Combinatorics Answers

1. (a)

$$\textstyle\sum_{k=0}^{n} {}^nC_k = 2^n,$$

$$\textstyle\sum_{k=0}^{n}\sum_{j=0}^{k} {}^kC_j \ {}^nC_k = \sum_{k=0}^{n} 2^k {}^nC_k = \sum_{k=0}^{n} 2^k 1^{n-k} {}^nC_k = (2+1)^n = 3^n,$$

$$\textstyle\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{i=0}^{j} {}^jC_i \ {}^kC_j \ {}^nC_k = \sum_{k=0}^{n} 3^k {}^nC_k = \sum_{k=0}^{n} 3^k 1^{n-k} {}^nC_k = (3+1)^n = 4^n.$$

(b) $\#(\mathcal{P}X) = 2^n$,

$$\#(\{(b,c) : b \subseteq c \subseteq X\}) = 3^n$$

because to specify b, c we have to say, for each $x \in X$, whether it belongs to b, to $c \setminus b$, or to $X \setminus c$; thus we have three choices for each of n different points;

$$\#(\{(a,b,c) : a \subseteq b \subseteq c \subseteq X\}) = 4^n$$

because now we can choose, for each $x \in X$, whether $x \in a$, $x \in b \setminus a$, $x \in c \setminus b$ or $x \in X \setminus c$.

(c)(i) $\#(\mathcal{P}X) = \sum_{r=0}^{n} \#([X]^r) = \sum_{r=0}^{n} {}^nC_r$ if $\#(X) = n$.

(ii)

$$\begin{aligned}\#(\{(b,c) : b \subseteq c \subseteq X\}) &= \sum_{k=0}^{n}\sum_{j=0}^{k} \#(\{(b,c) : b \subseteq c \subseteq X,\ \#(c) = k,\ \#(b) = j\}) \\ &= \sum_{k=0}^{n}\sum_{j=0}^{k} {}^kC_j \#(\{c : c \subseteq X,\ \#(c) = k\}) \\ &= \sum_{k=0}^{n}\sum_{j=0}^{k} {}^kC_j \ {}^nC_k.\end{aligned}$$

(iii)

$$\begin{aligned}&\#(\{(a,b,c) : a \subseteq b \subseteq c \subseteq X\}) \\ &\qquad= \sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{i=0}^{j} \#(\{(a,b,c) : a \subseteq b \subseteq c \subseteq X,\ \#(c) = k,\ \#(b) = j,\ \#(a) = i\}) \\ &\qquad= \sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{i=0}^{j} {}^jC_i \#(\{(b,c) : b \subseteq c \subseteq X,\ \#(c) = k,\ \#(b) = j\}) \\ &\qquad= \sum_{k=0}^{n}\sum_{j=0}^{k} {}^jC_i \ {}^kC_j \ {}^nC_k.\end{aligned}$$

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2. (a) Assuming for the moment that $n \geq 2$,

$$\#(\{(a,b) : a \subseteq b \subseteq X,\ \#(a) = 2\}) = \tfrac{n(n-1)}{2}.2^{n-2}$$

(counting the number of doubleton sets a, which is ${}^nC_2 = \frac{n(n-1)}{2}$, and for each of them counting the number of possible b, which is 2^{n-2})

$$= \sum_{r=2}^{n} \#(\{(a,b) : a \subseteq b \subseteq X,\ \#(a) = 2,\ \#(b) = r\}$$

$$= \sum_{r=2}^{n} {}^nC_r\ {}^rC_2$$

(counting the number of sets b of size r, and for each of them counting the doubleton subsets)

$$= \sum_{r=2}^{n} \tfrac{r(r-1)}{2}\ {}^nC_r$$

$$= \sum_{r=0}^{n} \tfrac{r(r-1)}{2}\ {}^nC_r.$$

Multiplying by 2, we get

$$\textstyle\sum_{r=0}^{n} r(r-1)\ {}^nC_r = 2^{n-2}n(n-1)$$

as required. If $n = 0$ or $n = 1$ then

$$\textstyle\sum_{r=0}^{n} r(r-1)\ {}^nC_r = 0 = 2^{n-2}n(n-1).$$

(b)

$$\begin{aligned}\sum_{r=0}^{n} r^2\ {}^nC_r &= \sum_{r=0}^{n} r(r-1)\ {}^nC_r + \sum_{r=0}^{n} r\ {}^nC_r \\ &= 2^{n-2}n(n-1) + 2^{n-1}n \\ &= 2^{n-2}n(n+1).\end{aligned}$$

