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MA203 Real Analysis

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This course is intended as a first introduction to the methods required to prove the basic theorems of real analysis, particularly those involving continuous and differentiable functions. The theorems themselves will mostly have been presented in MA206, and should be more or less familiar. The techniques of proof, however, are likely to be completely new. They are difficult, and you must expect to have to work hard.

'Analysis' is one of the traditional subjects taught in university mathematics degrees – necessarily so, as it underlies not only most of modern pure mathematics but also large parts of modern applied mathematics – and nearly everywhere gives rise to special problems. Many students who are quite good at other kinds of mathematics find that analysis seems to be beyond them. I think that this is for two separate reasons. The first, and less important, one is that analysis deals with *inequalities*. It may well be that an actual majority of the formulae in the notes below will have at least one of the symbols $\le$, $\ge$, $<$ or $>$. These are not really difficult to handle, but they do require technical skills which you may not yet have practised enough. So you are going to have to spend some time working on these. But there is a much more essential difficulty to come to terms with. All the characteristic sentences of analysis have multiple quantifiers. 'Quantifiers' are the expressions $\forall$, 'for every', and $\exists$, 'there is'. Of course these appear everywhere in mathematics. But in analysis we get sentences like

'for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(x_0)| \le \epsilon$ for every $x \in [x_0 - \delta, x_0 + \delta]$'

with three quantifiers ('$\forall\,\epsilon \ldots \exists\,\delta \ldots \forall\,x \ldots$') one after the other; and I think some of the sentences later on may have as many as five. These demand some special ways of thinking, which is what this course is mostly about.

The real number system

The whole of this course will be concerned with the set $\mathbb{R}$ of 'real numbers'. I will therefore run as quickly as possible over the key properties of this system. All the facts here are supposed to be familiar, and I'm not going to prove any of them; I write them out just so that you can make sure you really do know the things I am going to assume from now on.

Addition For any two real numbers x and y, we have a real number $x + y$, and

$x + (y + z) = (x + y) + z$ for all real numbers x, y and z;
there is a real number 0 such that $x + 0 = 0 + x = x$ for every $x \in \mathbb{R}$;
for every $x \in \mathbb{R}$ there is a number $-x \in \mathbb{R}$ such that $x + (-x) = (-x) + x = 0$;
$x + y = y + x$ for all x, $y \in \mathbb{R}$.

Multiplication For any two real numbers x and y, we have a real number xy, and

$x(yz) = (xy)z$ for all real numbers x, y and z;
there is a real number 1 such that $x1 = 1x = x$ for every $x \in \mathbb{R}$;
for every $x \in \mathbb{R} \setminus \{0\}$ there is a number $\frac{1}{x} \in \mathbb{R}$ such that $x \cdot \frac{1}{x} = \frac{1}{x} \cdot x = 1$;
$xy = yx$ for all x, $y \in \mathbb{R}$;
$1 \ne 0$.

The distributive laws

$x(y + z) = xy + xz$ for all x, y, $z \in \mathbb{R}$;
$(x + y)z = xz + yz$ for all x, y, $z \in \mathbb{R}$.

The ordering of $\mathbb{R}$ We have a relation $\le$ on $\mathbb{R}$ such that

$x \le x$ for every $x \in \mathbb{R}$,

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if $x \le y$ and $y \le z$ then $x \le z$,
if $x \le y$ and $y \le x$ then $x = y$,
for every x, $y \in \mathbb{R}$, either $x \le y$ or $y \le x$.

$\le$, $+$ and $\times$ If x, y, $z \in \mathbb{R}$, then
if $x \le y$ then $x + z \le y + z$ and $x - z \le y - z$;
if $x \le y$ and $0 \le z$ then $xz \le yz$;
if $x \le y$ and $z \le 0$ then $yz \le xz$.

This list is not quite complete, and there is one further essential fact about real numbers ('Dedekind completeness') which will play a large part in this course. But I will deal with it later when we can give it the time it deserves.

Chains of inequalities It will often happen that we have a string of inequalities

$$a \le b \le c \le d \le \dots,$$

meaning '$a \le b$ and $b \le c$ and $c \le d$ and ...'. In this case, because $\le$ is transitive, we can say
because $a \le b$ and $b \le c$, $a \le c$,
because $a \le c$ and $c \le d$, $a \le d$,
and so on. It sometimes happens that the chain returns to its starting point, as in

$$a \le b \le c \le d \le e \le a.$$

In this case, at the end, we get $a \le e$ and $e \le a$; so we must have $a = e$. But this means that we have $a \le d$ and $d \le a$, so $a = d$, and so on; in the end, we conclude that $a = b = c = d = e$.

Moduli For any $x \in \mathbb{R}$, we write

$$\begin{aligned} |x| &= x \text{ if } 0 \le x, \\ &= -x \text{ if } x \le 0; \end{aligned}$$

alternatively, we can define $|x|$ as $\max(x, -x)$, that is,

$$\begin{aligned} |x| &= x \text{ if } -x \le x, \\ &= -x \text{ if } x \le -x. \end{aligned}$$

Now the most important fact of all about moduli is the **triangle inequality**:

$$|x + y| \le |x| + |y| \text{ for all } x,\, y \in \mathbb{R}.$$

This is not the kind of thing I am going to spend the course teaching you how to prove, but the argument is instructive and fairly easy, so here it is: if x, $y \in \mathbb{R}$, then
$x \le |x|$ because $|x| = \max(x, -x)$,
so $x + y \le |x| + y$, because you can add anything to an inequality,
also $y \le |y|$, so $|x| + y \le |x| + |y|$,
since $x + y \le |x| + y \le |x| + |y$, $x + y \le |x| + |y|$.
Now $|-x| = \max(-x, -(-x)) = \max(-x, x) = |x|$, and similarly $|-y| = |y|$,
so $-(x + y) = (-x) + (-y) \le |-x| + |-y| = |x| + |y|$,
and since $|x + y|$ must be either $x + y$ or $-(x + y)$, we must have $|x + y| \le |x| + |y|$.
(The *proof* here is non-examinable. But the *fact* $|x + y| \le |x| + |y|$ is part of the survival kit for the course.)

Two more fundamental facts about moduli are

$$|xy| = |x||y| \text{ for all } x,\, y \in \mathbb{R},$$

$$|x| \ge 0 \text{ and } |x| = 0 \iff x = 0.$$

Manipulating moduli Apart from the three basic facts above, we are going to need quite a few others. Among these are

$$|x| - |y| \le |x - y| \text{ for all } x,\, y \in \mathbb{R}.$$

(To see this, note that $|x| = |(x-y)+y| \leq |x-y| + |y|$ and subtract $|y|$ from both sides.) In the same way

$$|y| - |x| \leq |y - x| = |x - y|.$$

But putting these together we have

$$\big||x| - |y|\big| = \max(|x| - |y|, |y| - |x|) \leq |x - y|.$$

Another on the list is

$$|x - y| = |x + (-y)| \leq |x| + |-y| = |x| + |y|.$$

A particularly important fact is that

$$\text{if } 0 < x \leq y \text{ then } \tfrac{1}{y} \leq \tfrac{1}{x};$$

this is because xy and $\frac{1}{xy}$ are both greater than 0, and if we multiply the inequality $x \leq y$ by the positive number $\frac{1}{xy}$ we get $\frac{1}{y} \leq \frac{1}{x}$. We may well need some others; I'll try to point them out as they arise.

What is a limit?

You should be familiar with the formula

$$\lim_{n\to\infty} x_n = a.$$

In elementary courses in calculus this is generally interpreted as

'for large enough n, $x_n \doteqdot a$.'

The trouble with this way of putting it is that it doesn't tell us how close x_n has to be to a to count as 'approximately equal' to a. In any particular application to a practical problem, we shall probably have a definite idea of how close we need to be. If you are measuring wallpaper, a millimeter is probably good enough; if you are designing a computer chip, it certainly isn't. If we are trying to devise a framework which we shall be able to use in *any* circumstances, we have to leave room for someone to tell us how close counts as close. That is, we need a parameter ϵ, set by somebody else, to specify that '$x_n \doteqdot a$' means '$|x_n - a| \leq \epsilon$'. Of course ϵ must be strictly greater than 0; '$|x_n - a| \leq 0$' just means '$x_n = a$', and the whole idea of the limit $\lim_{n\to\infty} x_n$ is that we don't suppose that any particular value of x_n will be exactly equal to the limit, except by accident.

So saying that '$\lim_{n\to\infty} x_n = a$' means that

for any acceptable error $\epsilon > 0$, $|x_n - a| \leq \epsilon$ for all n large enough.

How large will 'large enough' be? Of course this will depend on ϵ. But we are claiming that there is *some* number M such that $|x_n - a| \leq \epsilon$ whenever $n \geq M$.

Let me write this out again in a more concentrated form.

$$\lim_{n\to\infty} x_n = a$$

means

$$\forall\, \epsilon > 0\ \exists\, n_0 \in \mathbb{N}\ \forall\, n \geq n_0,\ |x_n - a| \leq \epsilon.$$

This is a typical definition from elementary analysis. It is really quite difficult to get hold of. We have three quantifiers $\forall$, $\exists$, $\forall$; these have to be put in the right order; and any variation of any symbol in the whole formula is liable to make it wrong.

The Analysis Game I believe that one way, and a useful way, of coping with these sentences is to treat them as specifying a game. The formula

$$\lim_{n\to\infty} x_n = a, \quad \forall\, \epsilon > 0\ \exists\, n_0 \in \mathbb{N}\ \forall\, n \geq n_0,\ |x_n - a| \leq \epsilon$$

means

for every $\epsilon > 0$, $\exists\, n_0 \in \mathbb{N}\ \forall\, n \geq n_0,\ |x_n - a| \leq \epsilon$.

So if I claim that $\lim_{n\to\infty} x_n = a$, what I am saying is that for any $\epsilon > 0$ which *you* (not I) choose, there will be some n_0 such that $|x_n - a| \le \epsilon$ for every $n \ge n_0$. Now after you have chosen the ϵ, I only have to find one n_0 which will work; and I (not you) have the right to point to a number which I believe is good enough. But I am then claiming that '$|x_n - a| \le \epsilon$ for every $n \ge n_0$', so *you* have the right to challenge me by naming a particular $n \ge n_0$ and demanding that we check that $|x_n - a|$ is indeed at most ϵ.

Thus we have the idea of a *game* with four moves, in which

Player I says '$\lim_{n\to\infty} x_n = a$',
Player II chooses some ϵ, and s/he must choose $\epsilon > 0$,
Player I chooses some $n_0 \in \mathbb{N}$,
Player II chooses some $n \in \mathbb{N}$, and s/he must choose $n \ge n_0$,
and at the end of these four moves, the players look at $|x_n - a|$, and if $|x_n - a| \le \epsilon$ then Player I wins (because that's what s/he said would happen), while if $|x_n - a| \not\le \epsilon$ then Player II wins.

The actual statement '$\lim_{n\to\infty} x_n = a$' is *true* if, and only if, Player I can always win the game, whatever Player II does. Of course Player I will have to take care to play the right move when s/he comes to pick n_0; even if the original move '$\lim_{n\to\infty} x_n = a$' left him/her in a winning position, s/he can still bungle it by being careless with the second move.

Example Suppose Player I starts with

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Will s/he win? Player II can choose any ϵ s/he likes, and it's generally good tactics to pick something small; suppose s/he tries $\epsilon = \frac{1}{1000}$. Player I does a little quick thinking: 'at the end, I shall need $|\frac{1}{n} - 0| \le \epsilon = \frac{1}{1000}$; now $|\frac{1}{n} - 0|$ is just $\frac{1}{n}$; how can I be sure (remembering that it's Player II who gets to choose n) that s/he will choose n so that $\frac{1}{n} \le \frac{1}{1000}$? well, I'd better make sure that $n \ge 1000$ – can I do this? of course! I'll say that $n_0 = 1000$'. So s/he does just that: $n_0 = 1000$. Now Player II has another move; but whatever s/he does, s/he has to pick $n \ge 1000$, so that $|\frac{1}{n} - 0| = \frac{1}{n} \le \frac{1}{1000} = \epsilon$, and Player I will win.

This is what happens if Player II plays $\epsilon = \frac{1}{1000}$. Could s/he have done any better? The trouble is that Player I is liable to answer with $n_0 = \frac{1}{\epsilon}$. And if s/he does this, then for any $n \ge n_0$ we shall have $|\frac{1}{n} - 0| = \frac{1}{n} \le \frac{1}{n_0} = \epsilon$. Is there any way of stopping him/her from playing $n_0 = \frac{1}{\epsilon}$? Actually, there is, because if you look at the rules you see that n_0 has got to be an integer. So if we take $\epsilon = \frac{1}{1000\pi}$, Player I can't answer with $n_0 = \frac{1}{\epsilon} = 1000\pi$, because that's not an integer (it's somewhere between 3141 and 3142). But this does us no good, because Player I doesn't have to hit any particular mark exactly. If s/he just chooses *some* integer $n_0 \ge \frac{1}{\epsilon}$ (e.g., $n_0 = 4000$ if $\epsilon = \frac{1}{1000\pi}$), then for any $n \ge n_0$ we shall still have $\frac{1}{n} \le \frac{1}{n_0} \le \epsilon$, and Player I will still win in the end.

Note that Player I must absolutely not tell Player II what his/her second move is going to be before Player II has committed to a particular ϵ. Because if Player I gets lazy, and just says '$n_0 = 10^{100}$' without looking at what ϵ was, then Player II can say '$\epsilon = \exp(-2^{10})$' for his/her first move and then '$n = n_0$' for his/her second, and when they come to do the sums they will get

$$|\frac{1}{n} - 0| = \frac{1}{n} = 10^{-100} > e^{-300} > e^{-1024}$$

and Player II will win.

Example Suppose that Player I starts by saying that

$$\text{`}\lim_{n\to\infty} (-1)^n = 1\text{'}.$$

Will s/he win? Suppose that Player II again tries a moderately small $\epsilon > 0$ – e.g., $\epsilon = \frac{1}{1000}$, and Player I answers with his favourite $n_0 = 10^{100}$. Now Player II has to pick $n \geq n_0$. S/he does a bit of rough working:

$$|(-1)^n - 1| = |1 - 1| = 0 \leq \frac{1}{1000} \text{ if } n \text{ is even,}$$
$$= |(-1)^n - 1| = 2 \not\leq \frac{1}{1000} \text{ if } n \text{ is odd.}$$

So Player II wants to choose an *odd* number n. Can s/he do it? Yes: $n = 10^{100} + 1$ will do nicely, because the only rules she has to look out for are $n \in \mathbb{N}$, $n \geq n_0$. Is there anything Player I could have done to stop this? No, because whatever n_0 s/he picks, Player II just has to choose $n = n_0$ (if n_0 is odd) or $n = n_0 + 1$ (if n_0 is even) and win. So Player I scuppered his/her chances right at the beginning by saying that $\langle(-1)^n\rangle_{n\in\mathbb{N}}$ converges to 1; this is just not true.

Of course in this case, Player II didn't have to take ϵ nearly as small as $\frac{1}{1000}$. Any number strictly less than 2 would have done; so s/he could have started with $\epsilon = 1$, for instance, showing that Player I wasn't just wrong, but had completely misunderstood the sequence.

The first theorem I will now try to show how these ideas can be assembled into a rigorous proof of a familiar fact.

Theorem If $\lim_{n\to\infty} x_n = b$ and $\lim_{n\to\infty} y_n = c$ then $\lim_{n\to\infty} x_n + y_n = b + c$.

proof I do not think the ideas of this proof can really make sense without going through them at least twice, once in the order in which one remembers them, and once in the order in which one writes them out.

We are in the position of a Player I who has announced, as her first move,

$$\lim_{n\to\infty} x_n + y_n = b + c.$$

The theorem says that she can win from this position. If Player II believes that the theorem is true, I suppose he will resign. But if he is sceptical, he will choose an $\epsilon > 0$ and challenge Player I to find a successful move in reply. If we are to be sure that Player I really can win whatever Player II does, we are going to have to describe a *strategy* for Player I; a set of rules to tell Player I what to do in any of the positions which Player II can manoeuvre her into. Since we have no idea what Player II is going to say for his first move (except that it must be to choose some $\epsilon > 0$), the proof more or less has to begin with

Let $\epsilon > 0$.

At this point, Player I is going to have to think ahead. Because Player II has another move; and Player I has to find an n_0 so large that Player II's subsequent choice of n is certain to lose. So the proof is going to have to look like this:

Let $\epsilon > 0$. [Player II's first move.]
...
Take n_0 such that ... [Player I's second move.]

(Maybe we shall be able to specify a formula for n_0; but we can't count on this, and maybe it will have to be chosen by some seriously complicated process.) But we know what will come next, because we know that Player II will choose some $n \geq n_0$; and since we have no control over this at all, we shall just have to write it out like that:

Let $\epsilon > 0$. [Player II's first move.]
...
Take n_0 such that ... [Player I's second move.]
Let $n \geq n_0$. [Player II's second move.]

At this point, the players have finished their moves, and proceed to see who has won. Now the question they have to decide is

is $|(x_n + y_n) - (b + c)| \leq \epsilon$?

because Player I will win if the answer is 'yes', and lose if the answer is 'no'.

Everything I have written so far is just a matter of understanding the structure of the game. It's got nothing to do with the actual sequences involved. But we have come to the crunch. In order to be sure that

$|(x_n+y_n)-(b+c)| \le \epsilon$, we need to know something about the numbers we're looking at. And the **key fact** is that

$$|(x_n+y_n)-(b+c)| \le |x_n-b|+|y_n-c|;$$

this is just a simple application of the triangle inequality, because $|(x_n+y_n)-(b+c)| = |(x_n-b)+(y_n-c)| \le |x_n-b|+|y_n-c|$; the modulus of the sum is less than or equal to the sum of the moduli. So it will be good enough if we can find some reason to be sure that $|x_n-b|+|y_n-c| \le \epsilon$.

Why should this be so? It's helpful to try to think about what is supposed to be happening here. We are supposing that $\lim_{n\to\infty} x_n = b$ and that $\lim_{n\to\infty} y_n = c$ and that $n \ge n_0$, and presumably Player I chose n_0 to be large; in which case, surely, we shall have $x_n \doteq b$ and $y_n \doteq c$ in some sense. So maybe we can actually arrange that $|x_n-b| \le$ (something) and $|y_n-c| \le$ (something). What 'something' should we try, if we want the sum to be at most ϵ? There are two pieces, so if we take each of them to be $\frac{\epsilon}{2}$ that ought to work.

Thus our proof (growing slowly) might have a final line

$$|(x_n+y_n)-(b+c)| \le |x_n-b|+|y_n-c| \le \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon \qquad \text{[Player I wins.]}$$

But of course we are going to have to work at this a bit.

How can we justify this idea that

$$\text{if } n \ge n_0, \text{ then } |x_n-b| \le \tfrac{\epsilon}{2} \text{ and } |y_n-c| \le \tfrac{\epsilon}{2}?$$

Well, remember that we know that $\lim_{n\to\infty} x_n = b$, and also that $\frac{\epsilon}{2} > 0$. So we know that there is some integer – call it n_1 – such that $|x_n-b| \le \frac{\epsilon}{2}$ whenever $n \ge n_1$. (It might happen that the n_1 here will itself serve for the n_0 we're looking for. But we mustn't count on it, so we had better give it a new name.) Similarly, there is some $n_2 \in \mathbb{N}$ such that $|y_n-c| \le \frac{\epsilon}{2}$ for every $n \ge n_2$. Putting these into the framework of the proof we are building, it looks like this:

> Let $\epsilon > 0$. Then $\frac{\epsilon}{2} > 0$.
>
> There is an $n_1 \in \mathbb{N}$ such that $|x_n-b| \le \frac{\epsilon}{2}$ for every $n \ge n_1$.
>
> There is an $n_2 \in \mathbb{N}$ such that $|y_n-c| \le \frac{\epsilon}{2}$ for every $n \ge n_2$.
>
> Take n_0 such that ....
> Let $n \ge n_0$.
> ...
> So $|(x_n+y_n)-(b+c)| \le |x_n-b|+|y_n-c| \le \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon$.

What do we need to do to fill this in? Well, we are going to need some good reason why $|x_n-b| \le \frac{\epsilon}{2}$ whenever $n \ge n_0$. However, we picked n_1 so that $|x_n-b| \le \frac{\epsilon}{2}$ whenever $n \ge n_1$. So if $n_0 \ge n_1$, that will do fine. Next, we also need to know that $|y_n-c| \le \frac{\epsilon}{2}$ whenever $n \ge n_0$. Since we picked n_2 so that $|y_n-c| \le \frac{\epsilon}{2}$ whenever $n \ge n_2$, we shall be successful if $n_0 \ge n_2$. Thus what Player I needs to do is to choose some n_0 such that $n_0 \ge n_1$ and $n_0 \ge n_2$. The most straightforward way of doing this is to make n_0 actually the greater of the two numbers. So the proof becomes

> Let $\epsilon > 0$. Then $\frac{\epsilon}{2} > 0$.
>
> There is an $n_1 \in \mathbb{N}$ such that $|x_n-b| \le \frac{\epsilon}{2}$ for every $n \ge n_1$.
>
> There is an $n_2 \in \mathbb{N}$ such that $|y_n-c| \le \frac{\epsilon}{2}$ for every $n \ge n_2$.
>
> Take $n_0 = \max(n_1, n_2)$.
> Let $n \ge n_0$.

Then $n \geq n_1$ and $n \geq n_2$, so $|x_n - b| \leq \frac{\epsilon}{2}$ and $|y_n - c| \leq \frac{\epsilon}{2}$ and

$$|(x_n + y_n) - (b + c)| \leq |x_n - b| + |y_n - c| \leq \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.$$

Since this works whatever $\epsilon > 0$ and $n \geq n_0$ Player II chooses, Player I is sure to win and the theorem is true.

Remarks I took a great deal of time over this. The points I am trying to make are

you don't think of proofs in the order in which you finally write them out;

you aim to set up the *structure* of a proof before filling in very much of the detail.

It won't be clear to you yet, but the proof here has a number of features which are so common that they're worth putting in your bag of tricks:

It starts with 'Let $\epsilon > 0$'

(because the statement of the theorem is Player I's first move, and we have to give Player II a free choice). It does occasionally happen that it's worth taking a bit of space to clear the ground before we ask Player II what he wants to do, but it can never be actually wrong to get ϵ into view before we start thinking about Player I's strategy.

$n_0 = \max(n_1, n_2)$.

This is by no means a universal rule; but it is exceedingly common that when we come to choose n_0, we just make it the biggest integer which has yet appeared.

$|x_n + y_n - b - c| \leq |x_n - b| + |y_n - c|$.

When you want to prove that a *sum* of any kind is less than or equal to something, then on most (not all) occasions it's worth breaking it up into pieces and using the triangle inequality. Of course it's not always obvious which pieces to use. For instance, it's perfectly true that

$$|(x_n + y_n) - (b + c)| \leq |x_n - c| + |y_n - b|,$$

and this fact is no use at all. However, the original hypotheses '$\lim_{n\to\infty} x_n = b$, $\lim_{n\to\infty} y_n = c$' should be a hint that we want to keep the x_n with b and the y_n with c.

The second theorem I expect you feel that learning forty proofs like the one above will make this a pretty hard term. So it would, if you had to learn them all independently. But in fact they run so close together that for many of them we only have to remember odd clauses in which they differ from another one. I will try to show this by giving a result which looks very different from the one here, but uses almost exactly the same ideas, if we look at it in the right way.

This theorem will be the same result, but for real functions instead of for sequences. Now there is an extra complication here, so I pause for a pair of definitions.

Definitions (a) A **real function** is a function f such that $\operatorname{dom} f \subseteq \mathbb{R}$ and $f(x) \in \mathbb{R}$ for every $x \in \operatorname{dom} f$.

(The point here is that a very large proportion of the important functions of mathematics aren't defined everywhere, starting with $\frac{1}{x}$, undefined at 0. We have to have some way of coping with this, and in the last fifty years it's become generally accepted that we should specify the *domain* of every function at the moment when we introduce it.)

(b) If f is a real function and $a \in \mathbb{R}$, then '$\lim_{x\to\infty} f(x) = a$' means

$$\forall\, \epsilon > 0\ \exists\, M \in \mathbb{R}\ \forall\, x \geq M,\ x \in \operatorname{dom} f \text{ and } |f(x) - a| \leq \epsilon.$$

Compare this with the definition of '$\lim_{n\to\infty} x_n = a$':

$$\forall\, \epsilon > 0\ \exists\, n_0 \in \mathbb{N}\ \forall\, n \geq n_0,\ |x_n - a| \leq \epsilon.$$

The $n_0 \in \mathbb{N}$ has turned into $M \in \mathbb{R}$, the $n \geq n_0$ has turned into $x \geq M$, and x_n has turned into $f(x)$. And there is a new complication, because when I was dealing with sequences I took it for granted that x_n would be defined for every relevant n. But when dealing with general real functions, one can't take that sort of thing without checking carefully; and before saying '$|f(x) - a| \leq \epsilon$' we have to say '$f(x)$ is defined'.

(c) If f and g are real functions, then $f + g$ is the real function defined by saying

$$\operatorname{dom}(f+g) = \operatorname{dom} f \cap \operatorname{dom} g, \quad (f+g)(x) = f(x) + g(x) \text{ for every } x \in \operatorname{dom}(f+g).$$

(Of course the *idea* is in the formula $(f+g)(x) = f(x) + g(x)$. The domain $\operatorname{dom}(f+g)$ specified is just the set on which we can calculate both $f(x)$ and $g(x)$ and use the formula.)

Now the theorem is this:

Theorem If f and g are real functions and $\lim_{x\to\infty} f(x) = b$, $\lim_{x\to\infty} g(x) = c$ then $\lim_{x\to\infty}(f+g)(x) = b + c$.

proof What I am going to do is to go through the proof which worked for sequences, make as few changes as possible, and see if it still works. Here goes:

Let $\epsilon > 0$. Then $\frac{\epsilon}{2} > 0$.

No problems so far.

Let $\epsilon > 0$. Then $\frac{\epsilon}{2} > 0$.

There is an $n_1 \in \mathbb{N}$ such that $|x_n - b| \leq \frac{\epsilon}{2}$ for every $n \geq n_1$.

Change '$n_1 \in \mathbb{N}$' into '$M_1 \in \mathbb{R}$', 'x_n' into '$f(x)$', '$n \geq n_1$' into '$x \geq M_1$':

Let $\epsilon > 0$. Then $\frac{\epsilon}{2} > 0$.

There is an $M \in \mathbb{R}$ such that $|f(x) - b| \leq \frac{\epsilon}{2}$ for every $x \geq M_1$.

This won't quite do, because we are talking about $f(x)$ before we've said whether there is such a number; we had better change to

There is an $M_1 \in \mathbb{R}$ such that $x \in \operatorname{dom} f$ and $|f(x) - b| \leq \frac{\epsilon}{2}$ for every $x \geq M_1$.

Carry on with the next line of the original proof:

Let $\epsilon > 0$. Then $\frac{\epsilon}{2} > 0$.

There is an $M_1 \in \mathbb{R}$ such that $x \in \operatorname{dom} f$ and $|f(x) - b| \leq \frac{\epsilon}{2}$ for every $x \geq M_1$.

There is an $n_2 \in \mathbb{N}$ such that $|y_n - c| \leq \frac{\epsilon}{2}$ for every $n \geq n_2$.

Translate this line in the same way as the one above:

Let $\epsilon > 0$. Then $\frac{\epsilon}{2} > 0$.

There is an $M_1 \in \mathbb{R}$ such that $x \in \operatorname{dom} f$ and $|f(x) - b| \leq \frac{\epsilon}{2}$ for every $x \geq M_1$.

There is an $M_2 \in \mathbb{R}$ such that $x \in \operatorname{dom} g$ and $|g(x) - c| \leq \frac{\epsilon}{2}$ for every $x \geq M_2$.

Look at the next line:

Let $\epsilon > 0$. Then $\frac{\epsilon}{2} > 0$.

There is an $M_1 \in \mathbb{R}$ such that $x \in \operatorname{dom} f$ and $|f(x) - b| \leq \frac{\epsilon}{2}$ for every $x \geq M_1$.

There is an $M_2 \in \mathbb{R}$ such that $x \in \operatorname{dom} g$ and $|g(x) - c| \leq \frac{\epsilon}{2}$ for every $x \geq M_2$.

Take $n_0 = \max(n_1, n_2)$.

We want M's not n's this time:

Let $\epsilon > 0$. Then $\frac{\epsilon}{2} > 0$.

There is an $M_1 \in \mathbb{R}$ such that $x \in \operatorname{dom} f$ and $|f(x) - b| \leq \frac{\epsilon}{2}$ for every $x \geq M_1$.

There is an $M_2 \in \mathbb{R}$ such that $x \in \operatorname{dom} g$ and $|g(x) - c| \leq \frac{\epsilon}{2}$ for every $x \geq M_2$.

Take $M = \max(M_1, M_2)$.

Carry on:

Let $\epsilon > 0$. Then $\frac{\epsilon}{2} > 0$.

There is an $M_1 \in \mathbb{R}$ such that $x \in \operatorname{dom} f$ and $|f(x) - b| \le \frac{\epsilon}{2}$ for every $x \ge M_1$.

There is an $M_2 \in \mathbb{R}$ such that $x \in \operatorname{dom} g$ and $|g(x) - c| \le \frac{\epsilon}{2}$ for every $x \ge M_2$.

Take $M = \max(M_1, M_2)$.
Let $n \ge n_0$.
Then $n \ge n_1$ and $n \ge n_2$, so $|x_n - b| \le \frac{\epsilon}{2}$ and $|y_n - c| \le \frac{\epsilon}{2}$ and

$$|(x_n + y_n) - (b + c)| \le |x_n - b| + |y_n - c| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This becomes

Let $\epsilon > 0$. Then $\frac{\epsilon}{2} > 0$.

There is an $M_1 \in \mathbb{R}$ such that $x \in \operatorname{dom} f$ and $|f(x) - b| \le \frac{\epsilon}{2}$ for every $x \ge M_1$.

There is an $M_2 \in \mathbb{R}$ such that $x \in \operatorname{dom} g$ and $|g(x) - c| \le \frac{\epsilon}{2}$ for every $x \ge M_2$.

Take $M = \max(M_1, M_2)$.
Let $x \ge M$.
Then $x \ge M_1$ and $x \ge M_2$, so $x \in \operatorname{dom} f$ and $x \in \operatorname{dom} g$ and $|f(x) - b| \le \frac{\epsilon}{2}$ and $|g(x) - c| \le \frac{\epsilon}{2}$ and

$$|(f(x) + g(x)) - (b + c)| \le |f(x) - b| + |g(y) - c| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

There is only one thing missing: we want '$(f+g)(x)$' in the last line instead of '$f(x)+g(x)$'. They are nearly the same thing, except that before talking about '$(f+g)(x)$' we ought to explain why $x \in \operatorname{dom}(f+g)$. But this is easy:

Let $\epsilon > 0$. Then $\frac{\epsilon}{2} > 0$.

There is an $M_1 \in \mathbb{R}$ such that $x \in \operatorname{dom} f$ and $|f(x) - b| \le \frac{\epsilon}{2}$ for every $x \ge M_1$.

There is an $M_2 \in \mathbb{R}$ such that $x \in \operatorname{dom} g$ and $|g(x) - c| \le \frac{\epsilon}{2}$ for every $x \ge M_2$.

Take $M = \max(M_1, M_2)$.
Let $x \ge M$.
Then $x \ge M_1$ and $x \ge M_2$, so $x \in \operatorname{dom} f$ and $x \in \operatorname{dom} g$ and $x \in \operatorname{dom}(f+g)$. Also $|f(x) - b| \le \frac{\epsilon}{2}$ and $|g(x) - c| \le \frac{\epsilon}{2}$, so

$$|(f+g)(x) - (b+c)| = |(f(x) + g(x)) - (b + c)| \le |f(x) - b| + |f(y) - c| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

And there is the proof, complete.

The next theorem For a couple of lectures now I shall be producing variations on the arguments above. I am going to ask you to learn them. The way to learn them is by comparing and contrasting. You will find that each proof has one or two points in which it differs from the others, and many more points in which it is almost the same. For my first example of this, let me give another result for which you know the fact but not the proof.

Theorem If $\lim_{n\to\infty} x_n = b$ and $\lim_{n\to\infty} y_n = c$, $\lim_{n\to\infty} x_n y_n = bc$.

proof I shall try to keep as close as possible to the line used in the proof for $\lim_{n\to\infty} x_n + y_n = b + c$. Try this:

Let $\epsilon > 0$.
There is an $n_1 \in \mathbb{N}$ such that $|x_n - b| \le ??$ whenever $n \ge n_1$.

There is an $n_2 \in \mathbb{N}$ such that $|y_n - c| \leq ??$ whenever $n \geq n_2$.
Set $n_0 = \max(n_1, n_2)$.
If $n \geq n_0$, then $n \geq n_1$ and $n \geq n_2$ so $|x_n - b| \leq ??$ and $|y_n - c| \leq ??$ and

$$|x_n y_n - bc| \leq ?? \leq \epsilon.$$

Now you see that I've put ?? in place of $\frac{\epsilon}{2}$, because it's hardly credible that the same formula will work. And in the last line we have a completely new problem to solve, because at this point we can expect to know something about $x_n - b$ and $y_n - c$, but it's not at all clear how to turn this into something about $x_n y_n - bc$. So here we need a new fact, which is

$$x_n y_n - bc = (x_n - b)(y_n - c) + b(y_n - c) + c(x_n - b).$$

This means that if $x_n - b$ and $y_n - c$ are both practically zero, the right-hand-side will be a sum of three small pieces and the left-hand-side will also be nearly zero. But just how small do $x_n - b$ and $y_n - c$ have to be? At this point it makes things much easier to use a technical trick which will be very useful elsewhere. Since we don't know what to put in the line

There is an $n_1 \in \mathbb{N}$ such that $|x_n - b| \leq ??$ whenever $n \geq n_1$,

invent a name for it; η, say. Then those two lines will read

There is an $n_1 \in \mathbb{N}$ such that $|x_n - b| \leq \eta$ whenever $n \geq n_1$.
There is an $n_2 \in \mathbb{N}$ such that $|y_n - c| \leq \eta$ whenever $n \geq n_2$,

and later on we shall have

If $n \geq n_0$, then $n \geq n_1$ and $n \geq n_2$ so $|x_n - b| \leq \eta$ and $|y_n - c| \leq \eta$.

Now the point of this method is to be absolutely clear where η came from. This section is part of Player I's scheming after Player II has chosen ϵ. So is it *Player I* who chooses η, probably in a way depending on ϵ, as a preliminary to looking for n_1 and n_2 and putting these together to produce n_0. Suppose he does this. Then, in the last line, we shall have

$$|x_n y_n - bc| = |(x_n - b)(y_n - c) + b(y_n - c) + c(x_n - b)|.$$

Now there is an absolutely standard way of treating this. Whenever you have the modulus of a sum, and you want to show that it is less than or equal to something, the first thing to try is the sum of the moduli:

$$\begin{aligned} |x_n y_n - bc| &= |(x_n - b)(y_n - c) + b(y_n - c) + c(x_n - b)| \\ &\leq |(x_n - b)(y_n - c)| + |b(y_n - c)| + |c(x_n - b)|. \end{aligned}$$

Next, given the modulus of a product, it is usually right to rewrite it as the product of the moduli:

$$\begin{aligned} |x_n y_n - bc| &= |(x_n - b)(y_n - c) + b(y_n - c) + c(x_n - b)| \\ &\leq |(x_n - b)(y_n - c)| + |b(y_n - c)| + |c(x_n - b)| \\ &= |x_n - b||y_n - c| + |b||y_n - c| + |c||x_n - b|. \end{aligned}$$

Remember that by the time we've got this far, we have $|x_n - b| \leq \eta$ (whatever η may be) and $|y_n - c| \leq \eta$. So we get

$$\begin{aligned} |x_n y_n - bc| &= |(x_n - b)(y_n - c) + b(y_n - c) + c(x_n - b)| \\ &\leq |(x_n - b)(y_n - c)| + |b(y_n - c)| + |c(x_n - b)| \\ &= |x_n - b||y_n - c| + |b||y_n - c| + |c||x_n - b| \\ &\leq \eta \cdot \eta + |b|\eta + |c|\eta \\ &= \eta(\eta + |b| + |c|). \end{aligned}$$

We shall therefore be home, with a win for Player I, if she can arrange that $\eta(\eta + |b| + |c|) \leq \epsilon$. Well, if she can solve quadratics, she will actually be able to get $\eta(\eta + |b| + |c|) = \epsilon$. But, first, this is a bother, and, second, there are difficulties with proving that numbers have square roots; so I much prefer to use a more primitive argument. Remember that Player I chooses η immediately after seeing the ϵ chosen by Player II. Player I is allowed to choose any $\eta > 0$. (η really must be strictly greater than 0 or it may not be true that

$|x_n - b|$ and $|y_n - c|$ are less than or equal to η for enough n.) In particular, Player I is certainly allowed to insist that $\eta \leq 1$. Now if $\eta \leq 1$, we just need $\eta(1 + |b| + |c|) \leq \epsilon$, that is, $\eta \leq \frac{\epsilon}{1+|b|+|c|}$. So if Player I takes η to be $\min(1, \frac{\epsilon}{1+|b|+|c|})$, she should win.

Let's write the whole proof out on this basis.

Let $\epsilon > 0$. [Player I's first move.]

Set $\eta = \min(1, \frac{\epsilon}{1+|b|+|c|}) > 0$. [Player II is getting into position.]

There is an $n_1 \in \mathbb{N}$ such that $|x_n - b| \leq \eta$ whenever $n \geq n_1$.

There is an $n_2 \in \mathbb{N}$ such that $|y_n - c| \leq \eta$ whenever $n \geq n_2$.

Set $n_0 = \max(n_1, n_2)$. [Player I's second move.]

If $n \geq n_0$, [Player II's second move]

then $n \geq n_1$ and $n \geq n_2$ so $|x_n - b| \leq \eta$ and $|y_n - c| \leq \eta$ and

$$\begin{aligned}
|x_n y_n - bc| &= |(x_n - b)(y_n - c) + b(y_n - c) + c(x_n - b)| \\
&\leq |(x_n - b)(y_n - c)| + |b(y_n - c)| + |c(x_n - b)| \\
&= |x_n - b||y_n - c| + |b||y_n - c| + |c||x_n - b| \\
&\leq \eta \cdot \eta + |b|\eta + |c|\eta \\
&= \eta(\eta + |b| + |c|) \\
&\leq \eta(1 + |b| + |c|) && \text{because } \eta \leq 1 \\
&\leq \epsilon && \text{because } \eta \leq \tfrac{\epsilon}{1+|b|+|c|}.
\end{aligned}$$

So Player I wins and $\lim_{n\to\infty} x_n y_n = bc$.

More real functions: Definition Suppose that f is a real function and that a, $b \in \mathbb{R}$. Then '$\lim_{x\to a} f(x) = b$' means

for every $\epsilon > 0$ there is a $\delta > 0$ such that $f(x)$ is defined and $|f(x) - b| \leq \epsilon$ whenever $0 < |x - a| \leq \delta$.

What is the idea here? The claim is that $f(x) \doteq b$ whenever $x \doteq a$ and $x \neq a$. (Remember that when we consider $\lim_{x\to a} f(x)$, one value of f which we *never* look at is $f(a)$, even if it's defined.) As before, it's Player II who decides what '$f(x) \doteq b$' means, by choosing $\epsilon > 0$, and following this Player I decides what '$x \doteq a$' means, by choosing a $\delta > 0$; finally Player II gets to choose x. As in the definition of $\lim_{x\to\infty}$, we have to provide for the possibility that f is not defined everywhere, and I do this by insisting that Player I must choose a δ such that $f(x)$ is defined whenever $|x - a| \leq \delta$ (and $x \neq a$); if she can't (for instance, if $a = 0$ and $f(x) = \sqrt{x}$ for $x \geq 0$, but undefined for $x < 0$), then Player I is bound to lose, and I don't allow myself to say '$\lim_{x\to 0} \sqrt{x} = 0$'. (Of course I do allow '$\lim_{x\downarrow 0} \sqrt{x} = 0$', but that's a different game.)

Definition If f and g are real functions, then $f \times g$ is the real function defined by saying

$$\operatorname{dom}(f \times g) = \operatorname{dom} f \cap \operatorname{dom} g, \quad (f \times g)(x) = f(x)g(x) \text{ for } x \in \operatorname{dom}(f \times g).$$

I give these definitions so that we can have a theorem about the limit of a product of real functions, as follows.

Theorem If f and g are real functions, and $\lim_{x\to a} f(x) = b$ and $\lim_{x\to a} g(x) = c$, then $\lim_{x\to a}(f\times g)(x) = bc$.

proof This is a translation of the theorem on the product of sequences, just as the theorem $\lim_{x\to\infty}(f + g)(x) = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$ is a translation of the theorem on the sum of sequences. Let me write it out directly.

Let $\epsilon > 0$. Set $\eta = \min(1, \frac{\epsilon}{1+|b|+|c|}) > 0$. There is a $\delta_1 > 0$ such that $x \in \operatorname{dom} f$ and $|f(x) - b| \leq \eta$ whenever $0 < |x-a| \leq \delta_1$. There is a $\delta_2 > 0$ such that $x \in \operatorname{dom} g$ and $|g(x)-c| \leq \eta$ whenever $0 < |x-a| \leq \delta_2$.

Set $\delta = \min(\delta_1, \delta_2)$. If $0 < |x - a| \leq \delta$, then $0 < |x - a| \leq \delta_1$ and $0 < |x - a| \leq \delta_2$, so $x \in \operatorname{dom} f$ and $x \in \operatorname{dom} g$ and $x \in \operatorname{dom}(f \times g)$; also $|f(x) - b| \leq \eta$ and $|g(x) - c| \leq \eta$, so

$$\begin{aligned}|f(x)g(x) - bc| &= |(f(x) - b)(g(x) - c) + b(g(x) - c) + c(f(x) - b)| \\ &\leq |(f(x) - b)(g(x) - c)| + |b(g(x) - c)| + |c(f(x) - b)| \\ &= |f(x) - b||g(x) - c| + |b||g(x) - c| + |c||f(x) - b| \\ &\leq \eta^2 + |b|\eta + |c|\eta = \eta(\eta + |b| + |c|) \leq \eta(1 + |b| + |c|) \leq \epsilon.\end{aligned}$$

As this works for any $\epsilon > 0$, Player I can win whatever Player II does at his first move, and $\lim_{x\to a}(f\times g)(x) = bc$ is true.

Remark Look again at the translation. ϵ and η do exactly the same things as before. n_1 and n_2 turn into δ_1 and δ_2, and n_0 turns into δ. But observe that δ is the *minimum* of δ_1 and δ_2, while n_0 was the *maximum* of n_1 and n_2. This is because of the rule change concerning the next move. In the case of sequences, