Combinatorics Questions

1. (a) For $n \geq 0$, find $\sum_{k=0}^{n} {}^{n}C_k$, $\sum_{k=0}^{n} \sum_{j=0}^{k} {}^{k}C_j\ {}^{n}C_k$, $\sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{i=0}^{j} {}^{j}C_i\ {}^{k}C_j\ {}^{n}C_k$.
(b) Let X be a set of size n. Find $\#(\mathcal{P}X)$, $\#(\{(b,c) : b \subseteq c \subseteq X\})$, $\#(\{(a,b,c) : a \subseteq b \subseteq c \subseteq X\})$.
(c) Explain why the answers in (a) are related to those in (b). [AA]

2. (a) Show that $\sum_{r=0}^{n} r(r-1)\ {}^{n}C_r = 2^{n-2}n(n-1)$ for every $n \geq 0$. (*Hint:* find $\#(\{(a,b) : a \subseteq b \subseteq X$ and $\#(a) = 2\})$, where $\#(X) = n$.)
(b) Find $\sum_{r=0}^{n} r^2\ {}^{n}C_r$ for $n \geq 0$.
(c) Show that $\sum_{r=0}^{n} r(r-1)(r-2)\ {}^{n}C_r = 2^{n-3}n(n-1)(n-2)$ for every $n \geq 0$. (*Hint:* find $\#(\{(a,b) : a \subseteq b \subseteq X$ and $\#(a) = 3\})$, where $\#(X) = n$.)
(d) Find $\sum_{r=0}^{n} r^3\ {}^{n}C_r$ for $n \geq 0$.
[AB]

3. Suppose that $0 \leq j \leq k \leq n$ and that X is a set of size n. Set $R = \{(a,b) : a \subseteq b \subseteq X, \#(a) = j, \#(b) = k\}$. For $a \in [X]^j$ find $\#(\{b : (a,b) \in R\})$ and for $b \in [X]^k$ find $\#(\{a : (a,b) \in R\})$. Hence show that $\#(R) = {}^{n}C_k\ {}^{k}C_j = {}^{n}C_j\ {}^{n-j}C_{k-j} = {}^{n}C_j\ {}^{n-j}C_{n-k}$. [AC]

4. (a) For $n \geq 0$, write q_n for the number of subsets of a set of n members which have size divisible by 3. Find q_0, q_1, q_2, q_3. Show that $q_n = \frac{1}{3}(2^n + 2\,\mathcal{R}\mathrm{e}(1+\omega)^n)$ for every n, where $\omega = e^{2\pi i/3}$.
(b) For $n \geq 0$, write q'_n for the number of subsets of a set of n members which have size divisible by 4. Find q'_0, q'_1, q'_2, q'_3. Show that $q'_n = \frac{1}{4}(2^n + 2\,\mathcal{R}\mathrm{e}(1+i)^n)$ for every n. [AD]

5. For $n \geq 0$, let q^*_n be the number of subsets a of a set of size n such that $\#(a) \equiv 1 \bmod 3$. Find q^*_n for $n \leq 5$. Setting $\omega = e^{2\pi i/3}$, $\gamma_n = \mathcal{R}\mathrm{e}(\frac{1}{\omega}(1+\omega)^n)$, show that $\gamma_n = \frac{3}{2}q^*_n - \frac{1}{2}2^n$. Find γ_7. [AE]

6. For $n \geq 1$, let s_n be the number of ways of putting n Christmas cards into n envelopes in such a way that every card is in the wrong envelope. (a) Find s_1, s_2, s_3, s_4, s_5. (b) Show that $s_{n+2} = (n+1)(s_n + s_{n+1})$ for every $n \geq 1$. (c) Set $\alpha_n = s_n/n!$. Show that $\alpha_{n+2} = \frac{n+1}{n+2}\alpha_{n+1} + \frac{1}{n+2}\alpha_n$ for every $n \geq 1$. (d) Show that $\alpha_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$ for each $n \geq 1$, and hence find $\lim_{n\to\infty} \alpha_n$. [AF]

7. Show that if $n \geq 1$, $k \geq 0$ then the number of strings $(m_1, \ldots, m_n)$ of non-negative integers such that $m_1 + \ldots + m_n = k$ is just ${}^{k+n-1}C_k$. [AG]

8. Show that if $m \geq n \geq 1$ then $\sum_{i=n}^{m} {}^{i-1}C_{i-n} = {}^{m}C_n$. (*Hint:* show that ${}^{i-1}C_{i-n} = \#(\{a : a \subseteq \{1,\ldots,m\}, \#(a) = n, \max a = i\})$.) [AH]

