1	0	0	0	9	$\frac{9}{3}=3$	
a_1	0	0	$2<$	1	0	-1	1	0	4	$\frac{4}{2}=2\leftarrow$	
a_2	0	1	0	1	0	0	0	1	6	–	
w	1	$-1\downarrow$	0	-1	0	0	1	0	-6		$[\rho_0+\rho_2]$
s_1	0	$2<$	0	$-\frac{5}{2}$	1	$\frac{3}{2}$	$-\frac{3}{2}$	0	3	$\frac{3}{2}=1{\cdot}5<$	$[\rho_1-\frac{3}{2}\rho_2]$
s_2	0	0	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	2	–	$[\frac{1}{2}\rho_2]$
a_2	0	1	0	1	0	0	0	1	6	$\frac{6}{1}=6$	$[\rho_3]$
w	1	0	0	$-\frac{9}{4}\downarrow$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	0	$-\frac{9}{2}$		$[\rho_0+\frac{1}{2}\rho_2]$
x_1	0	1	0	$-\frac{5}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$-\frac{3}{4}$	0	$\frac{3}{2}$	<0	$[\frac{1}{2}\rho_1]$
x_2	0	0	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	2	$2/\frac{1}{2}=4$	$[\rho_2]$
a_2	0	0	0	$\frac{9}{4}<$	$-\frac{1}{2}$	$-\frac{3}{4}$	$\frac{3}{4}$	1	$\frac{9}{2}$	$\frac{9}{2}/\frac{9}{4}=2\leftarrow$	$[\rho_3-\frac{1}{2}\rho_1]$
w	1	0	0	0	0	0	1	1	0		$[\rho_0+\rho_3]$
x_1	0	1	0	0	$\frac{2}{9}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{5}{9}$	4		$[\rho_1+\frac{5}{9}\rho_3]$
x_2	0	0	1	0	$\frac{1}{9}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{9}$	1		$[\rho_2-\frac{2}{9}\rho_3]$
x_3	0	0	0	1	$-\frac{2}{9}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{4}{9}$	2		$[\frac{4}{9}\rho_3]$

This tableau is optimal for the solution of Phase 1, and we have a solution with $w=-a_1-a_2=0$, that is, with $a_1=a_2=0$.

Now we start again, knowing that we have a feasible solution with $a_1=a_2=0$, and can therefore delete the last two columns from the last tableau:

	w	x_1	x_2	x_3	s_1	s_2	RHS	ratio	
z	1	-2	-1	-1	0	0			
x_1	0	1	0	0	$\frac{2}{9}$	$\frac{1}{3}$	4		
x_2	0	0	1	0	$\frac{1}{9}$	$-\frac{1}{3}$	1		
x_3	0	0	0	1	$-\frac{2}{9}$	$-\frac{1}{3}$	2		
Put into canonical form									
z	1	0	0	0	$\frac{1}{3}$	0	11		$[\rho_0+2\rho_1+\rho_2+\rho_3]$
x_1	0	1	0	0	$\frac{2}{9}$	$\frac{1}{3}$	4		
x_2	0	0	1	0	$\frac{1}{9}$	$-\frac{1}{3}$	1		
x_3	0	0	0	1	$-\frac{2}{9}$	$-\frac{1}{3}$	2		

This tableau is optimal. It is also degenerate because the reduced cost corresponding to the non-basic variable s_2 is zero, so there are other optima.

6. Adding slack variables for the inequalities we have the tableaux

	z	x_1	x_2	x_3	s_1	s_2	s_3	RHS	ratio	
z	1	-2	1	$-8\downarrow$	0	0	0	0		
s_1	0	0	0	$2<$	1	0	0	1	$\frac{1}{2}=0{\cdot}5\leftarrow$	
s_2	0	2	-4	6	0	1	0	3	$\frac{3}{6}=0{\cdot}5$	
s_3	0	-1	3	4	0	0	1	2	$\frac{2}{4}=0{\cdot}5$	
z	1	-2	1	0	4	0	0	4		$[\rho_0+4\rho_1]$
x_3	0	0	0	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$		$[\frac{1}{2}\rho_1]$
s_2	0	2	-4	0	-3	1	0	0		$[\rho_2-3\rho_1]$
s_3	0	-1	3	0	-2	0	1	0		$[\rho_3-2\rho_1]$

This tableau is degenerate as s_2 and s_3 are basic variables both set at zero (so the vertex it refers to, which has $x_1=x_2=s_1=0$, also has $s_2=s_3=0$).

We can however continue the algorithm as follows:

	z	x_1	x_2	x_3	s_1	s_2	s_3	RHS	ratio	
z	1	$-2\downarrow$	1	0	4	0	0	4		$[\rho_0+4\rho_1]$
x_3	0	0	0	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$		$[\frac{1}{2}\rho_1]$
s_2	0	$2<$	-4	0	-3	1	0	0	$0\leftarrow$	$[\rho_2-3\rho_1]$
s_3	0	-1	3	0	-2	0	1	0	<0	$[\rho_3-2\rho_1]$
z	1	0	$-3\downarrow$	0	1	1	0	4		$[\rho_0+2\rho_2]$
x_3	0	0	0	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$		
x_1	0	1	-2	0	$-\frac{7}{2}$	$\frac{1}{2}$	0	0	<0	$[\frac{1}{2}\rho_2]$
s_3	0	0	$1<$	0	$-\frac{7}{2}$	$\frac{1}{2}$	1	0	$0\leftarrow$	$[\rho_3+\frac{1}{2}\rho_2]$
z	1	0	0	0	$-\frac{19}{2}\downarrow$	$\frac{5}{2}$	3	4		$[\rho_0+3\rho_3]$
x_3	0	0	0	1	$\frac{1}{2}<$	0	0	$\frac{1}{2}$	$1\leftarrow$	
x_1	0	1	0	0	$-\frac{17}{2}$	$\frac{3}{2}$	2	0	<0	$[\rho_2+2\rho_3]$
x_2	0	0	1	0	$-\frac{7}{2}$	$\frac{1}{2}$	1	0	<0	
z	1	0	0	19	0	$\frac{5}{2}$	3	$\frac{27}{2}$		$[\rho_2+19\rho_1]$
s_1	0	0	0	2	1	0	0	1		$[2\rho_1]$
x_1	0	1	0	17	0	$\frac{3}{2}$	2	$\frac{17}{2}$		$[\rho_2+17\rho_1]$
x_2	0	0	1	7	0	$\frac{1}{2}$	1	$\frac{7}{2}$		$[\rho_3+7\rho_1]$

to find the unique optimum solution $x_1^*=\dfrac{17}{2}$, $x_2^*=\dfrac{7}{2}$, $x_3^*=0$, $z^*=\dfrac{27}{2}$.

7. Adding slack variables the tableaux are

	z	x_1	x_2	s_1	s_2	RHS	ratio	
z	1	$-2\downarrow$	-1	0	0	0		
s_1	0	$1<$	-1	1	0	10	$10 \leftarrow$	
s_2	0	2	0	0	1	40	$\frac{40}{2}=20$	
z	1	0	$-3\downarrow$	2	0	20		$[\rho_0+2\rho_1]$
x_1	0	1	-1	1	0	10	<0	
s_2	0	0	$2<$	-2	1	20	$\frac{20}{2}=10\leftarrow$	$[\rho_2-2\rho_1]$
z	1	0	0	$-1\downarrow$	$\frac{3}{2}$	50		$[\rho_0+\frac{3}{2}\rho_2]$
x_1	0	1	0	0	$\frac{1}{2}$	20	$-$	$[\rho_1+\frac{1}{2}\rho_2]$
x_2	0	0	1	-1	$\frac{1}{2}$	10	<0	$[\frac{1}{2}\rho_2]$

At this point we observe that the s_1 column has a negative coefficient in the z-line and only non-positive coefficients below it. What this means is that from the vertex

$$s_1 = s_2 = 0,\ x_1 = 20,\ x_2 = 10$$

corresponding to the final tableau, the equations

$$x_1 + \frac{1}{2}s_2 = 20, \quad x_2 - s_1 + \frac{1}{2}s_2 = 10$$

put no constraint on raising s_1 to any value we please, since that can be compensated by raising x_2 simultaneously; and that as the equation $z - s_1 + \frac{3}{2}s_2 = 50$ has no x_2 term (because x_2 is a basic variable), raising s_1 will simultaneously raise z, without limit.

8. With slack and artificial variables we have constraints

$$\begin{aligned} x_1 - x_2 + s_1 - a_1 &= -1 \\ -x_1 - x_2 + s_2 - a_2 &= -3 \\ 2x_1 + x_2 + s_3 &\leq 4 \end{aligned}$$

and for Phase I we seek to maximise $w = -a_1 - a_2$ subject to $x_1,\ x_2,\ s_1,\ s_2,\ s_3,\ a_1,\ a_2 \geq 0$. The tableaux are

	w	x_1	x_2	s_1	s_2	s_3	a_1	a_2	RHS	ratio
w	1	0	0	0	0	0	1	1	0	
a_1	0	-1	1	-1	0	0	1	0	1	
a_2	0	1	1	0	-1	0	0	1	3	
s_3	0	2	1	0	0	1	0	0	4	

Put into canonical form:

	w	x_1	x_2	s_1	s_2	s_3	a_1	a_2	RHS	ratio	
w	1	0	$-2 \downarrow$	1	1	0	0	0	-4		$\rho_0 - \rho_1 - \rho_2$
a_1	0	-1	$1 <$	-1	0	0	1	0	1	$1 \leftarrow$	
a_2	0	1	1	0	-1	0	0	1	3	3	
s_3	0	2	1	0	0	1	0	0	4	4	
w	1	$-2 \downarrow$	0	-1	1	0	2	0	-2		$\rho_0 + 2\rho_1$
x_2	0	-1	1	-1	0	0	1	0	1	< 0	ρ_1
a_2	0	$2 <$	0	1	-1	0	-1	1	2	$1 \leftarrow$	$\rho_2 - \rho_1$
s_3	0	3	0	1	0	1	-1	0	3	1	$\rho_3 - \rho_1$
w	1	0	0	0	0	0	1	1	$0 <$		$\rho_0 + \rho_2$
x_2	0	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	2		$\rho_1 + \frac{1}{2}\rho_2$
x_1	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	1		$\frac{1}{2}\rho_2$
s_3	0	0	0	$-\frac{1}{2}$	$\frac{3}{2}$	1	$\frac{1}{2}$	$-\frac{3}{2}$	0		$\rho_3 - \frac{3}{2}\rho_2$

Thus we have a feasible point with $a_1 = a_2 = 0$.

Drop those columns and start again with $z = 3x_1 + x_2$.

	z	x_1	x_2	s_1	s_2	s_3	RHS	ratio	
z	1	-3	-1	0	0	0	0		
x_2	0	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	2		
x_1	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	1		
s_3	0	0	0	$-\frac{1}{2}$	$\frac{3}{2}$	1	0		

Put into canonical form:

	z	x_1	x_2	s_1	s_2	s_3	RHS	ratio	
z	1	0	0	1	$-2\downarrow$	0	5		$\rho_0 + 3\rho_2 + \rho_2$
x_2	0	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	2	< 0	
x_1	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	1	< 0	
s_3	0	0	0	$-\frac{1}{2}$	$\frac{3}{2} <$	1	0	$0 \leftarrow$ (degenerate)	
z	1	0	0	$\frac{1}{3}$	0	$\frac{4}{3}$	5		$\rho_0 + \frac{4}{3}\rho_3$
x_2	0	0	1	$-\frac{2}{3}$	0	$\frac{1}{3}$	2		$\rho_1 + \frac{1}{3}\rho_3$
x_1	0	1	0	$\frac{1}{3}$	0	$\frac{1}{3}$	1		$\rho_2 + \frac{1}{3}\rho_3$
s_2	0	0	0	$-\frac{1}{3}$	1	$\frac{2}{3}$	0		$\frac{2}{3}\rho_3$

This is Phase II optimal; $x_1^* = 1$, $x_2^* = 2$, $z^* = 5$.