(c) Assuming for the moment that $n \geq 3$,

$$\#(\{(a,b) : a \subseteq b \subseteq X,\ \#(a) = 3\}) = \tfrac{n(n-1)(n-2)}{6}.2^{n-3}$$

(counting the number of triplet sets a, which is ${}^nC_3 = \frac{n(n-1)(n-2)}{6}$, and for each of them counting the number of possible b, which is 2^{n-3})

$$= \sum_{r=3}^{n} \#(\{(a,b) : a \subseteq b \subseteq X,\ \#(a) = 3,\ \#(b) = r\}$$

$$= \sum_{r=3}^{n} {}^nC_r\ {}^rC_3$$

(counting the number of sets b of size r, and for each of them counting the triplet subsets)

$$= \sum_{r=3}^{n} \tfrac{r(r-1)(r-2)}{6}\ {}^nC_r$$

$$= \sum_{r=0}^{n} \tfrac{r(r-1)(r-2)}{6}\ {}^nC_r.$$

Multiplying by 6, we get

$$\textstyle\sum_{r=0}^{n} r(r-1)(r-2)\ {}^nC_r = 2^{n-3}n(n-1)(n-2)$$

as required. If $n \leq 2$ then

$$\textstyle\sum_{r=0}^{n} r(r-1)(r-2)\ {}^nC_r = 0 = 2^{n-3}n(n-1)(n-2).$$

(d)

$$\begin{aligned}\sum_{r=0}^{n} r^3\ {}^nC_r &= \sum_{r=0}^{n} r(r-1)(r-2){}^nC_r + 3\sum_{r=0}^{n} r^2\ {}^nC_r - 2\sum_{r=0}^{n} r\ {}^nC_r \\ &= 2^{n-3}n(n-1)(n-2) + 3.2^{n-2}n(n+1) - 2.2^{n-1}n \\ &= 2^{n-3}n^2(n+3).\end{aligned}$$

3. If $\#(a) = j$, then

$$\#(\{b : a \subseteq b \subseteq X,\ \#(b) = k\}) = \#(\{c : c \subseteq X \setminus a,\ \#(c) = k-j\} = {}^{n-j}C_{k-j};$$

so

$$\#(R) = \textstyle\sum_a \#(\{b : (a,b) \in R\}) = {}^nC_j\ {}^{n-j}C_{k-j} = {}^nC_j\ {}^{n-j}C_{n-k}.$$

If $\#(b) = k\}$, then

$$\#(\{a : a \subseteq b,\ \#(a) = j\}) = {}^kC_j,$$

so

$$\#(R) = \textstyle\sum_b \#(\{a : (a,b) \in R\}) = {}^nC_k\ {}^kC_j.$$

4. (a) $q_0 = 1$, $q_1 = 1$, $q_2 = 1$, $q_3 = 2$. In general,

$$\mathcal{R}\mathrm{e}(1+\omega)^n = \mathcal{R}\mathrm{e}(\textstyle\sum_{k=0}^n {}^nC_k\omega^k) = \sum_{k=0}^n {}^nC_k\,\mathcal{R}\mathrm{e}\,\omega^k.$$

Now if k is divisible by 3, $\mathcal{R}\mathrm{e}(\omega^k) = \omega^k = 1$, while for any other k $\mathcal{R}\mathrm{e}(\omega^k) = -\frac{1}{2}$. So in the sum we get a contribution 1 from each set with a multiple of 3 members, and $-\frac{1}{2}$ from any other; that is, a contribution 1 from each of q_n sets, and a contribution $-\frac{1}{2}$ from each of $2^n - q_n$ sets. Putting these together,

$$\mathcal{R}\mathrm{e}(1+\omega)^n = q_n - \frac{1}{2}(2^n - q_n) = \frac{3}{2}q_n - 2^{n-1},$$

and

$$q_n = \frac{1}{3}(2^n + 2\,\mathcal{R}\mathrm{e}(1+\omega)^n).$$

(b) $q'_0 = q'_1 = q'_2 = q'_3 = 1$. In general,

$$\mathcal{R}\mathrm{e}(1+i)^n = \mathcal{R}\mathrm{e}(\textstyle\sum_{k=0}^n {}^nC_k i^k) = \sum_{k=0}^n {}^nC_k\,\mathcal{R}\mathrm{e}\,i^k.$$

Now if k is divisible by 4, $\mathcal{R}\mathrm{e}(i^k) = i^k = 1$; if k is divisible by 2 but not by 4, $\mathcal{R}\mathrm{e}\,i^k = -1$; while if k is odd, $\mathcal{R}\mathrm{e}\,i^k = 0$. So in the sum we get a contribution 1 from each set with a multiple of 4 members, and -1 from any other set with an even number of members; that is, a contribution 1 from each of q'_n sets, and a contribution -1 from each of $2^{n-1} - q'_n$ sets, because just half the sets have an even number of members. Putting these together,

$$\mathcal{R}\mathrm{e}(1+i)^n = q'_n - (2^{n-1} - q'_n) = 2q_n - 2^{n-1},$$

and

$$q'_n = \frac{1}{4}(2^n + 2\,\mathcal{R}\mathrm{e}(1+i)^n).$$

5. (No answer written out.)

6. (a) $s_1 = 0$, $s_2 = 1$, $s_3 = 2$, $s_4 = 9$, $s_5 = 44$.

(b) In general, if we have $n + 2$ cards, there are $n + 1$ possible envelopes in which to put the first card. Now look at the card belonging to the envelope in which we put that first card. (α) We might put it into the envelope belonging to the first card. In that case we have n remaining cards to fit into envelopes, and there are s_n ways of getting them all wrong. (β) If we want to do anything else, it's as if we had matched that second card to that first envelope, and declared that on that matching we had to get all the $n + 1$ remaining cards wrong. So there will be s_{n+1} ways of doing it.
Thus $s_{n+2} = (n + 1)(s_n + s_{n+1})$.

(c) $\alpha_{n+2} = \frac{1}{(n+2)!}s_{n+2} = \frac{1}{(n+2)!}(n + 1)(s_n + s_{n+1}) = \frac{1}{(n+2)!}(n + 1)(n!\alpha_n + (n + 1)!\alpha_{n+1}) = \frac{1}{n+2}\alpha_n + \frac{n+1}{n+2}\alpha_{n+1}$.

(d) Induce on n. $\alpha_1 = s_1 = 0 = 1 - 1$, $\alpha_2 = \frac{1}{2}s_2 = \frac{1}{2} = 1 - 1 + \frac{1}{2}$. For the step to $n \geq 3$, we have

$$\alpha_n = \frac{1}{n}\alpha_{n-2} + \frac{n-1}{n}\alpha_{n-1}$$
$$= \frac{1}{n}\sum_{k=0}^{n-2}\frac{(-1)^k}{k!} + \frac{n-1}{n}\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}$$

(by the inductive hypothesis)

$$= \sum_{k=0}^{n-2}\frac{(-1)^k}{k!} + \frac{n-1}{n}\frac{(-1)^{n-1}}{(n-1)!}$$
$$= \sum_{k=0}^{n-2}\frac{(-1)^k}{k!} + \frac{(-1)^{n-1}}{(n-1)!} - \frac{1}{n}\frac{(-1)^{n-1}}{(n-1)!}$$
$$= \sum_{k=0}^{n}\frac{(-1)^k}{k!}.$$

Consequently

$$\lim_{n\to\infty}\alpha_n = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!} = \exp(-1) = \frac{1}{e}.$$

7. (No answer written out.)

8. (No answer written out.)

9. (a)(i) The **order** of G is $\#(V)$.

(ii) The **complement** $\overline{G}$ of G is the graph $(V, \overline{E})$ where $\overline{E}$ consists of just those edges between elements of V which are not in E.

(iii) A **path** in G is a string $(\, v_0 \quad v_1 \quad \dots \quad v_n \,)$ of vertices such that v_i, v_{i+1} are linked by an edge for every $i < n$.

(iv) A path $(\, v_0 \quad \dots \quad v_n \,)$ is **simple** if $v_i \neq v_j$ whenever $0 \leq i < j \leq n$, except that we allow $v_0 = v_n$.

(v) A path $(\, v_0 \quad \dots \quad v_n \,)$ is **closed** if $v_0 = v_n$.

(vi) A **cycle** in G is a simple closed path of length at least 3.

(vii) A **component** in G is a non-empty set $C \subseteq V$ such that (α) any two vertices in C can be joined by a path (β) no member of C is linked to any member of $V \setminus C$ by an edge.

(viii) G is **connected** if any two vertices in V can be joined by a path.

(ix) The **degree** of a vertex $v \in V$ is the number of vertices liked to v by an edge.

(x) G is a **tree** if it is connected and has no cycles.

