Real Analysis Problems

1. (a) State the triangle inequality for real numbers.
(b) Use it to show that, for all real numbers x and y (i) $|x-y| \le |x|+|y|$ (ii) $|x|-|y| \le |x-y|$ (iii) $|x|-|x-y| \le |y|$ (iv) $||x|-|y|| \le |x-y|$. [AA]

2. (a) For a set $A \subseteq \mathbb{R}$, what is meant by (i) an upper bound of A (ii) $\sup A$?
(b) Give examples of sets A for which (i) $\sup A$ exists and belongs to A (ii) $\sup A$ exists and does not belong to A (iii) $\sup A$ does not exist.
(c) For sets $A,\, B \subseteq \mathbb{R}$ write $A+B=\{a+b : a \in A,\, b \in B\}$.
(i) Find $[0,1]+[-1,2]$, $[1,2]+[0,1[$, $]0,1[+]0,1[$. (Hint: they are all intervals; but you have to decide whether they are open or closed.)
(ii) Suppose that $A,\, B \subseteq \mathbb{R}$ have upper bounds a, b respectively. Show that $a+b$ is an upper bound of $A+B$.
(iii) Suppose that $A,\, B \subseteq \mathbb{R}$ and that c is an upper bound of $A+B$. Show that $x \le c-y$ whenever $x \in A$ and $y \in B$.
(iv) Suppose that $A,\, B \subseteq \mathbb{R}$ have suprema a, b respectively, and that c is an upper bound of $A+B$. Show that $a \le c-y$ for every $y \in B$, and therefore that $c-a$ is an upper bound of B.
(v) Show that if $A,\, B \subseteq \mathbb{R}$ have suprema a, b respectively, then $a+b$ is the least upper bound of $A+B$. [AB]

3. Which of the following are equivalent to the sentence '$a = \sup A$'? (You may take it for granted that $a \in \mathbb{R}$ and $A \subseteq \mathbb{R}$.)
(i) For every $y \in \mathbb{R}$, $y \ge a$ whenever y is an upper bound of A.
(ii) For every $y \in \mathbb{R}$, $y \in [a,\infty[$ iff y is an upper bound of A.
(iii) For every $y \in \mathbb{R}$, y is an upper bound of A whenever $y \ge a$.
(iv) A is not empty and for all $x \in A$, $y \in \mathbb{R}$, $x \le y$ iff $a \le y$.
(v) $\forall\, y \in \mathbb{R},\ a \le y \iff \forall x \in A,\ x \le y$.
For those which are *not* equivalent, give examples to show that they are not. [AC]

4. Find negations of the following sentences. (In each case, move any quantifiers '$\forall$' and '$\exists$' as far to the left as possible, and any negative assertions $\notin$, $\not\le$ as far to the right as you can; do not use $\not\forall$ and $\not\exists$. You may suppose throughout that A and B are subsets of $\mathbb{R}$.)

(i) $\forall\, x \in A,\ x \le 2$	(ii) $\exists\, x \in A,\ x > 0$
(iii) $\forall\, x \in A\ \forall\, y \in B,\ x \le y$	(iv) $\exists\, x \in A\ \exists\, y \in B,\ x \le y$
(v) $\forall\, x \in A\ \exists\, y \in B,\ x+y > 0$	(vi) $\exists\, x \in A\ \forall\, y \in B,\ x+y > 0$

[AD]

5. One of the following is true, and the other is false. Which is which? Justify your answer.
(i) if, for every $x \in A$, there is a $y \in B$ such that $x \le y$, then there is a $y \in B$ such that $x \le y$ for every $x \in A$.
(ii) if there is a $y \in B$ such that $x \le y$ for every $x \in A$, then, for every $x \in A$, there is a $y \in B$ such that $x \le y$. [AE]

6. Which of the following are true, and which are false?

(a) $\pi = 3.142$	(e) $-\pi > -3.142$
(b) $\pi \ge 3.142$	(f) if $a \le b$ and $c \le d$ then $a+c \le b+d$
(c) $\pi \le 3.142$	(g) if $a \le b$ and $c \le d$ then $a-d \le b-c$
(d) $-\pi < -3.142$	(h) if $a < b$ and $c < d$ then $a+c \le b+d$.