9. Let $G = (V,E)$ be a graph. (a) What is meant by (i) the *order* of G (ii) the *complement* $\overline{G}$ of G (iii) a *path* in G (iv) a *simple path* in G (v) a *closed path* in G (vi) a *cycle* in G (vii) a *component* of G (viii) saying that G is *connected* (ix) the *degree* $d(v)$ of a vertex v of G (x) saying that G is a *tree*.
(b) Show that if $V \neq \emptyset$ and $d(v) \geq 2$ for every $v \in V$, then G has a cycle.
(c) Show that $\sum_{v \in V} d(v) = 2\#(E)$.
(d) Show that if $V \neq \emptyset$ and G has no cycle then $\#(E) < \#(V)$.
(e) Show that if G is connected then $\#(E) \geq \#(V) - 1$.
(f) Show that if $V \neq \emptyset$ then the following are equivalent: (i) G is a tree; (ii) G is connected and $\#(V) \geq \#(E) + 1$; (iii) G has no cycle and $\#(V) \leq \#(E) + 1$. [BA]

10. Describe the 34 isomorphism types of graphs on a given set of 5 vertices, and say how many graphs there are of each type. Check that you have ${}^{10}C_k$ graphs with k edges for each k. [BB]

11. Describe the 5 isomorphism types of trees on a given set of 6 vertices, and say how many trees there are of each type. [BC]

12. (a) Let G be a finite graph with at least two vertices. Show that G has two vertices with the same degree.

(b) Explain how to construct an infinite graph with vertex set $\mathbb{N}$ in which the vertex n has degree n, for every n. [BD]

13. Let G be a graph of order n and with no triangles. Show that (i) if $n = 2k$ is even, then G has at most $k^2 = n^2/4 = \lfloor n^2/4 \rfloor$ edges (ii) if $n = 2k+1$ is odd, then G has at most $k(k+1) = (n^2-1)/4 = \lfloor n^2/4 \rfloor$ edges. (*Hint:* show that G has a vertex of degree not more than $\frac{n}{2}$, and induce on n.) [BE]

14. Let G be a graph with vertex set V and at least one edge. For $W \subseteq V$ let G_W be the restriction of G to W and $e(G_W)$ the number of edges of G_W. Show that there is a $W \subseteq G$ such that $e(G_W)+e(G_{V\setminus W}) < \frac{1}{2}e(G)$ and $\#(W) = \lfloor \frac{1}{2}\#(V) \rfloor$. (*Hint:* suppose that W is a random set of this size. Find the probability that any given edge of G is an edge of either G_W or $G_{V\setminus W}$, and hence calculate the expectation of $e(G_W)+e(G_{V\setminus W})$.) [BF]

15. Let G be a graph with vertices $v_1, \dots, v_n$, and suppose that $d(v_1) \le d(v_2) \le \dots \le d(v_n)$, where $d(v)$ is the degree of the vertex v. Suppose that $d(v_k) \ge k$ for $1 \le k < n - d(v_n)$. Show that G is connected. [BG]

16. Show that if G is any graph with complement $\overline{G}$, then at least one of G, $\overline{G}$ is connected. [BH]

17. (a) Define *Eulerian path* in a graph.

(b) Describe the isomorphism types of all those graphs of order 5 with Eulerian paths.

(c) Show that if G is a graph of odd order with an Eulerian path, and $\overline{G}$ is connected, then $\overline{G}$ has an Eulerian path. [CA]

18. (a) Explain what is meant by a *colouring* of a graph, the *colouring number* χG of a graph, and the *greedy algorithm* for colouring a graph.

(b) Show that for any graph G there is some ordering $v_1, \dots, v_n$ of its vertices such that the greedy algorithm, applied to that ordering, uses χG colours.

(c) Give an example of a graph G and an ordering $v_1, \dots, v_n$ of its vertices such that $d(v_1) \ge d(v_2) \ge \dots \ge d(v_n)$ but the greedy algorithm uses more colours than χG. [CB]

19. Let G be a graph of order n and $\overline{G}$ its complement.

(a) Let χG, $\chi\overline{G}$ be their colouring numbers. Show that $n \le \chi G.\chi\overline{G}$ and hence that $2\sqrt{n} \le \chi G + \chi\overline{G}$.

(b) Suppose that the vertices $v_1, \dots, v_n$ of G are ordered so that $d(v_1) \ge \dots \ge d(v_n)$, where $d(v)$ is the degree of the vertex v. Show that if the greedy colouring algorithm, applied to this ordering, uses k colours, then there must be some $i \le n$ such that $k \le \min(i, d(v_i)+1)$. (*Hint:* look at the first vertex which used the kth colour.) Suppose now that the greedy colouring algorithm is applied to the graph $\overline{G}$, but with the vertices taken in the reverse order, and uses l colours. Show that $l \le \min(n+1-j, n-d(v_j))$ for some j, and hence that $\chi G + \chi\overline{G} \le n+1$. [CD]

20. Show that a graph G is 2-colourable iff there are no cycles of odd length in G. [CE]

21. Consider the following graph G:

(No diagram available.)