(b) Start from any $v_0 \in V$; there is one because $V \neq \emptyset$. Because $d(v_0) > 0$ there is a $v_1 \in V$ such that v_0 is linked to v_0 by an edge. For $n \geq 1$, given that we have chosen v_{n-1} and v_n, let $v_{n+1} \in V$ be a vertex, other than v_{n-1}, linked to v_n by an edge; such exists because $d(v_n) \geq 2$. In this way we obtain a sequence $v_0, v_1, v_2, v_3, \dots$ such that v_i is linked by an edge to v_{i+1} for every i, but $v_i \neq v_{i+2}$ for every i.

Because V is finite, the v_i cannot all be distinct. Let v_j be the first which is equal to some preceding one. Then $v_0, \dots, v_{j-1}$ are all distinct and $v_j = v_i$ for some $i < j$. Now v_{i+1}, v_{i+2} are both distinct from v_i, so $j \geq i+3$ and $(\, v_i \quad \dots \quad v_{j-1} \quad v_j = v_i \,)$ is a cycle.

(c) Count the set

$$R = \{(v, e) : v \in V,\, e \in E,\, v \text{ is one end of } e\}.$$

Then

$$\begin{aligned}
\sum_{v \in V} d(v) &= \sum_{v \in V} \#(\{e : (v, e) \in R\}) \\
&= \#(R) \\
&= \sum_{e \in E} \#(\{v : (v, e) \in R\}) \\
&= 2\#(E).
\end{aligned}$$

(d) Induce on $\#(V)$. If $\#(V) = 1$ then of course $\#(E) = 0 < \#(V)$. For the inductive step to $\#(V) = n > 1$, if G has no cycle then there must be some $v \in V$ such that $d(v) \leq 1$, by (b). Consider the graph $G' = G - v$ with vertex set $V' = V \setminus \{v\}$ and with all the edges of G except that one (if there is one) ending at v. Then surely G' has no cycle, while its order $\#(V')$ is $n - 1$, so by the inductive hypothesis $\#(E') < \#(V')$, where E' is the set of edges of G'. But now

$$\#(E) \leq \#(E') + 1 < \#(V') + 1 = \#(V).$$

So the induction proceeds.

(e) Induce on $\#(E)$. If $\#(E)=0$ then G can have at most one vertex, so $\#(V)\le 1=\#(E)+1$. For the inductive step to $\#(E)=m\ge 1$, consider two cases. (i) If G has a cycle, $(\,v_0 \quad \dots \quad v_n\,)$, delete the edge between v_0 and v_1 to form a graph $G'=(V,E')$ with $\#(E')<\#(E)$. But G' is still connected. **P** If $v\in V$, there is a simple path from v to v_0 in G. If this is not itself a path in G', that's because it contains the link $(\,v_1 \quad v_0\,)$. But now we can replace this link by the path $(\,v_1 \quad \dots \quad v_n=v_0\,)$, which is a path in G'. Thus every vertex of G' is joined to v_0 by a path in G', and G' is connected. **Q** By the inductive hypothesis,

$$\#(V)\le\#(E')+1\le\#(E)+1.$$

(ii) If G has no cycle, then by (b) there is a $v\in G$ such that $d(v)\le 1$. Of course v is not the only vertex of G, because G has an edge, so $d(v)=1$, because v must be joined to another vertex by a path. As in (d), let $G'=(V',E')$ be the graph $G-v$, so that $\#(V')=\#(V)-1$ and $\#(E')=m-1$. Now again G' is connected. **P** If u, $w\in V'$ then they are joined by a simple path in G. But this simple path cannot pass through v, because $d(v)=1$, so any path using v must use its unique neighbour twice. So it is a path in G'. **Q** By the inductive hupothesis,

$$\#(V)=\#(V')+1\le\#(E')+2=\#(E)+1.$$

Thus in both cases the induction proceeds.

(f)(i)$\Rightarrow$(ii)+(iii) If G is a tree then $\#(E)<\#(V)$, by (d), and also $\#(E)\ge\#(V)-1$, by (e); so $\#(E)=\#(V)-1$ and both (ii) and (iii) are true.

(ii)$\Rightarrow$(i) Induce on $\#(E)$. If $\#(E)=0$ then of course G has no cycle and is a tree. For the inductive step to $\#(E)=m\ge 1$, **?** suppose, if possible, that G has a cycle $(\,v_0 \quad \dots \quad v_n\,)$. As in (e) above, consider the graph $G'=(V,E')$ obtained by deleting this edge. This is still connected, and $\#(V)\ge\#(E')+1$. But $\#(E')<m$, so by the inductive hypothesis G' is a tree and $\#(V)=\#(E')+1=\#(E)$, by the result just above; contradicting the hypothesis that $\#(V)\ge\#(E)+1$. **X** Thus the induction proceeds.