[AF]

7. In each of these possible positions in noughts-and-crosses, the X player moved first, and now must move again. In each case say what the result of the game will be if both players play correctly, and copy the position out with a further correct move by X.

*	*	O	draw
*	X	*	
*	*	*	

*	O	*	X wins
*	X	*	
*		*	

O		X	X wins
		*	
*		*	

*		X	X wins
O		*	

*	X		X wins
O			

[AG]

8. (a) In the Analysis Game, the first player says '$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$'. Who plays next? If both play correctly, will he win? If so, give a possible winning move.

(b) In the Analysis Game, the first player says '$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0{\cdot}01$'. Who plays next? If both play correctly, will he win? If so, give a possible winning move.

(c) In the Analysis Game, the first player says '$\lim_{x\to\infty} \sin x = 0$'. Who plays next? If both play correctly, will he win? If so, give a possible winning move. (You may take it for granted that sin behaves as you have been taught that it does.)

(d) In the Analysis Game, the first player says '$\langle \frac{n+1}{n} \rangle_{n\geq 1}$ converges'. Who plays next? If both play correctly, will he win? If so, give a possible winning move.

[BA]

9. For a real function f and real numbers a and b, what is meant by saying that $\lim_{x\downarrow a} f(x) = b$? Show from your definition that if $\lim_{x\downarrow a} f(x) = b$ and $\lim_{x\downarrow a} g(x) = c$ then $\lim_{x\downarrow a} f(x) + g(x) = b + c$.
[BB]

10. For a real function f and real numbers a and b, what is meant by saying that $\lim_{x\downarrow a} f(x) = b$? Show from your definition that if $\lim_{x\downarrow a} f(x) = b$ and $\lim_{x\downarrow a} g(x) = c$ then $\lim_{x\downarrow a} f(x)g(x) = bc$.
[BCA]

11. For a real function f and real numbers a and b, what is meant by saying that $\lim_{x\to a} f(x) = b$? Show from your definition that if $\lim_{x\to a} f(x) = b$ and $\lim_{x\to a} g(x) = c$ then $\lim_{x\to a} f(x)g(x) = bc$.
[BC]

12. Can you find a simple strategy which will give a correct move in noughts-and-crosses from any position?
[BDA]

13. Consider the following strategy for playing noughts-and-crosses.

If you have an open pair (that is, two of your symbols in a line with the third place empty), complete the pair to win

else if your opponent has one or more open pairs, block one (chosen at random)

else if the centre square is empty, fill it

else if one or more corners is empty, play in one (chosen at random)

else play anywhere

fi fi fi fi

(a) If you are first player and follow this strategy, will you always play correctly?

(b) If you are first player and you know your opponent will follow this strategy, can you be sure of beating him?
[BD]

14. Put the following nine sentences in such an order that they form a correct proof of the theorem that if $\langle x_n\rangle_{n\in\mathbb{N}} \to b$ and $\langle x_n\rangle_{n\in\mathbb{N}} \to c$, then $b = c$.

If $\epsilon = 0$ then $|b-c| = 0$ and $b = c$, so we can stop.
Also there is an $n_2 \in \mathbb{N}$ such that $|x_n - c| \le \epsilon$ for every $n \ge n_2$.
Set $\epsilon = \frac{1}{3}|b-c|$, so that $\epsilon \ge 0$.
Because $n_0 \ge n_1$, $|x_{n_0} - b| \le \epsilon$.
So $|b-c| \le |x_{n_0} - b| + |x_{n_0} - c| \le 2\epsilon = \frac{2}{3}|b-c|$.
Because $n_0 \ge n_2$, $|x_{n_0} - c| \le \epsilon$.
If $\epsilon > 0$, then there is an $n_1 \in \mathbb{N}$ such that $|x_n - b| \le \epsilon$ for every $n \ge n_1$.
Therefore $\frac{1}{3}|b-c| \le 0$ and $b = c$ in this case also.
Set $n_0 = \max(n_1, n_2)$.