9. The canonical Phase I tableau is

	w	x_1	x_2	s_1	s_2	s_3	a_1	a_2	RHS	ratio	
w	1	0	$-2\downarrow$	1	1	0	0	0	-4		
a_1	0	-1	$1 <$	-1	0	0	1	0	1	$1 \leftarrow$	
a_2	0	1	1	0	-1	0	0	1	3	3	
s_3	0	2	1	0	0	1	0	0	2	2	
w	1	$-2\downarrow$	0	-1	1	0	2	0	-2		$\rho_0 + 2\rho_1$
x_2	0	-1	1	-1	0	0	1	0	1	< 0	ρ_1
a_2	0	2	0	1	-1	0	-1	1	2	1	$\rho_2 - \rho_1$
s_3	0	$3 <$	0	1	0	1	-1	0	1	$\frac{1}{3} \leftarrow$	$\rho_3 - \rho_1$
w	1	0	0	$-\frac{1}{3}\downarrow$	1	$\frac{2}{3}$	$\frac{4}{3}$	0	$-\frac{4}{3}$		$\rho_0 + \frac{2}{3}\rho_3$
x_2	0	0	1	$-\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{4}{3}$	< 0	$\rho_1 + \frac{1}{3}\rho_3$
a_2	0	0	0	$\frac{1}{3}$	-1	$-\frac{2}{3}$	$-\frac{1}{3}$	1	$\frac{4}{3}$	4	$\rho_2 - \frac{2}{3}\rho_3$
x_1	0	1	0	$\frac{1}{3} <$	0	$\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$1 \leftarrow$	$\frac{1}{3}\rho_3$
w	1	1	0	0	1	1	1	0	-1		$\rho_0 + \rho_3$

This is Phase I optimal with $w^* < 0$, so the original LP is infeasible.

10. The canonical Phase I tableau is

	w	x_1	x_2	s_1	s_2	s_3	a_1	a_2	RHS	ratio	
w	1	0	$-2\downarrow$	1	1	0	0	0	-4		
a_1	0	-1	$1<$	-1	0	0	1	0	1	$1\leftarrow$	
a_2	0	1	1	0	-1	0	0	1	3	3	
s_3	0	2	-1	0	0	1	0	0	2	<0	
w	1	$-2\downarrow$	0	-1	1	0	2	0	-2		$\rho_0+2\rho_2$
x_2	0	-1	1	-1	0	0	1	0	1	<0	ρ_1
a_2	0	$2<$	0	1	-1	0	-1	1	2	$1\leftarrow$	$\rho_2-\rho_1$
s_3	0	1	0	-1	0	1	1	0	3	3	$\rho_3+\rho_1$
w	1	0	0	0	0	0	1	1	0		$\rho_0+\rho_2$
x_2	0	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	2		$\rho_1+\frac{1}{2}\rho_2$
x_1	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	1		$\frac{1}{2}\rho_2$
s_3	0	0	0	$-\frac{3}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	$-\frac{1}{2}$	2		$\rho_3-\frac{1}{2}\rho_2$

This is Phase I optimal; feasible point $x_1 = 1$, $x_2 = 2$, $s_3 = 2$.

Drop the last two columns and maximize $z = 3x_1 + x_2$.

	z	x_1	x_2	s_1	s_2	s_3	RHS	ratio	
z	1	-3	-1	0	0	0	0		
x_2	0	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	2		
x_1	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	1		
s_3	0	0	0	$-\frac{3}{2}$	$\frac{1}{2}$	1	2		

Put into canonical form:

z	1	0	0	1	$-2\downarrow$	0	5		$\rho_0+3\rho_2+\rho_1$
x_2	0	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	2	<0	
x_1	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	1	<0	
s_3	0	0	0	$-\frac{3}{2}$	$\frac{1}{2}<$	1	2	$4\leftarrow$	
z	1	0	0	$-5\downarrow$	0	4	13		$\rho_0+4\rho_3$
x_2	0	0	1	-2	0	1	4	<0	$\rho_1+\rho_3$
x_1	0	1	0	-1	0	1	3	<0	$\rho_2+\rho_3$
s_2	0	0	0	-3	1	2	4	<0	$2\rho_3$

As all the coefficients in the s_1 column are negative, we can increase s_1 as much as we like, while keeping $s_3 = 0$, $x_2 = 2s_1 + 4$ and $x_1 = s_1 + 3$; and $z = 5s_1 + 13$ is unbounded.

11. The tableaux from last week, stripped down, were:

	z	x_1	x_2	x_3	s_1	s_2	s_3	RHS	ratio	
z	1	-8	$-9\downarrow$	-5	0	0	0	0		
s_1	0	1	1	2	1	0	0	2	2	
s_2	0	2	$3<$	4	0	1	0	3	$1\leftarrow$	
s_3	0	6	6	2	0	0	1	8	$\simeq 1{\cdot}3$	
z		$-2\downarrow$		7		3				$[\sigma_0+3\sigma_2]$
s_1		$\frac{1}{3}$						1	3	$[\sigma_1-\frac{1}{3}\sigma_2]$
x_2		$\frac{2}{3}$						1	$1{\cdot}5$	$[\frac{1}{3}\sigma_2]$
s_3		$2<$						2	$1\leftarrow$	$[\sigma_3-2\sigma_2]$
z				1		1	1	11		$[\rho_0+\rho_3=\sigma_0+\sigma_2+\sigma_3]$
s_1								$\frac{2}{3}$		$[\rho_1-\frac{1}{6}\rho_3=\sigma_1-\frac{1}{6}\sigma_3]$
x_2								$\frac{1}{3}$		$[\rho_2-\frac{1}{3}\rho_3=\sigma_2-\frac{1}{3}\sigma_3]$
x_1								1		$[\frac{1}{2}\rho_3=\frac{1}{2}\sigma_3-\sigma_2]$

Optimum is $x_1^*=1$, $x_2^*=\dfrac{1}{3}$, $s_1^*=\dfrac{2}{3}$, $z^*=11$.

The transition matrices for the tableaux are

$$\begin{pmatrix}1 & -\frac{1}{3} & 0\\ 0 & \frac{1}{3} & 0\\ 0 & -2 & 1\end{pmatrix} \text{ and its inverse } \begin{pmatrix}1 & 1 & 0\\ 0 & 3 & 0\\ 0 & 6 & 1\end{pmatrix},$$

$$\begin{pmatrix}1 & 0 & -\frac{1}{6}\\ 0 & 1 & -\frac{1}{3}\\ 0 & 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1 & -\frac{1}{3} & 0\\ 0 & \frac{1}{3} & 0\\ 0 & -2 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & -\frac{1}{6}\\ 0 & 1 & -\frac{1}{3}\\ 0 & -1 & \frac{1}{2}\end{pmatrix} \text{ and its inverse } \begin{pmatrix}1 & 1 & 1\\ 0 & 3 & 2\\ 0 & 6 & 6\end{pmatrix}.$$

The z-rows $\sigma_0+3\sigma_2$ and $\sigma_0+\sigma_2+\sigma_3$ correspond to the vectors $\mathbf{y}=(0\ \ 3\ \ 0)$ and $\mathbf{y}=(0\ \ 1\ \ 1)$ respectively.

12. The ordinary tableau method gives us

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS	θ-ratio	
z	1	$-5\downarrow$	-4	-3	0	0	0	0		
x_4	0	$2<$	3	1	1	0	0	5	$\frac{5}{2}\leftarrow$	
x_5	0	4	1	2	0	1	0	11	$\frac{11}{4}$	
x_6	0	3	4	2	0	0	1	8	$\frac{8}{3}$	
z	1	0	$\frac{7}{2}$	$-\frac{1}{2}\downarrow$	$\frac{5}{2}$	0	0	$\frac{25}{2}$		$[z+\frac{5}{2}x_4]$
x_1	0	1	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{5}{2}$	5	$[\frac{1}{2}x_4]$
x_5	0	0	-5	0	-2	1	0	1		$[x_5-4x_1]$
x_6	0	0	$-\frac{1}{2}$	$\frac{1}{2}<$	$-\frac{3}{2}$	0	1	$\frac{1}{2}$	$1\leftarrow$	$[x_6-3x_1]$
z	1	0	3	0	1	0	1	13		$[z+x_6]$
x_1	0	1	$\frac{5}{2}$	0	2	0	-1	2		$[x_1-\frac{1}{2}x_3]$
x_5	0	0	-5	0	-2	1	0	1		$[x_5]$
x_3	0	0	-1	1	-3	0	2	1		$[2x_6]$

Compared with the ‘ordinary’ simplex method as I’ve presented it before, I’ve made a couple of changes here. First, I’ve put the z row at the top of each tableau. Next, in the comments in square brackets on the right, I’ve expressed all the transformations in terms of rows in the preceding tableau, instead of calculating the row for the entering variable first and working from that; and, third, instead of putting the entering variable in the top row of each new tableau, I’ve given it the place formerly taken by the leaving variable. The effect of these changes is to make the comments in square brackets correspond more transparently to some matrices we need to keep track of.

We can however omit a large proportion of the calculations, as follows:

	x_1	x_2	x_3	x_4	x_5	x_6	RHS	θ-ratio	
$\tilde{z}$	$-5\downarrow$	-4	-3	0	0	0	0		
$\tilde{x}_4$	$2<$	3	1	1	0	0	5	$\frac{5}{2}\leftarrow$	
$\tilde{x}_5$	4	1	2	0	1	0	11	$\frac{11}{4}$	
$\tilde{x}_6$	3	4	2	0	0	1	8	$\frac{8}{3}$	
z	0	$\frac{7}{2}$	$-\frac{1}{2}\downarrow$	$\frac{5}{2}$	0	0			$[z+\frac{5}{2}x_4]$
x_1			$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{5}{2}$	5	$[\frac{1}{2}x_4]$
x_5			0	-2	1	0	1		$[x_5-2x_4]$
x_6			$\frac{1}{2}<$	$-\frac{3}{2}$	0	1	$\frac{1}{2}$	$1\leftarrow$	$[x_6-\frac{3}{2}x_4]$
z	0	3	0	1	0	1	13		$[z+x_6=\tilde{z}+\tilde{x}_4+\tilde{x}_6]$
x_1				2	0	-1	2		$[x_1-x_6=2\tilde{x}_4-\tilde{x}_6]$
x_5				-2	1	0	1		$[x_5=-2\tilde{x}_4+\tilde{x}_5]$
x_3				-3	0	2	1		$[2x_6=-3\tilde{x}_4+2\tilde{x}_6]$

What have I done here? First, I just abandoned the 'z' column; there's nothing in it I can't work out from the rest. Next, I left out the x_1 and x_2 columns in the middle tableau, because I don't need them; as soon as I've calculated the z row, I know that the pivot will lie on the x_3 column, so that's all I need in order to find the θ-ratios and the rules for calculating the next tableau. In the bottom tableau, I've left out the whole left-hand half, because having found the z-row I know I'm at the optimum, and shan't need to repeat the process. The other thing I've left out is the RHS values in the z-rows, except for the last, because we don't need to know these until we come to the final result.

But look. How did I calculate the z-row in the bottom tableau? My formula $z+x_6$ referred to the x_6-row in the middle tableau, and I never did calculate the whole of that row. The method was this. The x_4-, x_5- and x_6-columns are complete. So I could use the formula $z'=z+x_6$ to calculate the coefficients 1, 0 and 1 in those columns of the new z-row. At this point, I remember how these z-rows are generated. Each new z-row is got by adding a multiple of some x-row to the old z-row. But every x-row in every tableau is got by adding and subtracting multiples of previous x-rows, so must be a linear combination of the x-rows in the top tableau, which I have labelled $\tilde{x}_4$, $\tilde{x}_5$ and $\tilde{x}_6$. Now this means that every z-row must be got from the original $\tilde{z}$-row by adding some linear combination of the $\tilde{x}$-rows. Which linear combination must it be? Well, since we know the x_4, x_5 and x_6 coefficients in the z-row, these must be the coefficients we need: in the bottom tableau we must have $z'=\tilde{z}+\tilde{x}_4+\tilde{x}_6$, because that's the only way of getting the coefficients 1, 0 and 1 we've found from the original expression $z'=z+x_6$. Having noticed that all the coefficients in this z-row are non-negative, so that I'm at the optimum, I calculate the RHS value for this row, also from the formula $z'=\tilde{z}+\tilde{x}_4+\tilde{x}_6$, because that's my optimum value z^*.