(iii)$\Rightarrow$(i) Induce on $\#(V)$. If $\#(V)=1$ then of course G is a tree. For the inductive step to $\#(V)=n>1$, we know from (b) that G has a vertex v of degree at most 1. Consider the graph $G'=G-v=(V',E')$, as in (d) and (e). Of course G' has no cycle, while

$$\#(V')=\#(V)-1\le\#(E)\le\#(E')+1;$$

so by the inductive hypothesis, G' is a tree and $\#(V')=\#(E')+1$. But this means that $\#(E)$ must be exactly $\#(E')+1$, so $d(v)=1$ in G. Now if u, $w\in V\setminus\{v\}=V'$, u and w are joined by a path in G', and therefore in G; while v is linked to some member of V' by the edge starting at v, so is joined to every member of V in G. Thus G also is connected, and the induction proceeds.

10. **case 1** No edges, 5 components; 1 graph.

(No diagram available.)

case 2 1 edge, 4 components; ${}^5C_2 = 10$ graphs.

(No diagram available.)

case 3 2 edges, 3 components.
case 3a The edges share a vertex: $5.{}^4C_2 = 30$ graphs.

(No diagram available.)

case 3b The edges don't share a vertex: $5.{}^4C_2/2 = 15$ graphs.

(No diagram available.)

(That makes $45 =^{10} C_2$ graphs with two edges.)
case 4 3 edges, 3 components; the edges must be in a triangle; ${}^5C_2 = 10$ graphs.

(No diagram available.)

case 5 3 edges, 2 components.
case 5a One of the components is a singleton. In this case the remaining four vertices form a connected graph with three edges, which must be a tree.
case 5a(i) The tree is a simple path. $5!/2 = 60$ graphs. (We can take the path in either direction.)

(No diagram available.)

case 5a(ii) The tree has a central division. $5.4 = 20$ graphs.

(No diagram available.)

case 5b One of the components is a doubleton, the other a triplet. 5 choices of the central vertex of the triple, ${}^4C_2$ choices of its companions, making $5.6 = 30$ graphs.

(No diagram available.)

(That makes $120 =^{10} C_3$ graphs with 3 edges.)
case 6 4 edges, 2 components.
case 6a One of the components is a singleton. In this case the other has 4 edges and 4 vertices.
case 6a(i) The 4-vertex component is a square. $5.3 = 15$ graphs. (Having picked the singleton, take any other vertex; there are three choices of the *opposite* vertex of the square.)

(No diagram available.)

case 6a(ii) The 4-vertex component is a triangle with a tag on. $5.4.3 = 60$ graphs.

(No diagram available.)

case 6b One component has 2 vertices and the other has 3. In this case they form a dumbbell and a triangle. ${}^5C_2 = 10$ graphs.

(No diagram available.)

case 7 4 edges, 1 component. This must be a tree.
case 7a A simple path. $5!/2 = 60$ graphs.

(No diagram available.)

case 7b A vertex of degree 3. $5.4.3 = 60$ graphs. (5 choices for the vertex of degree 3, 4 for its neighbour on the long arm, 3 for the end of the long arm.)

(No diagram available.)

case 7c A vertex of degree 4. 5 graphs.

(No diagram available.)

(That makes $210 = ^{10}C_4$ graphs with 4 edges.)
case 8 5 edges, 2 components. One component must be a singleton and the other is K_4 with one edge missing. $5.6 = 30$ graphs.

(No diagram available.)

case 9 5 edges, 1 component. This must have a cycle in it.
case 9a A simple cycle of length 5. $5!/10 = 12$ graphs. (Take all lists of the 5 vertices; divide by 5 because we can start anywhere along the cycle; divide by 2 because we can go in either direction.)

(No diagram available.)

case 9b A cycle of length 4 with one vertex off the side. $5.4.3 = 60$ graphs. (5 choices for the vertex at the side, 4 for its neighbour, 3 for the vertex opposite the neighbour.)

(No diagram available.)

case 9c A 3-cycle:
case 9c(i) The two other vertices attached to different points of the cycle. $^5C_2$ choices of the odd vertices, 3! choices of the others, $10.6 = 60$ graphs.

(No diagram available.)

case 9c(ii) The two other vertices attached in line. $5.4.3 = 60$ graphs.

(No diagram available.)

case 9c(iii) The two other vertices attached at the same point of the triangle. $5.^4C_2 = 30$ graphs.