(Collaboration on this question is strongly recommended.) [BE]

15. Show that if f and g are real functions and $\lim_{x\to a} g(x) = b$ and $\lim_{y\to b} f(y) = f(b) = c$ (so that f is defined near b and continuous at b), then $\lim_{x\to a} f(g(x)) = c$. [BF]

16. Show that if $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence converging to b, and f is a real function such that $\lim_{y\to b} f(y) = f(b) = c$, then $\lim_{n\to\infty} f(x_n) = c$. [CA]

17. A sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ is *non-increasing* if $x_{n+1} \le x_n$ for every $n \in \mathbb{N}$. *Without* quoting any theorem concerning non-decreasing sequences, show that a bounded non-increasing sequence is convergent. [CB]

18. (a) Show by induction on n that $(1+\eta)^n \ge 1 + n\eta$ for every $n \in \mathbb{N}$ and $\eta \ge -1$.

(b) Show that $\lim_{n\to\infty} x^n = 0$ whenever $|x| < 1$. (*Hint*: given $\epsilon > 0$, take $n_0 \ge \frac{1}{\epsilon(1-|x|)}$.)

(c) Show that $\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$ for every $n \in \mathbb{N}$ and every $x \in \mathbb{R} \setminus \{1\}$.

(d) Show that $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ whenever $|x| < 1$. [CC]

19. (a) Show that if $\langle x_n\rangle_{n\in\mathbb{N}}$ and $\langle y_n\rangle_{n\in\mathbb{N}}$ are bounded sequences, then $\langle x_n + y_n\rangle_{n\in\mathbb{N}}$ and $\langle x_n y_n\rangle_{n\in\mathbb{N}}$ are bounded.

(b) Give an example of a bounded sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ such that no x_n is zero and $\langle \frac{1}{x_n}\rangle_{n\in\mathbb{N}}$ is not bounded.

(c) Show that if $\langle x_n\rangle_{n\in\mathbb{N}}$ and $\langle y_n\rangle_{n\in\mathbb{N}}$ are non-decreasing sequences then $\langle x_n + y_n\rangle_{n\in\mathbb{N}}$ is non-decreasing.

(d) Show that if $\langle x_n\rangle_{n\in\mathbb{N}}$ and $\langle y_n\rangle_{n\in\mathbb{N}}$ are non-decreasing sequences and $x_n \ge 0$, $y_n \ge 0$ for every n, then $\langle x_n y_n\rangle_{n\in\mathbb{N}}$ is non-decreasing. [CD]

20. A sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ is *monotonic* if it is either non-increasing or non-decreasing.

(a) Give an example of a sequence which is both non-increasing and non-decreasing.

(b) If $\langle x_n\rangle_{n\in\mathbb{N}}$ and $\langle y_n\rangle_{n\in\mathbb{N}}$ are both monotonic, does $\langle x_n + y_n\rangle_{n\in\mathbb{N}}$ have to be monotonic? Justify your answer with a proof or a counter-example. [CE]

21. (a) Define *Cauchy sequence*, and state the General Principle of Convergence.

(b) Let $\langle x_n\rangle_{n\in\mathbb{N}}$ and $\langle y_n\rangle_{n\in\mathbb{N}}$ be two series such that $0 \le x_n \le y_n$ for every $n \in \mathbb{N}$. Set $s_n = \sum_{k=0}^{n} x_k$, $t_n = \sum_{k=0}^{n} y_k$ for every n.

(i) Show that $|s_m - s_n| \le |t_m - t_n|$ for all m, $n \in \mathbb{N}$. Hence show that if $\langle t_n\rangle_{n\in\mathbb{N}}$ is Cauchy, so is $\langle s_n\rangle_{n\in\mathbb{N}}$.