In the same way, having calculated the x_4-, x_5- and x_6-columns of the x_1-, x_5- and x_3-rows from the formulae $x_1'=x_1-x_6$, $x_5'=x_5$ and $x_3'=2x_6$, I can fill in the RHS values for these rows by observing that x_1' must be $2\tilde{x}_4-\tilde{x}_6$, x_5' must be $-2\tilde{x}_4+\tilde{x}_5$, and x_3' must be $-3\tilde{x}_4+2\tilde{x}_6$, because that's the only way of achieving the coefficients I know, given that every x-row is a linear combination of $\tilde{x}$-rows. In this way, I find my optimum $x_1^*=2$, $x_2^*=0$ and $x_3^*=1$.

[If we had not reached an optimum, of course, I should have to calculate not only the RHS value for each x-row, but also coefficients for a new column corresponding to a leaving variable, as pointed out by the z-row; and these calculations would all need to be done from the formulae expressing rows in terms of the $\tilde{x}$-rows of the original tableau. On the other hand, I might not bother to calculate the RHS value for the z-row, because I should expect to improve on it in the next tableau.]

Now look again at the expressions in square brackets at the right-hand edge of the table. The point about the new forms is that they can be interpreted as matrices. The three expressions

$$[\tfrac{1}{2}x_4], \quad [x_5 - 2x_4], \quad [x_6 = \tfrac{3}{2}x_4]$$

correspond to the matrix

$$\begin{array}{cc} & \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{array}{c} x_1 \\ x_5 \\ x_6 \end{array} & \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -2 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{pmatrix} \end{array},$$

while the expressions

$$[x_1 - x_6], \quad [x_5], \quad [2x_6]$$

correspond to the matrix

$$\begin{array}{cc} & \begin{array}{ccc} x_1 & x_5 & x_6 \end{array} \\ \begin{array}{c} x_1 \\ x_5 \\ x_3 \end{array} & \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{array}.$$

These matrices describe the row operations which turn one tableau into the next. For instance, looking at the first and second tableaux in the complete version, we have

$$\begin{array}{cc} & \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{array}{c} x_1 \\ x_5 \\ x_6 \end{array} & \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -2 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{pmatrix} \end{array} \begin{array}{cc} & \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \end{array} \\ \begin{array}{c} x_4 \\ x_5 \\ x_6 \end{array} & \begin{pmatrix} 2 & 3 & 1 & 1 & 0 & 0 & 5 \\ 4 & 1 & 2 & 0 & 1 & 0 & 11 \\ 3 & 4 & 2 & 0 & 0 & 1 & 8 \end{pmatrix} \end{array} = \begin{array}{cc} & \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \end{array} \\ \begin{array}{c} x_1 \\ x_5 \\ x_6 \end{array} & \begin{pmatrix} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{5}{2} \\ 0 & -5 & 0 & -2 & 1 & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{3}{2} & 0 & 1 & \frac{1}{2} \end{pmatrix} \end{array},$$

(I'm including the coefficients I didn't bother to calculate last time through), while in the second and third tableaux we get

$$\begin{array}{cc} & \begin{array}{ccc} x_1 & x_5 & x_6 \end{array} \\ \begin{array}{c} x_1 \\ x_5 \\ x_3 \end{array} & \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{array} \begin{array}{cc} & \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \end{array} \\ \begin{array}{c} x_1 \\ x_5 \\ x_6 \end{array} & \begin{pmatrix} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{5}{2} \\ 0 & -5 & 0 & -2 & 1 & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{3}{2} & 0 & 1 & \frac{1}{2} \end{pmatrix} \end{array} = \begin{array}{cc} & \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \end{array} \\ \begin{array}{c} x_1 \\ x_5 \\ x_3 \end{array} & \begin{pmatrix} 1 & \frac{5}{2} & 0 & 2 & 0 & -2 & 2 \\ 0 & -5 & 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & -3 & 0 & 2 \end{pmatrix} \end{array}.$$

If we multiply the two conversion matrices together (in the right order! the first matrix to act must go on the right) we get

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -2 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ -2 & 1 & 0 \\ -3 & 0 & 2 \end{pmatrix},$$

which is the right-hand side of the final tableau, because it is the matrix which converts to top tableau into the bottom tableau.

Now turn to the 'official' solutions as written out by Dr Owen. Each of the little tableaux there corresponds to some chunk of what I have written out above. Thus his matrices B_0^{-1}, B_1^{-1} and B_2^{-1} are the x_4-, x_5- and x_6-columns of each of my tableaux; these are the bits which have to be worked out fully. His $\mathbf{x}_{B_0}$, $\mathbf{x}_{B_1}$ and $\mathbf{x}_{B_2}$ are the RHS columns of each tableau. His $\mathbf{t}_1$, $\mathbf{t}_3$ vectors are the x_1-column of the first tableau and the x_3-column of the second tableau; these are the extra bits we need in order to work out the θ-ratios and proceed to the next tableau. His $-\mathbf{r}$ vectors are the working parts of the z-rows; in effect, he is not calculating the coefficients of each z-row corresponding to the basic variables of the current tableau, because he knows they're going to come out zero. His matrices

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 4 & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 & 0 \\ 1 & 0 & 0 \\ 4 & 0 & 1 \end{pmatrix}$$

are selections from the columns of the first tableau; they are the fragments of the first tableau which we need in order to do the filling-in part of the calculations in the later tableaux. His matrices E_1 and E_2 are the ones I have given above as those which convert each tableau into the next.

13. (a) If we set out to maximise $z = -2x_1 - 3x_2$ the tableaux are

	x	x_1	x_2	x_3	x_4	RHS	
z	1	2	3	0	0	0	
x_3	0	2	3	1	0	30	
x_4	0	-1	$-2 <$	0	1	-10	$\leftarrow -\frac{2}{1}, -\frac{3}{2}$
z	1	$\frac{1}{2}$	0	0	$\frac{3}{2}$	-15	$[z + \frac{3}{2}x_4]$
x_3	0	$\frac{1}{2}$	0	1	$\frac{3}{2}$	15	$[x_3 + \frac{3}{2}x_4]$
x_2	0	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	5	$[-\frac{1}{2}x_4]$

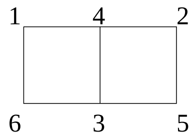
Since our final RHS column and z-row are non-negative, we have an optimal feasible solution $x_1^* = 0$, $x_2^* = 5$, $z^* = -15$ (that is, $2x_1 + 3x_2 = 15$).

(b) Maximising $z = -5x_1 - 6x_2$ we have

	x	x_1	x_2	x_3	x_4	RHS	
z	1	5	6	0	0	0	
x_3	0	-1	-1	1	0	-2	
x_4	0	$-4 <$	-1	0	1	-4	$\leftarrow -\frac{5}{4}, -\frac{6}{1}$
z	1	0	$\frac{19}{4}$	0	$\frac{5}{4}$	-5	$[z + \frac{5}{4}x_4]$
x_3	0	0	$-\frac{3}{4}$	1	$-\frac{1}{4} <$	-1	$[x_3 - \frac{1}{2}x_4] \leftarrow -\frac{19}{3}, -\frac{5}{1}$
x_1	0	1	$\frac{1}{4}$	0	$-\frac{1}{4}$	1	$[-\frac{1}{4}x_4]$
z	1	0	1	5	0	-10	$[z + 5x_3]$
x_4	0	0	3	-4	1	4	$[-4x_3]$
x_1	0	1	1	-1	0	2	$[x_1 - x_3]$

Once again, the RHS and z-rows are non-negative, so we have $x_1^* = 2$, $x_2^* = 0$, $z^* = -10$ (that is, $5x_1 + 6x_2 = 10$).

14. (a)



for a correct diagram, for a diagram as above with no crossing arcs.

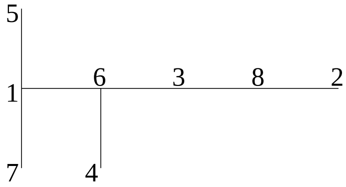
(b)

$$\begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

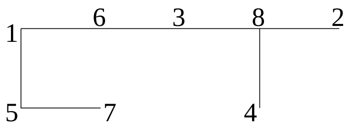
(c) 1 — 4 — 3, 1 — 6 — 3, 1 — 4 — 2 — 5 — 3

(d) 4 — 1 — 6 — 3 — 4 (or 4 — 2 — 5 — 3 — 4, or the reversal of either of these);
4 — 2 — 5 — 3 — 6 — 1 — 4 (or its reversal).

15. (a) accept (1,5)
accept (1,6)
accept (1,7)
reject (5,1) reject (5,7) reject (6,1) accept (6,3)
accept (6,4)
reject (7,1) reject (7,5) reject (3,6) accept (3,8)
reject (4,6) reject (4,8) accept (8,2)
stop (on count of nodes)

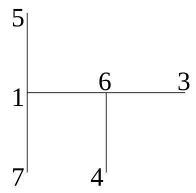


(b) accept (1,5)
reject (5,1) accept (5,7)
reject (7,1) reject (7,5) accept (1,6)
reject (6,1) accept (6,3)
reject (3,6) accept (3,8)
accept (8,2
reject (8,3) accept (8,4)
stop (on count of nodes)

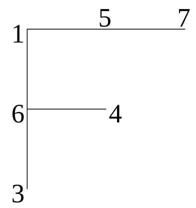


16. The nodes 2 and 8 become disconnected from the rest of the graph.

(a)



(b)



17. (a)

$$\begin{array}{c|c} & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

(b) 1 and 6 are sources, 8 is a sink.

(c) $1-2-5-8$ and $1-4-3-7-8$. (Tell me if you find other solutions!)

(d) $(2,5)$ and $(3,7)$. (Tell me if you find other solutions!)

18. (a) Cut the arcs $(1,2)$, $(1,4)$, $(3,5)$.
(b) The flow is $x_{12}=x_{25}=3$, $x_{14}=x_{45}=4$, $x_{13}=x_{35}=2$.
(c) Both are unique.

19. A possible table is

	c_{ij}	1	2	3	4
(1,2)	5	4	5	5	5
(1,3)	8	0	0	3	7
(2,4)	4	4	4	4	4
(2,5)	2	0	1	1	1
(3,4)	3	0	0	3	3
(3,5)	4	0	0	0	4
(4,5)	4	0	0	0	0
(4,6)	8	4	4	7	7
(5,6)	7	0	1	1	5

Using breadth-first searches at each stage of the algorithm, we have

Iteration 1 Links $1 \xrightarrow{5} 2$, $1 \xrightarrow{8} 3$, $2 \xrightarrow{4} 4$, $2 \xrightarrow{2} 5$, $4 \xrightarrow{8} 6$ provide a flow of 4 along $1-2-4-6$.

Iteration 2 Links $1 \xrightarrow{1} 2$, $1 \xrightarrow{8} 3$, $2 \xrightarrow{2} 5$, $3 \xrightarrow{3} 4$, $5 \xrightarrow{7} 6$ provide a flow of 1 along $1-2-5-6$.

Iteration 3 Links $1 \xrightarrow{8} 3$, $3 \xrightarrow{3} 4$, $3 \xrightarrow{4} 5$, $4 \xrightarrow{4} 6$ provide a flow of 3 along $1-3-4-6$.

Iteration 4 Links $1 \xrightarrow{5} 3$, $3 \xrightarrow{4} 5$, $5 \xrightarrow{1} 2$ (reverse of flow), $5 \xrightarrow{6} 6$ provide a flow of 4 along $1-3-5-6$.

Iteration 5 Link $1 \xrightarrow{1} 3$ only; cutset (1,2), (3,4), (3,5) has value $5+3+4=12$.

(check: flow from 1 is $5+7=12$; flow into 6 is $7+5=12$.)

I believe that the minimal cutset is unique; there are many alternative maximal flows, but I believe that all include the elements $x_{12}=5$, $x_{13}=7$, $x_{34}=3$ and $x_{35}=4$, as above.