(No diagram available.)

(This makes $252 = ^{10}C_5$ graphs with 5 edges.)
The rest are all complements of graphs with 4 or fewer edges, and may therefore be counted out from the list above. Spelling them out, we have
case 1' 10 edges, K_5; 1 graph.

(No diagram available.)

case 2' 9 edges; 10 graphs.

(No diagram available.)

case 3' 8 edges.
case 3a' The missing edges share a vertex: 30 graphs.

(No diagram available.)

case 3b' The missing edges don't share a vertex: 15 graphs.

(No diagram available.)

case 4' 7 edges, the missing edges in a triangle; that is, we have two vertices joined to an edge and the other three joined just to thos two; 10 graphs.

(No diagram available.)

case 5a(i)' 7 edges, the three missing ones forming a simple path; that is, we have a 5-cycle with two extra edges meeting at a vertex of the cycle. 60 graphs.

(No diagram available.)

case 5a(ii)' 7 edges, the three missing ones meeting at a vertex; that is, we have a K_4 with one extra vertex attached. 20 graphs.

(No diagram available.)

case 5b' 7 edges, two of the three missing edges meeting at a vertex, the other disjoint from both; that is, we have a 5-cycle with two extra edges not meeting at a vertex. 30 graphs.

(No diagram available.)

case 6a(i)' 6 edges, two triangles meeting at a point. 15 graphs.

(No diagram available.)

case 6a(ii)' 6 edges, a square with a diagonal and a tag attached to one of the ends of the diagonal. 60 graphs.

(No diagram available.)

case 6b' 6 edges, $K_{2,3}$. 10 graphs.

(No diagram available.)

case 7a' 6 edges, a 5-cycle with one extra edge. 60 graphs.

(No diagram available.)

case 7b' 6 edges, a square with a diagonal and a tag attached to one of the other vertices. 60 graphs.

(No diagram available.)

case 7c' 6 edges, K_4 and a singleton. 5 graphs.

(No diagram available.)

11. **case 1** A simple path. $6!/2 = 360$ trees.
case 2 A Y with a long stem. $6!/2 = 360$ trees.
case 3 A Y with a short stem. $6!/2 = 360$ trees.
case 4 A cross with one long arm. $6!/3 = 240$ trees.
case 5 An H. $6!/4 = 180$ trees.
(Total 1860 trees.)

12. (No answer written out.)

13. (No answer written out.)

14. Write $n = \#(V)$, $k = \lfloor \frac{1}{2}n \rfloor$, so that n is either $2k$ or $2k+1$; because G has an edge, $n \geq 2$ and $k \geq 1$. Given that e is an edge of G, the probability that e is an edge of G_W is the probability that both ends of e are in W, that is,

$$^{n-2}C_{k-2} /\ ^{n}C_k = \frac{(n-2)!}{(k-2)!(n-k)!} \frac{k!(n-k)!}{n!} = \frac{k(k-1)}{n(n-1)}.$$

Similarly, the probability that e is an edge of $G_{V \setminus W}$ is

$$^{n-2}C_{k} /\ ^{n}C_k = \frac{(n-2)!}{k!(n-k-2)!} \frac{k!(n-k)!}{n!} = \frac{(n-k)(n-k-1)}{n(n-1)}.$$

So the probability that e is an edge of one or the other is

$$\begin{aligned} \frac{1}{n(n-1)}(k(k-1) + (n-k)(n-k-1)) &= \frac{2k(k-1)}{2k(2k-1)} = \frac{k-1}{2k-1} \text{ if } n = 2k, \\ &= \frac{2k^2}{2k(2k+1)} = \frac{k}{2k+1} \text{ if } n = 2k+1. \end{aligned}$$

Now this means that the expectation of $e(G_W) + e(G_{V \setminus W})$ is either $\frac{k-1}{2k-1}e(G)$ or $\frac{k}{2k+1}e(G)$, and in either case is less than $\frac{1}{2}e(G)$. There must therefore be at least one W such that $e(G_W) + e(G_{V \setminus W}) < \frac{1}{2}e(G)$.

(This argument assumes implicitly that $k \geq 2$. But if $k = 1$ then n is either 2 or 3. If $n = 2$ we take W a singleton and get $e(G_W) + e(G_{V \setminus W}) = 0 < \frac{1}{2}e(G)$. If $n = 3$, then if G has 1 or 2 edges we can again pick W so that $e(G_W) + e(G_{V \setminus W}) = 0 < \frac{1}{2}e(G)$, while if G has 3 edges we can pick W sush that $e(G_W) + e(G_{V \setminus W}) = 1 < \frac{1}{2}e(G)$.)