(ii) Show that if $\langle y_n\rangle_{n\in\mathbb{N}}$ is summable, so is $\langle x_n\rangle_{n\in\mathbb{N}}$. [CF]

22. Give examples of

(i) a continuous function $f :]0,1[\to \mathbb{R}$ which is not bounded;
(ii) a bounded continuous function $f :]0,1[\to \mathbb{R}$ which does not attain its supremum;
(iii) a continuous function $f : [0,\infty[\to \mathbb{R}$ which is not bounded;
(iv) a bounded continuous function $f : [0,\infty[\to \mathbb{R}$ which does not attain its infimum.

In each case explain (with reference to theorems proved in lectures) why your function is continuous. In parts (ii) and (iv), say (without proof) what the supremum and infimum of your function are. [CG]

23. A real function f is *uniformly continuous* if

for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| \leq \epsilon$ whenever x, $y \in \mathrm{dom}(f)$ and $|x - y| \leq \delta$.

Show that if f and g are uniformly continuous real functions then $f + g$ is uniformly continuous. (*Hint*: use ideas from the proof that the sum of two continuous functions is continuous.) [CH]

24. Recall that a real function f is uniformly continuous if

for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| \leq \epsilon$ whenever x, $y \in \mathrm{dom}(f)$ and $|x - y| \leq \delta$.

Show that if f and g are uniformly continuous real functions then $f \circ g$ is uniformly continuous. [DA]

25. Show that, for any real function f and any a, $b \in \mathbb{R}$, $\lim_{x\to a} f(x) = b$ iff $\lim_{x\downarrow a} f(x) = \lim_{x\uparrow a} f(x) = b$. (*Hint*: you will probably need to set $\delta = \min(\delta_1, \delta_2)$.) [DB]

26. (a) Suppose that $\lim_{n\to\infty} x_n = b$, $\lim_{n\to\infty} y_n = c$ and that $x_n \leq y_n$ for every $n \in \mathbb{N}$. Show that $b \leq c$.
(b) If we know that $x_n < y_n$ for every n, can we conclude that $b < c$? [DC]

27. (a) For $\lambda \in \mathbb{R}$, consider the polynomial $f(x) = x^5 - 2x^4 + \lambda x^3 - 4$. Show that for any $c \in \mathbb{R}$ there are a, $b \in \mathbb{R}$ such that $a \leq b$ and $f(a) \leq c \leq f(b)$. (*Hint*: show that if $|x| \geq 1$ then $|f(x) - x^5| \leq (6 + |\lambda|)x^4$, so that if $x \geq |c| + 6 + |\lambda|$ then $f(x) \geq |c|$.)
(b) Now suppose that $f(x) = x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$. What changes must you make to the argument above to show that for any $c \in \mathbb{R}$ there are a, $b \in \mathbb{R}$ such that $a \leq b$ and $f(a) \leq c \leq f(b)$?
(c) Show that for any polynomial g of *odd* order there is an $x \in \mathbb{R}$ such that $g(x) = 0$. [DD]

28. Let f and g be real functions, and suppose that $x_0 \in \mathrm{dom}\, g$ is such that f is differentiable at $g(x_0)$ and $\lim_{x\to x_0} g(x) = g(x_0)$.
(a) Define a real function $h : \mathbb{R} \to \mathbb{R}$ by setting

$$\begin{aligned} h(x) &= \tfrac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} \text{ if } x \in \mathrm{dom}(f \circ g) \text{ and } g(x) \neq g(x_0) \\ &= f'(g(x_0)) \text{ for other } x \in \mathbb{R}. \end{aligned}$$

Show that $\lim_{x\to x_0} h(x) = f'(x_0)$.
(b) Hence show that if g is differentiable at x_0 then $f \circ g$ is differentiable at x_0. [DE]

29. Let $f : [a,b] \to \mathbb{R}$ be a continuous function, where $a \leq b$ in $\mathbb{R}$.
(a) Show that its set of values is a closed interval $[c,d]$. (*Hint*: you will need to use *two* of the theorems from lectures in Week 16.)
(b) Now suppose that f is injective. Show that *either* $f(x) \leq f(y)$ whenever $a \leq x \leq y \leq b$ *or* $f(x) \geq f(y)$ whenever $a \leq x \leq y \leq b$. (*Hint*: consider the cases $f(a) \leq f(b)$, $f(a) \geq f(b)$ separately.) [DF]

30. Suppose that $f : [a,b] \to \mathbb{R}$ is injective and that $f(x) \leq f(y)$ whenever $a \leq x \leq y \leq b$. Show that f^{-1} is continuous. [DG]

31. Recall that a real function f is uniformly continuous if

for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| \le \epsilon$ whenever x, $y \in \mathrm{dom}(f)$ and $|x - y| \le \delta$.