20. A possible table is

	c_{ij}	1	2	3	4	5
(1,2)	5	4	5	5	5	5
(1,3)	8	0	0	3	5	7
(2,4)	4	4	4	4	4	4
(2,5)	2	0	1	1	1	1
(3,4)	3	0	0	3	3	3
(3,5)	4	0	0	0	2	4
(4,5)	4	4	4	4	4	2
(4,6)	8	0	0	3	3	5
(5,6)	7	4	5	5	7	7

Iteration 1 Links $1 \xrightarrow{1} 2$, $1 \xrightarrow{8} 3$, $2 \xrightarrow{2} 5$, $3 \xrightarrow{3} 4$, $5 \xrightarrow{3} 6$ provide a flow of 1 along $1-2-5-6$.

Iteration 2 Links $1 \xrightarrow{8} 3$, $3 \xrightarrow{3} 4$, $3 \xrightarrow{4} 5$, $4 \xrightarrow{4} 2$ (reverse flow), $4 \xrightarrow{8} 6$ provide a flow of 3 along $1-3-4-6$.

Iteration 3 Links $1 \xrightarrow{5} 3$, $3 \xrightarrow{4} 5$, $5 \xrightarrow{1} 2$ (reverse flow), $5 \xrightarrow{4} 4$ (reverse flow), $5 \xrightarrow{2} 6$ provide a flow of 2 along $1-3-5-6$.

Iteration 4 Links $1 \xrightarrow{3} 3$, $3 \xrightarrow{2} 5$, $5 \xrightarrow{1} 2$ (reverse flow), $5 \xrightarrow{4} 4$ (reverse flow), $4 \xrightarrow{5} 6$ provide a flow of 2 along $1-3-5-$(reverse flow) $4-6$.

Iteration 5 Link $1 \xrightarrow{1} 3$ only; cutset (1,2), (3,4), (3,5).

(The solution *must* make it clear at which point the flow from 4 to 5 is deleted, and why.)

21.

c_{ij}		1	2	3
(1,2*in*)	2	2	2	2
(1,3*in*)	5	0	3	4
(2*in*,2*out*)	3	2	2	3
(3*in*,3*out*)	4	0	3	4
(2*out*,3*in*)	2	0	0	0
(2*out*,4)	4	2	2	3
(3*out*,2*in*)	2	0	0	1
(3*out*,4)	3	0	3	3

Iteration 1 Links $1 \xrightarrow{2} 2in$, $1 \xrightarrow{5} 3in$, $2 \xrightarrow{3} 2out$, $3in \xrightarrow{4} 3out$, $2out \xrightarrow{4} 4$ give a flow of 2 along $1 - 2in - 2out - 4$.

Iteration 2 Links $1 \xrightarrow{5} 3in$, $3 \xrightarrow{4} 3out$, $3out \xrightarrow{2} 2in$, $3out \xrightarrow{3} 4$ give a flow of 3 along $1 - 3in - 3out - 4$.

Iteration 3 Links $1 \xrightarrow{2} 3in$, $3in \xrightarrow{1} 3out$, $3out \xrightarrow{2} 2in$, $2in \xrightarrow{1} 2out$, $2out \xrightarrow{2} 4$ give a flow of 1 along $1 - 3in - 3out - 2in - 2out - 4$.

Iteration 4 Only $1 \xrightarrow{1} 3in$ available; cutset (1,2*in*), (3*in*,3*out*) (that is, the link (1,2) and the transshipment node 3).

22. **(a)** $\nabla f_1(\mathbf{x}) = \begin{pmatrix} 2x_1 - 3x_2 \\ -3x_1 + 6x_2 \end{pmatrix}$, $\mathbf{H}f_1(\mathbf{x}) = \begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix}$.
Stationary point when $2x_1 - 3x_2 = -3x_1 + 4x_2 = 0$, i.e., $x_1 = x_2 = 0$.
$\det \mathbf{H}f_1(\mathbf{0}) = 3 > 0$, leading coefficient $2 > 0$ so this is a local minimum.

(b) $\nabla f_2(\mathbf{x}) = \begin{pmatrix} 4x_1 - 5x_2 \\ -5x_1 + 4x_2 \end{pmatrix}$, $\mathbf{H}f_2(\mathbf{x}) = \begin{pmatrix} 4 & -5 \\ -5 & 4 \end{pmatrix}$.
Stationary point when $4x_1 - 5x_2 = -5x_1 + 4x_2 = 0$, i.e., $x_1 = x_2 = 0$.
$\det \mathbf{H}f_2(\mathbf{0}) = -9 < 0$ so this is a saddle point.

(c) $f_3(\mathbf{x}) = x_1x_2 - x_1^2x_2 - x_1x_2^2$,

$$\nabla f_3(\mathbf{x}) = \begin{pmatrix} x_2 - 2x_1x_2 - x_2^2 \\ x_1 - x_1^2 - 2x_1x_2 \end{pmatrix} = \begin{pmatrix} x_2(1 - 2x_1 - x_2) \\ x_1(1 - x_1 - 2x_2) \end{pmatrix},$$

$$\mathbf{H}f_3(\mathbf{x}) = \begin{pmatrix} -2x_2 & 1 - 2x_1 - 2x_2 \\ 1 - 2x_1 - 2x_2 & -2x_1 \end{pmatrix}.$$

Stationary points when
either $x_2 = 0$, $x_1 = 0$
or $x_2 = 0$, $1 - x_1 - 2x_2 = 0$, i.e., $x_1 = 1$, $x_2 = 0$
or $1 - 2x_1 - x_2 = 0$, $x_1 = 0$, i.e., $x_1 = 0$, $x_2 = 1$
or $1 - 2x_1 - x_2 = 0$, $1 - x_1 - 2x_2 = 0$, i.e., $x_1 = x_2 = \frac{1}{3}$.

$\mathbf{H}f_3(\mathbf{0}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has determinant $-1 < 0$ so this is a saddle point.

$\mathbf{H}f_3\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & -2 \end{pmatrix}$ has determinant $-1 < 0$ so this is a saddle point.

$\mathbf{H}f_3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -1 & 0 \end{pmatrix}$ has determinant $-1 < 0$ so this is a saddle point.

$\mathbf{H}f_3\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} \end{pmatrix}$ has determinant $\frac{1}{3} > 0$ and leading coefficient $-\frac{2}{3} < 0$ so this is a local maximum.

(d) $\nabla f_4(\mathbf{x}) = \begin{pmatrix} \cos(x_1+x_2) \\ \cos(x_1+x_2) \end{pmatrix}$.

$\mathbf{H}f_4(\mathbf{x}) = \begin{pmatrix} -\sin(x_1+x_2) & -\sin(x_1+x_2) \\ -\sin(x_1+x_2) & -\sin(x_1+x_2) \end{pmatrix}$.

Stationary points when $\cos(x_1+x_2) = 0$, i.e., x_1+x_2 is an odd multiple of $\frac{\pi}{2}$.

At such stationary points, either $f_4(\mathbf{x}) = 1$ (when $x_1+x_2 = 2k\pi + \frac{\pi}{2}$ for some integer k) and $\mathbf{x}$ is a local maximum, or $f_4(\mathbf{x}) = -1$ (when $x_1+x_2 = 2k\pi - \frac{\pi}{2}$ for some integer k) and $\mathbf{x}$ is a local minimum.

(The determinant of $\mathbf{H}f_4$ is always 0, so tells us nothing.)

(e) $\nabla f_5(\mathbf{x}) = \begin{pmatrix} 4x_1+2x_2+2x_3 \\ 2x_1+6x_2+2x_3 \\ 2x_1+2x_2+4x_3 \end{pmatrix} = 2\begin{pmatrix} 2x_1+x_2+x_3 \\ x_1+3x_2+x_3 \\ x_1+x_2+2x_3 \end{pmatrix}$.

$\mathbf{H}f_5(\mathbf{x}) = 2\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

Stationary point when $2x_1+x_2+x_3 = x_1+3x_2+x_3 = x_1+x_2+2x_3 = 0$, i.e., at $\mathbf{0}$.

$\mathbf{H}f_5(\mathbf{0})$ has determinant $2^3 \cdot 7 > 0$; its principal minor $2\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ has determinant $2^2 \cdot 4 > 0$; and its leading coefficient is $2 \cdot 2 > 0$. So $\mathbf{0}$ is a local minimum.

(Alternative method: observe that $f_5(\mathbf{x}) = (x_1+x_2)^2 + (x_2+x_3)^2 + (x_1+x_3)^2 + x_2^2$, so is zero at $\mathbf{0}$ and strictly positive everywhere else.)

(f)

$$\nabla f_6(\mathbf{x}) = \begin{pmatrix} -2x_1x_3e^{-x_1^2-x_2^2} \\ -2x_2x_3e^{-x_1^2-x_2^2} \\ e^{-x_1^2-x_2^2} \end{pmatrix} = e^{-x_1^2-x_2^2}\begin{pmatrix} -2x_1x_3 \\ -2x_2x_3 \\ 1 \end{pmatrix}.$$

$$\mathbf{H}f_6(\mathbf{x}) = \begin{pmatrix} (4x_1^2x_3-2x_3)e^{-x_1^2-x_2^2} & 4x_1x_2x_3e^{-x_1^2-x_2^2} & -2x_1e^{-x_1^2-x_2^2} \\ 4x_1x_2x_3e^{-x_1^2-x_2^2} & (4x_2^2x_3-2x_3)e^{-x_1^2-x_2^2} & -2x_2e^{-x_1^2-x_2^2} \\ -2x_1e^{-x_1^2-x_2^2} & -2x_2e^{-x_1^2-x_2^2} & 0 \end{pmatrix}$$

$$= 2e^{-x_1^2-x_2^2}\begin{pmatrix} 2x_1^2x_3-x_3 & 2x_1x_2x_3 & -x_1 \\ 2x_1x_2x_3 & 2x_2^2x_3-x_3 & -x_2 \\ -x_1 & -x_2 & 0 \end{pmatrix}.$$

No stationary points, because $\frac{\partial f_6}{\partial x_3}$ is never 0.

23. $L(\mathbf{x},\lambda) = x_1^3 - x_2^4 - \lambda(x_1^2 + 2x_2^2 - 4)$, so our equations are

$$\begin{aligned}\frac{\partial L}{\partial x_1} &= 3x_1^2 - 2\lambda x_1 = 0,\\ \frac{\partial L}{\partial x_2} &= -4x_2^3 - 4\lambda x_2 = 0,\\ x_1^2 + 2x_2^2 &= 4.\end{aligned}$$

So we have

$$\text{either } x_1 = 0 \text{ or } \lambda = \frac{3}{2}x_1,$$

$$\text{either } x_2 = 0 \text{ or } \lambda = -x_2^2.$$

case 1 If $x_1 = 0$ then $x_2 = \pm\sqrt{2}$, $f = -4$.
case 2 If $x_2 = 0$ then $x_1 = \pm 2$, $f = \pm 8$.
case 3 If $\frac{3}{2}x_1 = -x_2^2$ then $x_1^2 - 3x_1 - 4 = 0$, that is, $(x_1 - 4)(x_1 + 1) = 4$ and x_1 is either 4 or -1; but as $x_1^2 \leq 4$, the first is impossible, and $x_1 = -1$, $x_2 = \pm\sqrt{\frac{3}{2}}$, $f = -1 - \frac{9}{4} = -\frac{13}{4}$.

The global maximum is at $x_1 = 2$, $x_2 = 0$ and the global minimum at $x_1 = -2$, $x_2 = 0$.

24. (a) f_1 is not coercive. There are many ways of seeing this. (i) We can express it as $(x_1 - 2x_2)(x_1 - x_2)$, so that it is zero if either $x_1 = 2x_2$ or $x_1 = x_2$, and either can be true for large $\|\mathbf{x}\|$. (ii) Its only stationary point is at $\mathbf{0}$, and the Hessian there is $\begin{pmatrix} 1 & -3 \\ -3 & 4 \end{pmatrix}$, which has determinant less than 0. This is therefore a saddle point, and f_1 has no local minimum, let alone a global minimum.

(b) f_2 is coercive, because it is $(x_1 - x_2)^2 + 2x_2^2$, which is large if $|x_2|$ is large.