(i) Is there a 3-colouring of G such that a and b have the same colour? (ii) Is there a 3-colouring of G such that a and b have different colours? (iii) Is there a 3-colouring of G such that a and c have the same colour? (iv) Is there a 3-colouring of G such that a and c have different colours? Justify your answers. How many colourings of G are there with a given set of three colours? [CF]

22. Describe all the isomorphism types of graphs of order 5 which are not 3-colourable. [CG]

23. Let G be a graph of order n.

(a) For $i \leq n$ let c_i be the number of ways of colouring G with i colours, using every colour at at least one vertex. Set $p_G(x) = c_0 + \sum_{i=1}^{n} c_i \frac{x(x-1)(x-2)\dots(x-i+1)}{i!}$ (if $n = 0$ then $p_G(x) = c_0 = 1$); this is the **chromatic polynomial** of G. Show that for any integer $k \geq 0$, $p_G(k)$ is the number of ways of colouring G with k colours, if we do not insist on using every colour.

(b) Show that if e is any edge between vertices v and w of G, then $p_G(x) = p_{G'}(x) - p_{G^*}(x)$, where $G' = G - \{e\}$ is the graph with n vertices obtained by deleting the edge e from G, and G^* is the graph with $n-1$ vertices obtained by collapsing v and w to a single point joined to every vertex linked in G to either v or w.

(c) Suppose that $p_G(x) = \sum_{i=0}^{n} a_i x^i$. Show that (i) $(-1)^{n-i} a_i \geq 0$ for $i \leq n$ (ii) $a_n = 1$ (iii) if $n \geq 1$, then $a_{n-1} = -m$, where m is the number of edges of G (iv)* if $n \geq 2$, then $a_{n-2} = \frac{1}{2}m(m-1) - l$, where l is the number of triangles in G. (*Hint:* induce on the number of edges of G, using (b).)

(d) Show that if $H_1, \dots, H_k$ are the components of G then $p_G(x) = \prod_{j=1}^{k} p_{H_j}(x)$. [CH]

24. (a) What is a *Hamiltonian circuit* in a graph?

(b) Describe all the isomorphism types of graphs of order 5 which have Hamiltonian circuits. [CI]

25. State and prove *Euler's criterion* for a finite graph G to have an Eulerian path. [CJ]

26. Draw the skeletons of the five regular solids. What are the colouring numbers of these five graphs? Justify your answers. [CK]

27. (a) What is a *planar* graph? State Euler's theorem relating the numbers of vertices, edges and faces of a connected graph drawn in the plane.

(b) Draw the skeletons of the five regular solids; in each case count the vertices, faces and edges of your diagram.

(c) Show that in any planar graph there is a vertex of degree at most 5. Give an example of a planar graph in which every vertex has degree at least 5.

(d) Show that the complete graph K_5 is not planar.

(e) What is the bipartite graph $K_{3,3}$? Show that it is not planar.

(f) State *Kuratowski's theorem* characterizing planar graphs. [DB]

28. (a) Starting from the complete graph K_6 on six vertices, delete four edges. Out of the ${}^{15}C_4$ ways of doing this, how many produce non-planar graphs?

(b) Show that for every graph G of order 6, either G or its complement is planar. [DC]

29. Describe the fourteen isomorphism types of the non-planar graphs with a given set of six vertices, and say how many graphs of each type there are. [DE]

30. Let $G = (V, E)$ be a graph, a and b two vertices of G not joined by an edge, and $P_1, \dots, P_n$ independent simple paths from a to b in G. Let us say that a set $S \subseteq V$ is 'good' if

$a \in S$, $b \notin S$;

for each $i \leq n$, the vertices of P_i belonging to S form an initial segment of P_i, containing at least one vertex other than a;

writing v_i for the last vertex of P_i belonging to S, then for every $x \in S \setminus \{v_1, \dots, v_n\}$ there are independent paths Q_x, $Q_{x1}, \dots, Q_{xn}$, all within S, such that Q runs from a to x and Q_{xi} runs from a to v_i for each i.

(Allow Q_x to be the trivial path if $x = a$.) Show that if S is a 'good' set of vertices then

either there is an $x \in S$ joined by an edge to b

or $\{v_1, \dots, v_n\}$ separate a from b

or there is a strictly larger 'good' set $S' \supseteq S$.