(a) Show that the function $x \mapsto x : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.

(b) Show that the function $x \mapsto x^2 : \mathbb{R} \to \mathbb{R}$ is not uniformly continuous. (*Hint*: take $\epsilon = 1$ and $x = \frac{1}{\delta}$.)

(c) Show that $|\sin^2 b - \sin^2 a| \le 2|b - a|$ for all a, $b \in \mathbb{R}$. (You may assume that sin is differentiable and that $|\sin x| \le 1$, $|\sin'(x)| \le 1$ for every x.) Show that $\sin^2$ is uniformly continuous.

(d) Show that if f and g are bounded uniformly continuous real functions then $f \times g$ is uniformly continuous. [EA]

32. (a) Use the Intermediate Value Theorem to show that for every $x \in [0, \infty[$ there is a $y \in [0, \infty[$ such that $y^2 = x$. Show that this y is unique.

(b) For $x \in [0, \infty[$ let $\sqrt{x}$ be the unique number in $[0, \infty[$ such that $(\sqrt{x})^2 = x$. Show that

(i) if $0 \le x \le y \le x + \delta$ then $|\sqrt{y} - \sqrt{x}| \le \sqrt{\delta}$;

(ii) $\sqrt{} : [0, \infty[\to [0, \infty[$ is uniformly continuous. [EB]

33. Rewrite the following, completed and corrected to form a proof that if $\lim_{x \downarrow x_0} f(x) = \lim_{x \downarrow x_0} g(x) = 0$ and $\lim_{x \downarrow x_0} \frac{f'(x)}{g'(x)} = b$ then $\lim_{x \downarrow x_0} \frac{f(x)}{g(x)} = b$.

Let $\epsilon > 0$. Let $\delta_0 \ge 0$ be such that $\frac{f'(x)}{g'(x)}$ is defined and $|\frac{f'(x)}{g'(x)} - b| \le \dots$ whenever $x \le x_0 + \delta_0$. Note that this implies that $f'(x)$ and $g'(x)$ are defined, and that $g'(x) \ne 0$, for $x_0 < x$; so that $f(x)$ and $g(x)$ are defined for $x_0 < x \le x_0 + \delta_0$, and g is continuous at every point of $]x_0, x_0 + \delta_0]$. By Rolle's theorem, $g(x)$ cannot be zero for more than one value of x in $]x_0, x_0 + \delta_0]$. So there is a $\delta \in]0, \delta]$ such that $g(x) \ne 0$ for $x_0 < x \le x_0 + \delta$.

Take x such that $x_0 < x \le x_0 + \delta$. Then $\lim_{y \downarrow x_0} \frac{f(x) - f(y)}{g(x) - g(y)} = \dots$. Let $\delta_1 > 0$ be such that $\frac{f(x) - f(y)}{g(x) - g(y)}$ is defined and $|\frac{f(x) - f(y)}{g(x) - g(y)} - \dots| \le \dots$ whenever $x_0 < y \le x_0 + \delta_1$. Since we cannot have $g(y) = g(x)$ for $x_0 < y \le x_0 + \delta_1$, $x_0 + \delta_1 > x$. Now examine $y = x_0 + \delta_1$. Because $y < x$ and f and g are differentiable (therefore continuous) at every point of $[y, x]$, $\dots$ tells us that there is a $z \in]x_0, y[$ such that $\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{g'(z)}{f'(z)}$. So

$$|\frac{f(x)}{g(x)} - b| \le |\frac{f(x)}{g(x)} - \frac{f(x) - f(y)}{g(x) - g(y)}| + |\frac{f'(z)}{g'(z)} - b| \le \dots \le \epsilon.$$

As ϵ is arbitrary, $\lim_{x \downarrow 0} \frac{f(x)}{g(x)} = b$. [EC]

34. Suppose that $\langle x_n \rangle_{n \in \mathbb{N}}$, $\langle y_n \rangle_{n \in \mathbb{N}}$ are two series, of which $\langle y_n \rangle_{n \in \mathbb{N}}$ is absolutely summable; set $s_n = \sum_{k=0}^{n} x_k$ for each n.