(c) f_3 is not coercive; if we set $x_1 = \alpha^3$, $x_2 = -\alpha^2$ we can get $f_3(\mathbf{x}) = 0$ while $\|\mathbf{x}\| = \sqrt{|\alpha|^6 + |\alpha^4|}$ is as large as we like.

(d) f_4 is not coercive, because if $x_1 = x_2 = x_3 = \alpha$ then $f_4(\mathbf{x}) = -\alpha^4$.

(e) f_5 is coercive, because all the terms $\cos x_2x_3$, etc., are between -1 and 1, so that if $\alpha \geq 1$ and $\|\mathbf{x}\| \geq (\alpha + 1)\sqrt{3}$ (so that at least one of $|x_1|$, $|x_2|$, $|x_3|$ is greater than or equal to $\alpha + 1$) then $f(\mathbf{x}) \geq \alpha^2$.

25. We have

$$L(\mathbf{x},\lambda) = x_1x_2 + x_3 - \lambda(x_1^2 + x_2^2 + x_3^2 - c),$$

so that the equations become

$$x_2 - 2\lambda x_1 = 0, \quad x_1 - 2\lambda x_2 = 0, \quad 1 - 2\lambda x_3 = 0.$$

The first two have the solutions

$$x_1 = x_2 = 0 \;\; or\; \lambda = \pm\tfrac{1}{2}.$$

case 1 If $x_1 = x_2 = 0$ then $f = x_3 = \pm\sqrt{c}$.

case 2a If $\lambda = \frac{1}{2}$ then $x_3 = 1$; this is only possible if $c \geq 1$, and in this case $x_1 = x_2$ is $\pm\sqrt{(c-1)/2}$, so that $x_1x_2 = \frac{1}{2}(c-1)$ and $f = \frac{1}{2}(c+1)$.

case 2b If $\lambda = -\frac{1}{2}$ then $x_3 = -1$; this is only possible if $c \geq 1$, and in this case $x_1 = -x_2$ is $\pm\sqrt{(c-1)/2}$, so that $x_1x_2 = -\frac{1}{2}(c-1)$ and $f = -\frac{1}{2}(c+1)$.

Thus

if $c \leq 1$, the minimum is at $x_1 = x_2 = 0$, $x_3 = -\sqrt{c} = f^*$;

if $c > 1$, the minima are at $x_1 = \pm\sqrt{(c-1)/2}$, $x_2 = -x_1$, $x_3 = -1$, $f^* = -\frac{1}{2}(c+1)$.

For the derivative, we have

$\dfrac{df^*}{dc} = -\dfrac{1}{2\sqrt{c}}$ if $0 < c < 1$,

$\dfrac{df^*}{dc} = -\dfrac{1}{2}$ if $c > 1$,

$\dfrac{df^*}{dc} = -\dfrac{1}{2}$ if $c = 1$.

26. The equations are

$$1 - 2\lambda_1 x_1 = 0, \quad 1 - 2\lambda_1 x_2 - 2\lambda_2 x_2 = 0, \quad 1 - 2\lambda_2 x_2 = 0.$$

So none of x_1, x_2, x_3 can be zero,

$$\lambda_1 = \frac{1}{2x_1}, \quad \lambda_2 = \frac{1}{2x_2}, \quad 1 - \frac{x_2}{x_1} - \frac{x_2}{x_3};$$

multiplying by x_1x_3, $x_1x_3 - x_2(x_3 + x_1) = 0$, as required.

When $\gamma = \delta$, we know that $x_1^2 = x_3^2$ so $x_1 = \pm x_3$. But as $x_1x_3 \neq 0$ we cannot have $x_1 + x_3 = 0$, so

$$x_1 = x_3, \quad 2x_2x_1 = x_1^2, \quad x_1 = 2x_2,$$

$$5x_2^2 = \gamma, \quad x_2 = \pm\sqrt{\gamma/5}, \quad f^* = f^*(\gamma,\gamma) = 5x_2 = \pm\sqrt{5\gamma}.$$

Since we are looking for the minimum, it is $f^*(\gamma,\gamma) = -\sqrt{5\gamma}$ when $x_1 = x_3 = -2\sqrt{\gamma/5}$, $x_2 = -\sqrt{\gamma/5}$.

Now recall that $\frac{\partial f^*}{\partial\gamma} = \lambda_1^* = \frac{1}{2x_1} = -\frac{1}{4}\sqrt{\frac{5}{\gamma}}$.

[Of course $\frac{d}{d\gamma}(-\sqrt{5\gamma}) = -\frac{1}{2}\sqrt{\frac{5}{\gamma}}$; this is because $\frac{d}{d\gamma}f^*(\gamma,\gamma) = \frac{\partial}{\partial\gamma}f^*(\gamma,\delta) + \frac{\partial}{\partial\delta}f^*(\gamma,\delta)$.]

27. (a) f is coercive and continuous and g is continuous, so $F = \{\mathbf{x} : g(\mathbf{x}) = 0\}$ is closed and there is an $\mathbf{x} \in F$ at which f is minimal.

f does not have a maximal value because (for instance) $\begin{pmatrix} \alpha \\ -\frac{1}{\alpha} \\ 0 \end{pmatrix} \in F$ for every $\alpha \neq 0$, while $f\begin{pmatrix} \alpha \\ -\frac{1}{\alpha} \\ 0 \end{pmatrix} \geq \alpha^2$.

(b)(i) The equations $\dfrac{\partial f}{\partial x_i} - \lambda\dfrac{\partial g}{\partial x_i} = 0$ become

$$2x_1^* - \lambda x_2^* - \lambda x_3^* = 0, \quad 10x_2^* - \lambda x_1^* - \lambda x_3^* = 0, \quad 16x_3^* - \lambda x_1^* - \lambda x_2^* = 0,$$

and convert immediately into the equations given.

(ii) Since we certainly cannot have every x_i^* equal to zero, the equation

$$\begin{pmatrix} 2 & -\lambda & -\lambda \\ -\lambda & 10 & -\lambda \\ -\lambda & -\lambda & 16 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

has a non-trivial solution, and the matrix must have zero determinant. But

$$\det \begin{pmatrix} 2 & -\lambda & -\lambda \\ -\lambda & 10 & -\lambda \\ -\lambda & -\lambda & 16 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -2\lambda^3 - 28\lambda^2 + 320 = -2\lambda(\lambda+4)(\lambda^2 + 10\lambda - 40)$$

is zero iff $\lambda = -4$ or $-5 \pm \sqrt{65}$.

(iii) Because $g(\mathbf{x}^*) = 0$, at least one of the products $x_i^* x_j^*$ is less than 0, so that x_i^* and x_j^* have opposite signs.

Suppose, if possible, that $x_1^{*2} < x_2^{*2}$. Set $x_1 = x_2^*$, $x_2 = x_1^*$, $x_3 = x_3^*$; then $g(\mathbf{x}) = g(\mathbf{x}^*) = 0$, while

$$f(\mathbf{x}^*) - f(\mathbf{x}) = x_1^{*2} + 5x_2^{*2} - x_2^{*2} - 5x_1^{*2} = 4(x_2^2 - x_1^{*2}) > 0$$

and f is not minimal at $\mathbf{x}^*$. So this can't happen, and $x_1^{*2} \geq x_2^{*2}$. Similarly, $x_2^{*2} \geq x_3^{*2}$. Since at least two of the x_j^* are non-zero, x_1^* and x_2^* are non-zero.

Since $|x_1^*| \geq |x_2^*| \geq |x_3^*|$, both $x_1^* + x_2^*$ and $x_1^* + x_3^*$ must have the same sign as x_1^*, or be zero. Since x_1^*, x_2^* and x_3^* cannot all be of the same sign, we must have x_1^* on one side of zero and x_2^*, x_3^* on the other. But as $0 \neq x_1^* = \frac{1}{2}\lambda(x_2^* + x_3^*)$, $\lambda < 0$.

(iv) Eliminating x_1 from the equations

$$10x_2^* - \lambda x_1^* - \lambda x_3^* = 0, \quad 16x_3^* - \lambda x_1^* - \lambda x_2^* = 0,$$

we get $(16+\lambda)x_3^* = (10+\lambda)x_2^*$. Since x_2^* is non-zero, so is x_3^*; since x_2^* and x_3^* have the same sign, so do $16+\lambda$ and $10+\lambda$ (and these are both non-zero). But this means that $\lambda \notin [-16, -10]$.

(c) By (b-iii), $\lambda \neq -5 + \sqrt{65}$; by (b-iv), $\lambda \neq -5 - \sqrt{65}$. So (b-ii) tells us that $\lambda = -4$. Accordingly

$$2x_1^* + 4x_2^* + 4x_3^* = 0, \quad 10x_2^* + 4x_1^* + 4x_3^* = 0, \quad 16x_3^* + 4x_1^* + 4x_2^* = 0,$$

so

$$x_1^* = -6x_3^*, \quad x_2^* = 2x_3^*;$$

since $g(\mathbf{x}^*) = 0$, $x_1^* = \pm\frac{3}{2}$, $x_2^* = \mp\frac{1}{2}$, $x_3^* = \mp\frac{1}{4}$ and $f(\mathbf{x}^*) = 4$.

There are just two possible solutions for $\mathbf{x}^*$.

28. (a) Start: $f(1) \simeq 2{\cdot}6209$, $f(4) \simeq 2{\cdot}0391$.

a_n	b_n	u	$f(u)$	v	$f(v)$
1	4	2·4500*	0·1393*	2·5500*	0·0598*
2·4500	4	3·1750*	0·1767*	3·2750*	0·3017*
2·4500	3·2750	2·8125*	−0·0265*	2·9125*	−0·0091*
2·4500	2·9125	2·6312*	0·0135*	2·7312*	−0·0197*
2·6312	2·9125	2·7219*	−0·0177*	2·8219*	−0·0261*
2·7219	2·9125				

Final interval $[2{\cdot}7219, 2{\cdot}9125]$.
$(f(2{\cdot}8172) \simeq -0{\cdot}0263.)$

(b) $f'(x) = 1 - 3\sin x$.
Starting from $f'(1) \simeq -1{\cdot}5244$, $f'(4) \simeq 3{\cdot}2704$,

a_n	b_n	u	$f'(u)$
1	4	2·5*	−0·7954*
2·5	4	3·25*	1·3246*
2·5	3·25	2·875*	0·2097*
2·5	2·875	2·6875*	−0·3159*
2·6875	2·875		

Final interval $[2{\cdot}6875, 2{\cdot}875]$.
$(f(2{\cdot}7812) \simeq -0{\cdot}0261.)$

(c) $x_{k+1} = \frac{x_{k-1}f'(x_k) - x_k f'(x_{k-1})}{f'(x_k) - f'(x_{k-1})}$.

x_k	$f'(x_k)$
1·0000	−1·5244*
4·0000	3·2704*
1·9538*	−1·7827*
2·6757*	−0·3478*
2·8506*	0·1393*
2·8006*	−0·0034*

so $x = 2{\cdot}8006$ will serve.
(The actual minimum is $x^* = \pi - \arcsin\frac{1}{3} \simeq 2{\cdot}8018$.)