[DF]

31. State *Menger's theorem* (for undirected graphs) and the *Marriage Lemma* (in defect form). Prove the Marriage Lemma from Menger's theorem. [EA]

32. Let X and Y be finite sets, and $R \subseteq X \times Y$ a relation. Show that there is a surjection from X to Y covered by R iff $\#(R^{-1}[J]) \geq \#(J)$ for every $J \subseteq Y$. [EB]

33. Let X and Y be finite sets, and suppose that $R \subseteq X \times Y$ is such that $\#(R[\{x\}]) \geq k \geq \#(R^{-1}[\{y\}]$ for all $x \in X$, $y \in Y$, where $k \geq 1$. Show that there is an injection $f : X \to Y$ covered by R. (*Hint*: for any $I \subseteq X$, count $R \cap (I \times Y)$.) [EC]

34. Let V be a set with $n \geq 1$ members, and $0 \leq r < \frac{n}{2}$. Show that there is an injective function $f : [V]^r \to [V]^{r+1}$ such that $a \subseteq f(a)$ for every $a \in [V]^r$. (*Hint:* find $\#(R[\{a\}])$, $\#(R^{-1}[\{b\}]$ for $a \in [V]^r$, $b \in [V]^{r+1}$, where $R = \{(a, b) : a \subseteq b \subseteq X,\ \#(a) = r,\ \#(b) = r + 1\}$.) [ED]

35. Let G be a graph which is *k-regular*, where $k \geq 1$, that is, such that the degree of every vertex of G is k. Show that G is expressible as a disjoint union of cycles, if for once you allow 2-cycles, that is, cycles $(\,x \quad y \quad x\,)$ where x and y are joined by an edge. (*Hint:* show that there is a permutation ρ of the vertices of G such that x, $\rho(x)$ are joined by an edge for every vertex x.) [EE]

36. Let X and Y be finite sets, and suppose that $R \subseteq X \times Y$ is such that $\#(R[\{x\}]) = \#(R^{-1}[\{y\}]) = k \geq 1$ for all $x \in X$, $y \in Y$. Show that there are bijections $f_1, \dots , f_k$ from X to Y such that $R = \bigcup_{i \leq k}\{(x, f_i(x)) : x \in X\}$. (*Hint:* Show first that $\#(X) = \#(Y)$. Show that there is some injection $f_k \subseteq R$, and consider $R \setminus f_k$.) [EF]

37. Let X and Y be finite sets, $f : X \to Y$ an injective function, and $R \subseteq X \times Y$ a relation covering f. Show that there is an $x \in X$ such that whenever $(x, y) \in R$ there is an injective function $f_y : X \to Y$, covered by R, with $f_y(x) = y$. (*Hint:* If $\#(R[I]) > \#(I)$ for every non-empty $I \subseteq X$, show that any $x \in X$ has this property. Otherwise take x to belong to some minimal I such that $\#(R[I]) = \#(I)$.) [EG]

38. Let $\mathbf{A}$ be a *doubly stochastic* square matrix, that is, an $n \times n$ matrix such that $a_{ij} \geq 0$ for all i, j and $\sum_{i=1}^n a_{ik} = \sum_{j=1}^n a_{kj} = 1$ for every $k \leq n$. Show that there is a permutation ρ of $\{1, \dots , n\}$ such that $a_{i,\rho(i)} > 0$ for evey $i \leq n$. [EH]

39. (a) Define *partially ordered set*, *totally ordered set.*
(b) State and prove *Dilworth's theorem.* [EI]

40. Let V be a set with $n \geq 1$ members, and set $r = \lfloor \frac{n}{2} \rfloor$. Show that the partially ordered set $\mathcal{P}V$ can be expressed as a union of ${}^nC_r$ totally ordered subsets, but not fewer. (*Hint:* recall that for $k < \frac{n}{2}$ there is an injective function $f : [V]^k \to [V]^{k+1}$ such that $a \subseteq f(a)$ for $a \in [V]^k$. Show that for $\frac{n}{2} \leq k < n$ there is a surjective function $f : [V]^k \to [V]^{k+1}$ such that $a \subseteq f(a)$ for $a \in [V]^k$.) [EJ]

41. (a) Describe a graph G of order 5 such that neither G nor its complement has a triangle.
(b) Show that if G is a graph of order 6 then either G or its complement must have a triangle. [EK]

42. State and prove *Ramsey's theorem* on graphs with vertex set $\mathbb{N}$. [EL]

43. State the *Marriage Lemma* (in its simple form). Prove it directly (without using Menger's theorem). [EM]

44. (a) What is meant by saying that simple paths $P_1, \ldots, P_n$ between two vertices a, b of a graph G are *independent*?

(b) What is meant by saying that a set S of vertices of a graph G *separates* two vertices of G?

(c) State *Menger's theorem.*

(d) For each of the following graphs, with marked vertices a and b, find (i) a largest possible set of independent paths from a to b (ii) a smallest possible set separating a from b.

(No diagram available.)

(No diagram available.)

[EN]