(a) Show that if $|x_n| \le |y_n|$ for every $n \in \mathbb{N}$, then $\langle x_n \rangle_{n \in \mathbb{N}}$ is summable. (*Hint*: show that $\langle s_n \rangle_{n \in \mathbb{N}}$ is a Cauchy sequence.)

(b) Show that if, for some $n_1 \in \mathbb{N}$ and $M \ge 0$, $|x_n| \le M|y_n|$ for every $n \ge n_1$, then $\langle x_n \rangle_{n \in \mathbb{N}}$ is summable.

(c) Show that if $\lim_{n \to \infty} \frac{x_n}{y_n}$ is defined in $\mathbb{R}$ then $\langle x_n \rangle_{n \in \mathbb{N}}$ is summable. [ED]

35. Let $\langle x_n \rangle_{n \in \mathbb{N}}$ be a sequence such that $\lim_{n \to \infty} |\frac{x_{n+1}}{x_n}| = b < 1$.

(a) Set $c = \frac{1}{2}(1 + b)$. Show that there are $n_0 \in \mathbb{N}$ and $M \ge 0$ such that $|x_n| \le Mc^n$ for every $n \ge n_0$. (*Hint*: show that there is an n_0 such that $|x_{n+1}| \le c|x_n|$ for every $n \ge n_0$.)

(b) Show that $\langle x_n \rangle_{n \in \mathbb{N}}$ is absolutely summable. (*Hint*: use question ED above and question CC from Problem Sheet 3.) [EE]

36. (a) Let f be a real function such that f' is defined and continuous on $[a, a+h]$, where $a \in \mathbb{R}$ and $h > 0$, and f'' is defined on $]a, a+h[$.

(i) Show that there is an $h_1 \in \,]0, h[$ such that $\frac{f(a+h)-f(a)-hf'(a)}{h^2} = \frac{f'(a+h_1)-f'(a)}{2h_1}$. (*Hint*: apply Cauchy's Mean Value Theorem to $\frac{g(h)-g(0)}{h^2-0^2}$ where $g(x) = f(a+x) - f(a) - xf'(a)$.)

(ii) Show that there is an $h_2 \in \,]0, h_1[$ such that $\frac{f(a+h)-f(a)-hf'(a)}{h^2} = 2f''(a+h_2)$.

(b) Now suppose that f'' is defined and continuous on $[a, a+h]$ and f''' is defined on $]a, a+h[$. Show that there is an $h_3 \in \,]0, h[$ such that $f(a+h) = f(a) + hf'(a) + \frac{1}{2}h^2 f''(a) + \frac{1}{6}h^3 f'''(a+h_3)$. (*Hint*: use $g(x) = f(a+x) - f(a) - xf'(a) - \frac{1}{2}x^2 f''(a)$.) [EF]

37. (a) Define the expressions

$$\lim_{x\to a} f(x) = b, \quad \lim_{x\downarrow a} f(x) = b, \quad \lim_{x\uparrow a} f(x) = b$$

where f is a real function.

(b) Show from your definitions that
(i) $\lim_{x\to a} f(x) = b$ iff $\lim_{x\downarrow a} f(x) = b$ and $\lim_{x\uparrow a} f(x) = b$;
(ii) if $\lim_{\downarrow a} f(x) = b$ and $\lim_{x\downarrow a} g(x) = c$ and $f(x) \geq g(x)$ for every $x \in \operatorname{dom} f \cap \operatorname{dom} g$, then $b \geq c$.