29. $\nabla f(\mathbf{x}) = \begin{pmatrix} 2x_1 - x_3 \\ 2x_2 \\ -x_1 + 2x_3 \end{pmatrix}$; $\nabla f(\mathbf{x}_0) = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$, so the line of first search is $\begin{pmatrix} 1-2t \\ 1-2t \\ t \end{pmatrix}$, with

$$f(\mathbf{x}_0 - t\nabla f(\mathbf{x}_0)) = (1-2t)^2 - (1-2t)t + (1-2t)^2 + t^2 = 2 - 9t + 11t^2.$$

This is minimal when $t = \frac{9}{22}$, so

$$\mathbf{x}_1 = \tfrac{1}{22}\begin{pmatrix} 4 \\ 4 \\ 9 \end{pmatrix} \simeq \begin{pmatrix} 0{\cdot}1818 \\ 0{\cdot}1818 \\ 0{\cdot}4091 \end{pmatrix},$$

$$\nabla f(\mathbf{x}_1) = \tfrac{1}{22}\begin{pmatrix} -1 \\ 8 \\ 14 \end{pmatrix},$$

$$\begin{aligned} f(\mathbf{x}_1 - t\nabla f(\mathbf{x}_1)) &= \tfrac{1}{484}((4+t)^2 - (4+t)(9-14t) + (4-8t)^2 + (9-14t)^2) \\ &= \tfrac{1}{484}(77 - 261t + 275t^2) \end{aligned}$$

which is minimal when $t = \frac{261}{550}$, so that

$$\begin{aligned} \mathbf{x}_2 &= \tfrac{1}{22}\begin{pmatrix} 4 \\ 4 \\ 9 \end{pmatrix} - \tfrac{261}{22\cdot 550}\begin{pmatrix} -1 \\ 8 \\ 14 \end{pmatrix} \\ &= \tfrac{1}{22\cdot 550}\begin{pmatrix} 2461 \\ 112 \\ 1296 \end{pmatrix} \simeq \begin{pmatrix} 0{\cdot}2034 \\ 0{\cdot}0093 \\ 0{\cdot}1071 \end{pmatrix}. \end{aligned}$$

30. $f'(x) = 4x^3 - 18x$, $f''(x) = 12x^2 - 18$ so $x_{n+1} = x_n - \frac{4x_n^3 - 18x_n}{12x_n^2 - 18}$.

(i)

x_n	$f(x_n)$	$f'(x_n)$	$f''(x_n)$
0·94	−7·1716	−13·5977*	−7·3968*
−0·8983*	−6·6115	13·2700*	−8·3163*
0·6973*	−4·1401	−11·1957*	−12·1646*
−0·2230*	−0·4451	3·9699*	−17·4032*
0·0051*	−0·0002	−0·0918*	−17·9997*
0·0000	0·0000.		

Terminate at 0·0000.

(ii)

x_n	$f(x_n)$	$f'(x_n)$	$f''(x_n)$
0·95	−7·3080	−13·6705*	−7·1700*
−0·9566*	−7·3987	13·7175*	−7·0184*
0·9979*	−7·9702	−13·9872*	−6·0511*
−1·3136*	−12·5529	14·5780*	2·7077*
−6·6976*	1608·5380	−1081·2183*	520·2990*
−4·6196*	263·3465	−311·1794*	238·0839*
−3·3125*	21·6489	−85·7675*	113·6753*
−2·5580*	−16·0737	−20·9105*	60·5232*
−2·2126*	−20·0937	−3·4992*	40·7446*
−2·1267	−20·2500.		

Terminates at −2·1267.

(iii)

x_n	$f(x_n)$	$f'(x_n)$	$f''(x_n)$
0·96	−7·4451	−13·7411*	−6·9408*
−1·0198*	−8·2776	14·1138*	−5·5213*
1·5365*	−15·6738	−13·1475*	10·3295*
2·8093*	−8·7435	38·1179*	76·7055*
2·3124*	−19·5326	7·8341*	46·1638*
2·1427*	−20·2417	0·7796*	37·0915*
2·1216	−20·2500.		

Terminates at 2·1216.

31. $\nabla f(\mathbf{x}) = \begin{pmatrix} 4x_1^3 - 2x_1x_2 \\ -x_1^2 + 2x_2 \end{pmatrix}, \quad \mathbf{H}f(\mathbf{x}) = \begin{pmatrix} 12x_1^2 - 2x_2 & -2x_1 \\ -2x_1 & 2 \end{pmatrix},$

$$\mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad f(\mathbf{x}) = 1,$$

$$\nabla f(\mathbf{x}) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{H}f(\mathbf{x}) = \begin{pmatrix} 10 & -2 \\ -2 & 2 \end{pmatrix},$$

$$\mathbf{x}_1 = \begin{pmatrix} 0.6250 \\ 0.1250 \end{pmatrix}, \quad f(\mathbf{x}) \simeq 0.1194,$$

$$\nabla f(\mathbf{x}) \simeq \begin{pmatrix} 0.8203 \\ -0.1406 \end{pmatrix}, \quad \mathbf{H}f(\mathbf{x}) \simeq \begin{pmatrix} 4.4375 & -1.2500 \\ -1.2500 & 2 \end{pmatrix},$$

$$\mathbf{x}_2 \simeq \begin{pmatrix} 0.4247 \\ 0.0701 \end{pmatrix}, \quad f(\mathbf{x}) \simeq 0.0248,$$

$$\nabla f(\mathbf{x}) \simeq \begin{pmatrix} 0.2468 \\ -0.0401 \end{pmatrix}, \quad \mathbf{H}f(\mathbf{x}) \simeq \begin{pmatrix} 2.0240 & -0.8494 \\ -0.8494 & 2 \end{pmatrix},$$

$$\mathbf{x}_3 \simeq \begin{pmatrix} 0.2865 \\ 0.0315 \end{pmatrix}, \quad f(\mathbf{x}) \simeq 0.0051,$$

$$\nabla f(\mathbf{x}) \simeq \begin{pmatrix} 0.0760 \\ -0.0191 \end{pmatrix}, \quad \mathbf{H}f(\mathbf{x}) \simeq \begin{pmatrix} 0.9222 & -0.5731 \\ -0.5731 & 2 \end{pmatrix},$$

$$\mathbf{x}_4 \simeq \begin{pmatrix} 0.1934 \\ 0.0144 \end{pmatrix}, \quad f(\mathbf{x}) \simeq 0.0011,$$

$$\nabla f(\mathbf{x}) \simeq \begin{pmatrix} 0.0234 \\ -0.0087 \end{pmatrix}, \quad \mathbf{H}f(\mathbf{x}) \simeq \begin{pmatrix} 0.4202 & -0.3869 \\ -0.3869 & 2 \end{pmatrix},$$

$$\mathbf{x}_5 \simeq \begin{pmatrix} 0.1306 \\ 0.0065 \end{pmatrix}, \quad f(\mathbf{x}) \simeq 0.0002,$$

$$\nabla f(\mathbf{x}) \simeq \begin{pmatrix} 0.0072 \\ -0.0040 \end{pmatrix}, \quad \mathbf{H}f(\mathbf{x}) \simeq \begin{pmatrix} 0.1915 & -0.2611 \\ -0.2611 & 2 \end{pmatrix},$$

$$\mathbf{x}_6 \simeq \begin{pmatrix} 0.0881 \\ 0.0030 \end{pmatrix}, \quad f(\mathbf{x}) \simeq 0.0000,$$

$$\nabla f(\mathbf{x}) \simeq \begin{pmatrix} 0.0022 \\ -0.0018 \end{pmatrix}, \quad \mathbf{H}f(\mathbf{x}) \simeq \begin{pmatrix} 0.0873 & -0.1763 \\ -0.1763 & 2 \end{pmatrix}.$$

(Note how the method does not bring x_1 to zero quickly, but does get x_2 and f and ∇f close to zero.)

32. $f'(x) = -e^{-x} + 2x$, $f''(x) = e^{-x} + 2$.

(a) Starting from $f'(-1) \simeq -4{\cdot}7183$, $f'(1) \simeq 1{\cdot}6321$,

a_n	b_n	u	$f'(u)$
-1	1	0	-1
0	1	0·5	0·3935
0	0·5	0·25	$-0{\cdot}2788$
0·25	0·5	0·375	0·0627
0·25	0·375		

($f(0{\cdot}3125) \simeq 0{\cdot}8293$.)

(b)

x_k	$f'(x_k)$
-1	$-4{\cdot}7183$
1	1·6321
0·4860	0·3569
0·3421	$-0{\cdot}0260$
0·3519	0·0004

so $x = 0{\cdot}3519$ will serve.

(c)

x_n	$f(x_n)$	$f'(x_n)$	$f''(x_n)$
0	1	-1	2
0·5	0·8565	0·3935	2·6487
0·3514	0·8272	$-0{\cdot}0008$	2·4211
0·3518	0·8272	0·0001	2·4216

so $x = 0{\cdot}3518$ will do.

(The minimum is $x^* \simeq 0{\cdot}3517$ to four decimal places.)

33. First iteration, starting from $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $f = 1$, $\nabla f = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$:

t	x_1	x_2	f	t'	x_1'	x_2'	f'
0	1	1	1	1	−1	0	1
0·4888	0·0224	0·5112	0·2610	0·5112	−0·0224	0·4888	0·2387
0·7332	−0·4665	0·2668	0·0605	0·7556	−0·5112	0·2444	0·0642
0·6110	−0·2220	0·3890	0·1346	0·6334	−0·2668	0·3666	0·1134
0·6721	−0·3443	0·3279	0·0827	0·6945	−0·3890	0·3055	0·0700
0·7027	−0·4054	0·2973	0·0665	0·7250	−0·4501	0·2750	0·0609
0·7180	−0·4359	0·2820	0·0621	0·7403	−0·4806	0·2597	0·0608

Interval of uncertainty (in the linear parameter t) is $[0{\cdot}7180, 0{\cdot}7556]$, midpoint 0·7368.

Second iteration, starting from $\mathbf{x}_1 = \begin{pmatrix} -0{\cdot}4735 \\ 0{\cdot}2632 \end{pmatrix}$, $f = 0{\cdot}0605$, $\nabla f = \begin{pmatrix} -0{\cdot}1755 \\ 0{\cdot}3022 \end{pmatrix}$:

t	x_1	x_2	f	t'	x_1'	x_2'	f'
0	−0·4735	0·2632	0·0605	2	−0·1226	−0·3412	0·1218
0·9285	−0·3106	−0·0174	0·0113	1·0715	−0·2855	−0·0606	0·0153
0·4642	−0·3921	0·1229	0·0198	0·6073	−0·3670	0·0797	0·0138
0·6963	−0·3514	0·0528	0·0115	0·8394	−0·3263	0·0095	0·0104
0·8124	−0·3310	0·0177	0·0104	0·9555	−0·3059	−0·0255	0·0118

Interval of uncertainty in parameter t is $[0{\cdot}6963, 0{\cdot}9555]$, midpoint 0·8259.

Third iteration, starting from $\mathbf{x}_2 = \begin{pmatrix} -0{\cdot}3286 \\ 0{\cdot}0136 \end{pmatrix}$, $f = 0{\cdot}0104$, $\nabla f = \begin{pmatrix} -0{\cdot}1330 \\ -0{\cdot}0807 \end{pmatrix}$:

t	x_1	x_2	f	t'	x_1'	x_2'	f'
0	−0·3286	0·0136	0·0104	2	−0·0626	0·1751	0·0300
0·8393	−0·2170	0·0814	0·0050	1·1607	−0·1743	0·1073	0·0092
0·4197	−0·2728	0·0475	0·0043	0·7410	−0·2301	0·0735	0·0043
0·2098	−0·3007	0·0306	0·0063	0·5312	−0·2580	0·0565	0·0039

Interval of uncertainty in parameter t is $[0{\cdot}2098, 0{\cdot}7410]$, midpoint 0·4754.

Fourth iteration, starting from $\mathbf{x}_3 = \begin{pmatrix} -0{\cdot}2654 \\ 0{\cdot}0520 \end{pmatrix}$, $f = 0{\cdot}0040$, $\nabla f = \begin{pmatrix} -0{\cdot}0472 \\ 0{\cdot}0336 \end{pmatrix}$:

t	x_1	x_2	f	t'	x_1'	x_2'	f'
0	−0·2654	0·0520	0·0040	4	−0·0767	−0·0824	0·0073
1·5683	−0·1914	−0·0007	0·0014	2·4317	−0·1507	−0·0297	0·0021
0·7841	−0·2284	0·0257	0·0020	1·6476	−0·1877	−0·0033	0·0014

Interval of uncertainty in parameter t is $[0{\cdot}7841, 2{\cdot}4317]$, midpoint 1·6079.

Fifth iteration, starting from $\mathbf{x}_4 = \begin{pmatrix} -0{\cdot}1896 \\ -0{\cdot}0020 \end{pmatrix}$, $f = 0{\cdot}0014$, $\nabla f = \begin{pmatrix} -0{\cdot}0280 \\ -0{\cdot}0399 \end{pmatrix}$:

t	x_1	x_2	f	t'	x_1'	x_2'	f'
0	−0·1896	−0·0020	0·0014	2	−0·1336	0·0779	0·0050

Interval of uncertainty in parameter t is $[0, 2]$, midpoint 1

$\mathbf{x}_5 = \begin{pmatrix} -0{\cdot}1616 \\ 0{\cdot}0379 \end{pmatrix}$, $f = 0{\cdot}0011$, $\nabla f = \begin{pmatrix} -0{\cdot}0046 \\ 0{\cdot}0498 \end{pmatrix}$.

(Note how the quartic term x_1^4 and the quadratic term x_2^2 lead the approximations to converge rapidly in the second coordinate but slowly in the first.)

34. $f''(x) = 2 + 6cx + 12x^2 = 12(x + \frac{c}{4})^2 + (2 - \frac{3c^2}{4})$, so f is convex iff $2 \geq \frac{3c^2}{4}$, that is, $c^2 \leq \frac{8}{3}$, that is, $|c| \leq \sqrt{\frac{8}{3}}$.

35. (i) $\mathbf{H}f = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, which is positive definite (its principal minors have determinants 2, 3 and 4), so f is convex.

(ii) $\mathbf{H}f = \begin{pmatrix} 12x_1^2 & 3x_2^2 \\ 3x_2^2 & 6x_1x_2 + 12x_2^2 \end{pmatrix}$ has determinant $144x_1^2x_2^2 + 72x_1^3x_2 - 9x_2^4$ which is sometimes negative, so $\mathbf{H}f$ cannot always be positive semi-definite and f is not convex.

(iii) $\mathbf{H}f = e^{x_1-x_2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, which is positive semi-definite (its eigenvalues are 0 and $2e^{x_1-x_2}$), so f is convex. [*Note to mathematicians*: what is happening here is that $f = g \circ h$, where $h(\mathbf{x}) = x_1 - x_2$ and $g(x) = e^x$ are convex *and g is monotonic increasing.*]

36. (i) $\mathbf{H}f = 2e^{x_1+x_2}(e^{x_1+x_2} - 1)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ has eigenvalues 0 and $4e^{x_1+x_2}(e^{x_1+x_2} - 1)$; the latter can be less than 0, so the matrix is not always positive semi-definite, and the function is not convex. [*Note to mathematicians*: the functions $\mathbf{x} \mapsto e^{x_1+x_2}$, $x \mapsto (x-2)^2$ are convex, but the latter is not monotonic increasing.]

(ii) $\mathbf{H}f = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. [This is not positive definite; its principal minors have determinants 1, 1 and 0.] Its eigenvalues are given by the equation

$$\begin{aligned} 0 &= \det\begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{pmatrix} \\ &= (1-\lambda)(1-\lambda)(2-\lambda) - (1-\lambda) - (1-\lambda) = (1-\lambda)(2 - 3\lambda + 2\lambda^2 - 2) \\ &= -\lambda(\lambda-1)(\lambda-3), \end{aligned}$$

so that the eigenvalues 0, 1 and 3 are all non-negative, $\mathbf{H}f$ is positive semi-definite and f is convex.

37. (a)(i) Yes. If $\mathbf{x}$, $\mathbf{y} \in C_1 \cap C_2$ and $t \in [0,1]$ then

$$\mathbf{x},\ \mathbf{y} \in C_1, \text{ which is convex, so } t\mathbf{x} + (1-t)\mathbf{y} \in C_1,$$

$$\mathbf{x},\ \mathbf{y} \in C_2, \text{ which is convex, so } t\mathbf{x} + (1-t)\mathbf{y} \in C_2,$$

and $t\mathbf{x} + (1-t)\mathbf{y} \in C_1 \cap C_2$; as $\mathbf{x}$, $\mathbf{y}$ and t are arbitrary, $C_1 \cap C_2$ is convex.

(ii) No. For example, try $C_1 = [0,1]$, $C_2 = [2,3]$; these are both convex subsets of $\mathbb{R}$ but their union is not convex.

(b)(i) Yes. If $\mathbf{x}$, $\mathbf{y} \in C$ and $t \in [0,1]$, then

$$\begin{aligned}(f+g)(t\mathbf{x} + (1-t)\mathbf{y}) &= f(t\mathbf{x} + (1-t)\mathbf{y}) + g(t\mathbf{x} + (1-t)\mathbf{y}) \\ &\le tf(\mathbf{x}) + (1-t)f(\mathbf{y}) + tg(\mathbf{x}) + (1-t)g(\mathbf{y})\end{aligned}$$

(because f and g are convex)

$$= t(f+g)(\mathbf{x}) + (1-t)(f+g)(\mathbf{y});$$

as $\mathbf{x}$, $\mathbf{y}$ and t are arbitrary, $f+g$ is convex.

(ii) No. For instance, if we set $C = \mathbb{R}$, $f(x) = x^2$, $a = -1$ then af is not convex.

(iii) Yes. If $\mathbf{x}$, $\mathbf{y} \in C$ and $t \in [0,1]$, then
— if $f(t\mathbf{x} + (1-t)\mathbf{y}) \le g(t\mathbf{x} + (1-t)\mathbf{y})$, then

$$\max(f,g)(t\mathbf{x} + (1-t)\mathbf{y}) = g(t\mathbf{x} + (1-t)\mathbf{y}) \le tg(\mathbf{x}) + (1-t)g(\mathbf{y})$$

(because g is convex)

$$\le t\max(f,g)(\mathbf{x}) + (1-t)\max(f,g)(\mathbf{y});$$

— if $g(t\mathbf{x} + (1-t)\mathbf{y}) \le f(t\mathbf{x} + (1-t)\mathbf{y})$, then

$$\max(f,g)(t\mathbf{x} + (1-t)\mathbf{y}) = f(t\mathbf{x} + (1-t)\mathbf{y}) \le tf(\mathbf{x}) + (1-t)f(\mathbf{y})$$

(because f is convex)

$$\le t\max(f,g)(\mathbf{x}) + (1-t)\max(f,g)(\mathbf{y});$$

thus $\max(f,g)(t\mathbf{x}+(1-t)\mathbf{y}) \le t\max(f,g)(\mathbf{x})+(1-t)\max(f,g)(\mathbf{y})$ in both cases. As $\mathbf{x}$, $\mathbf{y}$ and t are arbitrary, $\max(f,g)$ is convex.

(iv) No. For example, if $C = \mathbb{R}$, $f(x) = e^x$, $g(x) = e^{-x}$ then $\min(f,g)(x) = e^{-|x|}$ is not convex.

(v) No. For example, if $C = \mathbb{R}$, $f(x) = g(x) = x^2 - 1$ then $f \times g$ is not convex (it takes a greater value at 0 than at either ± 1.)

38. (a) A *convex program* is a problem of the form
minimise $f(\mathbf{x})$
subject to $\mathbf{x} \in C$
and $g_1(\mathbf{x}), \ldots, g_k(\mathbf{x}) \leq 0$
where $C \subseteq \mathbb{R}^n$ is a convex set and f, $g_1, \ldots, g_k$ are convex functions defined (at least) on C.
In this case the *dual problem* is
maximise $\min_{\mathbf{x} \in C} L(\mathbf{x}, \boldsymbol{\lambda})$
subject to $\lambda_1, \ldots, \lambda_k \leq 0$,
where

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \lambda_1 g_1(\mathbf{x}) - \ldots - \lambda_k g_k(\mathbf{x})$$

for $\mathbf{x} \in C$, $\lambda \in \mathbb{R}^k$.

(b) We can restate (P) as
(P) minimise $x_1^2 + 2x_2^2 + 3x_3^2$
for $\mathbf{x} \in C$
subject to the constraints $-x_1 - x_2 + 2 \leq 0$, $-x_2 - x_3 + 1 \leq 0$.
The function $f(\mathbf{x}) = x_1^2 + 2x_2^2 + 3x_3^2$ has $\mathbf{H}f = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ which has eigenvalues 2, 4 and 6, so is positive definite, and f is convex. The set $C = \mathbb{R}^3$ is certainly convex. The functions $g_1(\mathbf{x}) = -x_1 - x_2 + 2$, $g_2(\mathbf{x}) = -x_2 - x_3 + 1$ are convex because their Hessians are zero. So (P) is a convex program.

(c) The dual problem is
(D) maximise $\min_{\mathbf{x} \in \mathbb{R}^3} x_1^2 + 2x_2^2 + 3x_3^2 - \lambda_1(-x_1 - x_2 + 2) - \lambda_2(-x_2 - x_3 + 1)$
subject to $\boldsymbol{\lambda} \leq \mathbf{0}$.
For any fixed $\boldsymbol{\lambda}$, we have to minimise

$$x_1^2 + \lambda x_1 + 2x_2^2 + (\lambda_1 + \lambda_2)x_2 + 3x_3^2 + \lambda_2 x_3 - 2\lambda_1 - \lambda_2.$$

The minimum will be achieved when $x_1 = -\frac{1}{2}\lambda_1$, $x_2 = -\frac{1}{4}(\lambda_1 + \lambda_2)$ and $x_3 = -\frac{1}{6}\lambda_2$, and its value there is

$$\begin{aligned} h(\boldsymbol{\lambda}) &= -\frac{1}{4}\lambda_1^2 - \frac{1}{8}(\lambda_1 + \lambda_2)^2 - \frac{1}{12}\lambda_2^2 - 2\lambda_1 - \lambda_2 \\ &= -\frac{3}{8}\lambda_1^2 - \frac{5}{24}\lambda_2^2 - \frac{1}{4}\lambda_1\lambda_2 - 2\lambda_1 - \lambda_2. \end{aligned}$$

To find the maximum of this (ignoring the constraint $\boldsymbol{\lambda} \leq \mathbf{0}$ for the moment), differentiate:

$$\begin{aligned} \frac{\partial h}{\partial \lambda_1} &= 0 = -\frac{3}{4}\lambda_1 - \frac{1}{4}\lambda_2 - 2 \\ \frac{\partial h}{\partial \lambda_2} &= 0 = -\frac{1}{4}\lambda_1 - \frac{5}{12}\lambda_2 - 1, \end{aligned}$$

which give us $\lambda_1 = -\frac{7}{3}$, $\lambda_1 = -1$. (To check that this is indeed a maximum, look at $\mathbf{H}h = -\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{5}{12} \end{pmatrix}$; since $\det \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{5}{12} \end{pmatrix} = \frac{1}{4} > 0$, $\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{5}{12} \end{pmatrix}$ is positive definite, $\mathbf{H}h$ is negative definite and any stationary point of h must be a maximum.) Since the maximum found has $\boldsymbol{\lambda} \leq \mathbf{0}$, this is the solution of (D), with

$$\lambda_1 = -\frac{7}{3},\ \lambda_2 = -1,\ x_1 = \frac{7}{6},\ x_2 = \frac{5}{6},\ x_3 = \frac{1}{6},$$

and value $\frac{17}{6}$. Since $\mathbf{x}$ is (necessarily) an interior point of $C = \mathbb{R}^3$, this is also the solution to (P), with value $\frac{17}{6}$.

39. (a) We can express (P) as

$$\text{minimise } (x_1+1)^2 + x_2^2 \text{ for } \mathbf{x} \in C \text{ subject to } -x_1 - 2x_2 + 2 \le 0.$$

As $f(\mathbf{x}) = (x_1+1)^2 + x_2^2$ and $g(\mathbf{x}) = -x_1 - 2x_2 + 2$ are convex functions (their Hessians being $\mathbf{H}f = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\mathbf{H}g = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, which are both positive semi-definite), and C is a convex set, this is a convex program.

(b) The dual problem is now

$$\text{(D) find } \max_{\lambda \le 0} \min_{\mathbf{x} \in C} L(\mathbf{x}, \lambda)$$

where $L(\mathbf{x}, \lambda) = (x_1+1)^2 + x_2^2 - \lambda(-x_1 - 2x_2 + 2)$.

(c) We have

$$\begin{aligned} L &= x_1^2 + 2x_1 + 1 + x_2^2 + \lambda x_1 + 2\lambda x_2 - 2\lambda \\ &= (x_1 + 1 + \tfrac{1}{2}\lambda)^2 + (x_2 + \lambda)^2 - \tfrac{5}{4}\lambda^2 - 3\lambda. \end{aligned}$$

(i) If $\lambda \le 0$ and $1 + \frac{1}{2}\lambda \ge 0$, that is, $-2 \le \lambda \le 0$, $h(\lambda) = \min_{\mathbf{x} \in C} L(\mathbf{x}, \lambda)$ is attained when $x_1 = 0$ and $x_2 = -\lambda$, and takes the value $1 - 2\lambda - \lambda^2 = 2 - (\lambda+1)^2$; this is maximal for $\lambda = -1$ and takes the value 2.

(ii) If $\lambda \le -2$ then $h(\lambda) = \min_{\mathbf{x} \in C} L(\mathbf{x}, \lambda)$ is attained when $x_1 = -1 - \frac{1}{2}\lambda$ and $x_2 = -\lambda$, and takes the value $-\frac{5}{4}\lambda^2 - 3\lambda = -\frac{5}{2}(\lambda + \frac{6}{5})^2 + \frac{9}{5}$; this is maximal when $\lambda = -2$ and takes the value 1.

Putting these together, $\max_{\lambda \le 0} \min_{\mathbf{x} \in C} L(\mathbf{x}, \lambda) = 2$, attained when $\lambda = -1$, $x_1 = 0$, $x_2 = 1$.

(d) Our Lagrangean is now

$$L(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) = (x_1+1)^2 + x_2^2 - \lambda_1(x_1 - u_1^2) - \lambda_2(x_2 - u_2^2) - \lambda_3(x_1 + 2x_2 - 2 - u_3^2).$$

Equating the partial derivatives to zero, we get

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= 0 = 2x_1 + 2 - \lambda_1 - \lambda_3 \\ \frac{\partial L}{\partial x_2} &= 0 = 2x_2 - \lambda_2 - 2\lambda_3 \\ \frac{\partial L}{\partial u_1} &= 0 = 2\lambda_1 u_1 \\ \frac{\partial L}{\partial u_2} &= 0 = 2\lambda_2 u_2 \\ \frac{\partial L}{\partial u_3} &= 0 = 2\lambda_3 u_3. \end{aligned}$$

(i) If $u_1 = 0$ then $x_1 = 0$ so $x_2 \ge 1$ (because $x_1 + 2x_2 \ge 2$) and $u_2 \ne 0$, $\lambda_2 = 0$, $\lambda_3 = 2x_2 \ne 0$, $u_3 = 0$, $x_1 + 2x_2 = 2$ so $x_1 = 0$, $x_2 = 1$ and $(x_1+1)^2 + x_2^2 = 2$.

(ii) If $u_1 \ne 0$ then $\lambda_1 = 0$, $\lambda_3 = 2x_1 + 2 > 0$, $u_3 = 0$, $x_1 = 2 - 2x_2$.

(α) If $u_2 = 0$ then $x_2 = 0$, $x_1 = 2$, $(x_1+1)^2 + x_2^2 = 9$.

(β) If $u_2 \ne 0$ then $\lambda_2 = 0 = \lambda_1$ so $2x_1 + 2 = 2x_2$; together with $x_1 = 2 - 2x_2$ we get $x_2 = 1$, $x_1 = 0$ as in case (i).

Thus $\min_{\mathbf{x} \in C, x_1 + 2x_2 \ge 2} (x_1+1)^2 + x_2^2 = 2$, attained when $x_1 = 0$, $x_2 = 1$.

(e) The solution $\lambda = -1$, $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ provided by (c) (in fact, by part (i) of (c)) is a feasible point for (P) with $g(\mathbf{x}) = 0$, so at this point $h(\lambda) = f(\mathbf{x})$ and both must be at the optimal values.

40. (a) Express the constraints in the form $1 - x_1 \leq 0$, $x_1^2 + x_2^2 - 1 \leq 0$. $f(\mathbf{x}) = x_2$ and $g_1(\mathbf{x}) = 1 - x_1$ are convex because they are linear, so have zero Hessians; $g_2(\mathbf{x}) = x_1^2 + x_2^2 - 1$ is convex because its Hessian is the positive definite matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

(b) If $x_1 \geq 1$ and $x_2^2 + x_2^2 \leq 1$ then $x_1 = 1$, $x_2 = 0$, so we have a unique feasible point $\mathbf{x}^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and this must be the optimum, with $f(\mathbf{x}) = 0$.

(c) The dual problem is
(D) maximise $h(\boldsymbol{\lambda}) = \min_{\mathbf{x} \in \mathbb{R}^2} L(\mathbf{x}, \boldsymbol{\lambda})$
subject to $\boldsymbol{\lambda} \leq \mathbf{0}$,

where

$$L(\mathbf{x}, \boldsymbol{\lambda}) = x_2 - \lambda_1(x_1^2 + x_2^2 - 1) - \lambda_2(1 - x_1).$$

(d) When $\lambda_1 = 0$, $L(\mathbf{x}, \boldsymbol{\lambda}) = x_2 - \lambda_2(1 - x_1)$ is unbounded below (since x_2 can be anything), so $h(\boldsymbol{\lambda})$ has to be interpreted as $-\infty$.

When $\lambda_1 < 0$,

$$L = -\lambda_1(x_1 - \frac{\lambda_2}{2\lambda_1})^2 + \frac{\lambda_2^2}{4\lambda_1} - \lambda_1(x_2 - \frac{1}{2\lambda_1})^2 + \frac{1}{4\lambda_1} + \lambda_1 - \lambda_2$$

which takes its minimal when $x_1 = \frac{\lambda_2}{2\lambda_1}$, $x_2 = \frac{1}{2\lambda_1}$ with

$$\begin{aligned} h(\boldsymbol{\lambda}) &= \frac{\lambda_2^2}{4\lambda_1} - \lambda_2 + \frac{1}{4\lambda_1} + \lambda_1 \\ &= \frac{1}{4\lambda_1}(1 + (\lambda_2 - 2\lambda_1)^2) \\ &< 0. \end{aligned}$$

If we take $\lambda_2 = 2\lambda_1$ and let $\lambda_1 \to -\infty$, we see that $h(\boldsymbol{\lambda}) \doteq 0$, so that $\sup_{\boldsymbol{\lambda} \leq 0} h(\boldsymbol{\lambda}) = 0$.

41. (a) Setting $f(x) = e^x(2 + \sin x)$, $f'(x) = e^x(2 + \sin x + \cos x)$ and $f''(x) = e^x(2 + 2\cos x) \geq 0$ for every $x \in \mathbb{R}$, so f is convex. Setting $g(x) = e^x$, $g''(x) = e^x \geq 0$ for every $x \in \mathbb{R}$, so g is convex. Thus (P) is a convex program.

(b) The dual problem is
(D) maximise $\min_{x \in \mathbb{R}} e^x(2 + \sin x) - \lambda e^x$
for $\lambda \leq 0$.

(c) If $\lambda = 0$, then $L(x, \lambda) = e^x(2 + \sin x)$ is greater than 0 for every x and converges to 0 as $x \to -\infty$, so

$$h(0) = \inf_{x \in \mathbb{R}} e^x(2 + \sin x) = 0,$$

but the infimum is never attained.

If $\lambda < 0$, then $L = e^x(2 + \sin x - \lambda)$, $\frac{\partial L}{\partial x} = e^x(2 + \sin x + \cos x - \lambda)$ is never 0. Since $L \geq e^x \geq 0$ for every x, and $\lim_{x \to \infty} L(x, \lambda) = 0$, we again have $h(\lambda) = 0$.

(d) 0 is not the optimal value for the primal (P), because the condition '$e^x \leq 0$' is never attained; there is no feasible point.

42. (a) C is convex because if $\mathbf{x}$, $\mathbf{y} \in C$, $\alpha \in [0,1]$ and $\mathbf{z} = \alpha\mathbf{x} + (1-\alpha)\mathbf{y}$ then
either one of x_2, y_2 is greater than 0 and $0 < \alpha < 1$ so $z_2 = \alpha x_2 + (1-\alpha)y_2 > 0$ and $\mathbf{z} \in C$
or $\alpha \in \{0,1\}$ so $\mathbf{z}$ is equal to one of $\mathbf{x}$, $\mathbf{y}$ and belongs to C
or $\mathbf{x} = \mathbf{y} = \mathbf{0}$ so $\mathbf{z} = \mathbf{0} \in C$.

(b) The dual problem is
(D) maximise $h(\lambda) = \min_{x\in C} x_1 - \lambda x_2$
subject to $\lambda \leq 0$.

(c) The solution to (P) is $\mathbf{x}^* = \mathbf{0}$. But in (D) we have $h(\lambda) = -\infty$ for every λ, because we can take x_1 arbitrarily far to the left.

43. (a) $\mathbf{H}g = \frac{2}{x_1^3}\begin{pmatrix} x_2^2 & -x_1x_2 \\ -x_1x_2 & x_1^2 \end{pmatrix}$ has eigenvalues 0 and $\frac{2(x_1^2+x_2^2)}{x_1^3}$ which are non-negative if $x_1 > 0$, which is true everywhere on C, so g is convex. Since C is also convex, (P) is a convex program.
Since

$$\{\mathbf{x} : \mathbf{x} \in C,\, g(\mathbf{x}) \leq 0\} = \{\mathbf{x} : x_2 = 0,\, x_1 \geq 1\},$$

the optimum for (P) is $\mathbf{x}^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $f(\mathbf{x}^*) = 1$.

(b) The dual problem is
(D) maximise $\min_{\mathbf{x}\in C} x_1 - \lambda\frac{x_2^2}{x_1}$
for $\lambda \leq 0$.

Now if $\lambda \leq 0$, $\mathbf{x} \in C$ then $x_1 - \lambda\frac{x_2^2}{x_1} \geq 0$, so $h(\lambda) \geq 0$. If $\lambda = 0$, then $h(0) = \inf_{\mathbf{x}\in C} x_1 = 0$. If $\lambda < 0$, then, given any $\epsilon > 0$, we can set

$$x_1 = \frac{1}{\epsilon}, \quad x_2 = \frac{\epsilon}{2\sqrt{-\lambda}}$$

to get a point $\mathbf{x} \in C$ such that $L(\mathbf{x},\lambda) = \epsilon$; so $h(\lambda) = \inf_{\mathbf{x}\in C} L(\mathbf{x},\lambda)$ is again 0.

44. (a) C_0 is a right circular cone with apex $\mathbf{0}$ and axis $x_1 = x_2 = 0$. So it is convex and $C = C_0 \cap \{\mathbf{x} : x_1 \geq -1\}$ is also convex, and (P) is a convex program.

(b) If $\mathbf{x} \in C$, then $|x_2| \leq x_3$, so that $x_2 + x_3 \geq 0$, and if $x_2 + x_3 = 0$ then $x_2 = -x_3$, $x_1 = 0$. Thus the optimal value for (P) is 0, being the value of f at any feasible point.

(c) The dual problem is
(D) maximise $h(\lambda) = \inf_{\mathbf{x}\in C} x_1 - \lambda(x_2 + x_3)$ for $\lambda \leq 0$.

(d) If $\lambda = 0$ then $h(0) = \inf_{\mathbf{x}\in C} x_1$ because (by the definition of C) $x_1 \geq -1$ for every $\mathbf{x} \in C$, while (for instance) $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \in C$.

If $\lambda < 0$ then $x_1 - \lambda(x_2 + x_3) \geq x_1 \geq -1$ for every $\mathbf{x} \in C$, since $x_1 + x_3 \geq 0$ for every $\mathbf{x} \in C$. Thus $h(\lambda) \geq -1$. On the other hand, for any $\alpha \geq 0$, $\begin{pmatrix} -1 \\ -\alpha \\ \sqrt{\alpha^2+1} \end{pmatrix} \in C$, so that

$$h(\lambda) \leq -1 - \lambda(-\alpha + \sqrt{\alpha^2+1}) \to -1$$

as $\alpha \to \infty$, and $h(\lambda) = -1$.

[To find $\lim_{\alpha\to\infty} \sqrt{\alpha^2+1} - \alpha$, note that $\alpha^2 + 1 \leq (\alpha + \frac{1}{2\alpha})^2$, so $\sqrt{\alpha^2+1} - \alpha \leq \frac{1}{2\alpha}$ for every $\alpha > 0$.]