15. **?** Suppose, if possible, that the component C of G containing v_n was not the whole of V. Set $k = \#(V \setminus C) = n - \#(C)$. Of course C contains every vertex linked to v_n by an edge, so $\#(C) \geq 1 + d(v_n)$ and $k \leq n - 1 - d(v_n)$. Take the largest j such that $v_j \in V \setminus C$. Then $j \geq k$, so $d(v_j) \geq d(v_k) \geq k$. But this means that the component of G containing v_j has at least $k+1$ members, which is impossible, because it is included in $V \setminus C$. **X**

16. If G is not connected, let C be a component of G. Then for any $v \in C$, $w \in V \setminus C$ there is no edge from v to w in G, so there is an edge from v to w in $\overline{G}$. But this means that for any u, $u' \in V$ there is a path of length at most 2 from u to u' in $\overline{G}$, and $\overline{G}$ must be connected.

17. (a) An **Eulerian path** in a graph is a path which uses every edge exactly once.

(b)

(No diagram available.)

(No diagram available.)

(No diagram available.)

(c) Let V be the common vertex set of G and $\overline{G}$; set $n = \#(V)$, so that n is odd. For each $v \in V$, $d_{\overline{G}}(v) = n - 1 - d_G(v)$ is even iff $d_G(v)$ is even. Because G has an Eulerian path, at most two vertices have odd degree in G; consequently at most two vertices have odd degree in $\overline{G}$; because $\overline{G}$ is connected, it has an Eulerian path.

18. (a) For any graph $G=(V,E)$, a *colouring* of G is a function $f:V\to C$, where C is a set of 'colours', such that $f(u)\neq f(v)$ whenever u, $v\in V$ are linked by an edge.

The *colouring number* of G is the smallest size of any set C such that there is a colouring $f:V\to C$.

If V is listed in order as $\{v_1,\dots,v_n\}$ then the *greedy algorithm* for colouring G (corresponding to that ordering of the vertices) is the function $f:V\to\mathbb{N}$ which sets $f(v_1)=1$ and generally, for every $j\le n$, sets

$$f(v_j)=\min\{k:k\in\mathbb{N},\ k\neq f(v_i)\text{ whenever }i<j\text{ and }v_i\text{ is linked to }v_j\text{ by an edge}\},$$

that is, v_j is coloured with the first colour which is not forbidden by the previous assignments of $f(v_i)$ for $i<j$.

(b) We have a colouring with χG colours; number the colours $1,\dots,\chi G$, so that the colouring is a function $g:V\to\{1,\dots,\chi G\}$. Order the vertices as $v_1,\dots,v_n$ in such a way that $g(v_1)\le g(v_2)\le\dots\le g(v_n)$. Let $f:V\to\mathbb{N}$ be the corresponding greedy algorithm. Then I claim that $f(v_k)\le g(v_k)$ for every $k\le n$. **P** Of course $f(v_1)=1\le g(v_1)$, so the induction starts. For the inductive step to $k>1$, if $i<k$ and v_i is linked to v_k by an edge then $f(v_i)\le g(v_i)\le g(v_k)$; but also $g(v_i)\neq g(v_k)$, because g is a colouring, so $g(v_i)<g(v_k)$ and $f(v_i)<g(v_k)$. This means that $g(v_k)$ is a permitted colour for v_k under the colouring f. Since $f(v_k)$ is the *first* permitted colour for v_k, $f(v_k)\le g(v_k)$ and the induction proceeds.

(c) Suppose that G has six vertices and six edges, arranged in a hexagon. Then $\chi G=2$, since we can colour alternate vertices red and the other three vertices green. But if we put the vertices in the wrong order, so that v_1 and v_2 are opposite each other (and this is allowed by the rule $d(v_1)\ge d(v_2)\ge\dots$, since $d(v)=2$ for every v) then the greedy algorithm will assign the same colour to both, and now there is no way of colouring the rest with only two colours in all.

19. (No answer written out.)

20. (No answer written out.)

21. (i) Yes.

(No diagram available.)

(ii) Yes.

(No diagram available.)

(iii) Yes, as shown by either of the above.

(iv) No. If we start from the centre, then the assignment of colours to the central vertex, and one of its neighbours, is arbitrary (except that they must have different colours), but then the other three vertices of the central diamond have colours forced by the first two assignments. Now we have two choices left for the colour of the vertex a; if we give it the same colour as the central vertex, we get the diagram of (i), forced by looking at the other outer vertices clockwise round from a; if we give a the other allowed colour, and trace round the outer diamond anticlockwise, we get the pattern in (ii). In either case c is given the same colour as a.

So we have three choices for the colour of the central vertex, two for one of its neighbours, and two for the vertex a, making 12 possibilities in all.

22.

(No diagram available.)

23. (No answer written out.)

24. (No answer written out.)

25. (No answer written out.)

26. (No answer written out.)

27. (No answer written out.)

28. 30+60+90+60=240.

29. **case 1** K_6; one graph.

case 2 The graph includes K_5 but not K_6, that is, there is a vertex with some of the edges to that vertex missing, but all other edges are present.

case 2a One edge missing; ${}^6C_2 = 15$ graphs.

case 2b Two edges missing; $6.{}^5C_2 = 60$ graphs.

case 2c Three edges missing; $6.{}^5C_3 = 60$ graphs.

case 2d Four edges missing; $6.{}^5C_4 = 30$ graphs.

case 2e Five edges missing; 6 graphs.

case 3 The graph includes $K_{3,3}$ but not K_5.

case 3a Two edges are missing, not meeting at the same vertex; ${}^6C_2.{}^4C_2/2 = 45$ graphs.

case 3b Three edges are missing. In this case, because we have $K_{3,3}$ in the graph, these three edges cannot be of the form $(a \quad b), (c \quad d), (e \quad f)$; nor can they be in a path $(a \quad b \quad c \quad d)$. Nor can they share a vertex, since that is case 2c above. So we have two types:

case 3b(i) $(a \quad b \quad c)$ and $(d \quad e)$; 15 choices for $(d \quad e)$, 4 choices for the missing vertex, 3 for the middle one of $(a \quad b \quad c)$, making 180 graphs.

case 3b(ii) $(a \quad b \quad c \quad a)$. ${}^6C_3 = 20$ choices of the set $\{a, b, c\}$, so 20 graphs.

case 3c Four edges missing. In the 3-3 split called for by $K_{3,3}$, there isn't room for all the edges to be missing on the same side, so we have two possibilities:

case 3c(i) Three edges missing on one side, one on the other; ${}^6C_3 = 20$ choices of the triplet in which all internal edges are missing, 3 choices of the remaining edge, making 60 graphs.

case 3c(ii) Two edges missing on each side. ${}^6C_2$ choices of the two vertices each missing two edges, then ${}^4C_2$ choices of the vertices to join them on each side, making $15.6 = 90$ graphs.

case 3d Five edges missing; must be three on one side, two on the other; ${}^6C_3$ choices of the side on which all internal edges are missing, then 3 choices of the edge to leave in on the other side, making $20.3 = 60$ graphs.

case 3e Six edges missing; ${}^6C_3/2 = 10$ graphs. (Divide by 2 because you can't tell which side is which.)

case 4 Now suppose that neither K_5 nor $K_{3,3}$ is a subgraph. The only way in which the graph can be non-planar is to have K_5 topologically embedded in it; that is, the graph is of the form K_5 with an extra vertex embedded in the edge. Now any additional edge will produce a copy of $K_{3,3}$ or K_5. So we have 6 choices for the 'interpolated' vertex and ${}^5C_2$ choices for its two neighbours, making $6.10 = 90$ graphs.

30. (No answer written out.)

31. (No answer written out.)

32. (No answer written out.)

33. (No answer written out.)

34. (No answer written out.)

35. (No answer written out.)

36. (No answer written out.)

37. (No answer written out.)

38. (No answer written out.)

39. (No answer written out.)

40. (No answer written out.)

41. (No answer written out.)

42. (No answer written out.)

43. (No answer written out.)

44. (a) $P_1, \ldots, P_n$ are *independent* if no vertex, other than a and b, is used by more than one of them

(b) S *separates* a from b in $G = (V, E)$ if $S \subseteq V \setminus \{a, b\}$ and every path from a to b in G passes through a vertex in S.

(c) *Menger's theorem*: in any finite graph $G = (V, E)$, if a and b are distinct vertices not linked by an edge, then $s = p$, where

$$s = \min\{\#(S) : S \subseteq V \setminus \{a, b\} \text{ separates } a \text{ from } b\}$$

$$p = \max\{k : \text{there are independent simple paths } P_1, \ldots, P_k \text{ from } a \text{ to } b \text{ in } G\}.$$

(d)

(No diagram available.)