(c) Show that if $\lim_{x\to 0} f(x) = 0$ and $g : [-1, 1] \to [-1, 1]$ is any function then $\lim_{x\to 0} f(x)g(x) = 0$. [GA]

38. (a) What is meant by saying that a real function is *continuous*?

(b) Show from your definition that
(i) if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\langle x_n \rangle_{n\in\mathbb{N}}$ is a convergent sequence then $\langle f(x_n) \rangle_{n\in\mathbb{N}}$ is convergent;
(ii) the function $x \mapsto |x| : \mathbb{R} \to \mathbb{R}$ is continuous.

(c) What is the *infimum* of a subset of $\mathbb{R}$? Which subsets of $\mathbb{R}$ have infima? Give examples of (i) a set $A_1 \subseteq \mathbb{R}$ with no infimum (ii) a set $A_2 \subseteq \mathbb{R}$ with an infimum belonging to A_2 (iii) a set $A_3 \subseteq \mathbb{R}$ with an infimum not belonging to A_3.

(d) Complete and correct the following to render it a proof that any continuous function $f : [a, b] \to \mathbb{R}$, where $a \leq b$ in $\mathbb{R}$, is bounded.

> Set $A = \ldots$. Then $a \in A$ and A is bounded above by b. Accordingly $c = \inf A$ is defined and $c \in [a, b]$. There is therefore a $\delta > 0$ such that $|f(y) - f(c)| \leq \ldots$ whenever $y \in \operatorname{dom} f$ and $|y - c| \leq \delta$. Set $z = \min(b, c + \delta)$. Then $z \in A$. To see this, note that there is an $x \in A$ such that $\ldots$. Now $[a, z] = [a, x] \cup [x, z]$. Because $x \in A$, there is an M such that $|f(y)| \geq M$ for every $y \in \ldots$; because $[x, z] \subseteq \operatorname{dom} f \cap [c - \delta, c + \delta]$, $|f(y)| \leq |f(c)| + 1$ for every $y \in [x, z]$; so $|f(y)| \leq \ldots$ for every $y \in [a, z]$, and f is bounded on $[a, z]$. This means that $z \leq c$. So $z = b = c$ and $b \in A$; which is what we set out to prove.

[GB]

39. (a) Show that any bounded non-decreasing sequence is convergent.

(b) Define *Cauchy sequence*. Show that
(i) every convergent sequence is Cauchy;
(ii) every Cauchy sequence is bounded.

(c) Complete the following sentences and arrange them in such an order that they form a correct proof that any Cauchy sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ is convergent.

1. For each $n \in \mathbb{N}$, $A_n = \ldots$ is a non-empty subset of $[-M, M]$, so has an infimum y_n say, with $-M \leq y_n \leq x_n \leq M$.
2. There is an $n_0 \in \mathbb{N}$ such that $|x_i - x_n| \leq \epsilon$ whenever $i, n \geq n_0$.
3. Let $\epsilon > 0$.
4. Because $\langle x_n \rangle_{n\in\mathbb{N}}$ is Cauchy, it is bounded; let M be such that $-M \leq x_n \leq M$ for every $n \in \mathbb{N}$.
5. As ϵ is arbitrary, $\lim_{n\to\infty} x_n - y_n = 0$.
6. Thus $\langle y_n \rangle_{n\in\mathbb{N}}$ is a bounded non-decreasing sequence and has a limit y say.
7. For each $n \in \mathbb{N}$, $A_{n+1} \ldots A_n$ so y_n is a lower bound of A_{n+1} and $y_n \ldots y_{n+1}$.
8. If $n \geq n_0$, $A_n \subseteq [x_n - \epsilon, x_n + \epsilon]$ so $y_n \geq x_n - \epsilon$ and $|x_n - y_n| = x_n - y_n \leq \ldots$.
9. So $\lim_{n\to\infty} x_n = 0 + y = y$ and $\langle x_n \rangle_{n\in\mathbb{N}}$ is convergent.

[GC]

40. (a) What is meant by saying that a real function is *differentiable*?

(b) Show from your definition that the product of two differentiable functions is differentiable.

(c) State Rolle's theorem and the Mean Value Theorem, and prove the latter.

(d) Show that if $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function and $f' : \mathbb{R} \to \mathbb{R}$ is bounded, then there is an $M \geq 0$ such that $|f(x)| \leq M(1 + |x|)$ for every $x \in \mathbb{R}$.

(In this question, standard theorems concerning limits may be used without proof, but should be clearly stated.)

[GD]

41. Recall that a real function f is *uniformly continuous* if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| \leq \epsilon$ whenever $x, y \in \operatorname{dom} f$ and $|x - y| \leq \delta$.

Rearrange the following sentences to form a proof that if $a \leq b$ in $\mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ is continuous, it is uniformly continuous.

1. Let $\epsilon > 0$.
2. Set $A = \{x : a \leq x \leq b, \exists\delta > 0, |f(y) - f(y')| \leq \epsilon$ whenever $y, y' \in [a, x]$ and $|y - y'| \leq \delta\}$.
3. So $z = \sup A$ is defined and $a \leq z \leq b$.
4. $a \in A$ because if $y, y' \in [a, a]$ then $|f(y) - f(y') = 0$.
5. Because $z = \sup A$, there is an $x_1 \in A \cap [z - \frac{1}{2}\delta_0, z]$.
6. Because $z \in [a, b]$, f is continuous at z; let $\delta_0 > 0$ be such that $|f(y) - f(z)| \leq \epsilon$ whenever $y \in [a, b] \cap [z - \delta_0, z + \delta_0]$.
7. Set $\delta = \min(\frac{1}{2}\delta_0, \delta_1) > 0$.
8. Let $\delta_1 > 0$ be such that $|f(y) - f(y')| \leq \frac{1}{2}\epsilon$ whenever $y, y' \in [a, x_1]$ and $|y - y_1| \leq \delta_1$.
9. A is bounded above by b.
10. If $\min(y, y') \geq z - \delta_0$, then $|f(y) - f(y')| \leq |f(y) - f(z)| + |f(y') - f(z)| \leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$.
11. If $\min(y, y') \leq z - \delta_0$ then $\min(y, y') \leq x - \frac{1}{2}\delta_0$; since $|y - y'| \leq \frac{1}{2}\delta_0$, both y and y' belong to $[a, x]$; since $|y - y'| \leq \delta_1$, $|f(y) - f(y')| \leq \epsilon$.
12. Suppose that $y, y' \in [a, \min(b, z + \delta_0)]$ and $|y - y'| \leq \delta$.
13. Since z is an upper bound for A, $\min(b, z + \delta_0) \leq z$, so in fact $z = b$ and $b \in A$.
14. Since y and y' are arbitrary, $\min(b, z + \delta_0) \in A$.
15. This means that there is a $\delta > 0$ such that $|f(y) - f(y')| \leq \epsilon$ whenever $y, y' \in [a, b]$ and $|y - y'| \leq \delta$. As ϵ is arbitrary, f is uniformly continuous.

[GE]

42. (a) Define the expressions $\lim_{x\to\infty} f(x) = b$, $\lim_{x\to-\infty} f(x) = b$.

(b) What is meant by saying that a real function is *continuous*? Show from your definition that the composition of continuous functions is continuous.

(c) For which of the following sets, if any, is there an unbounded continuous function $f : I \to \mathbb{R}$? Give an example in each case in which one exists.
(i) $I = [1, \infty[$.
(ii) $I =]-1, 1]$.
(iii) $I = [-2, 2]$.

(d) Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are both defined in $\mathbb{R}$, then f is bounded. [GF]

43. (a) What is meant by saying that a real function is *differentiable*?

(b) Show that a differentiable function is continuous. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ which is not differentiable.

(c) State Rolle's theorem and the Mean Value Theorem, and prove the latter from the former.

(d) Let $f :]-1, 1[\to \mathbb{R}$ be a continuous function. Suppose that $\lim_{x\to 0} f'(x) = b$. Show that $f'(0)$ is defined and equal to b. [GG]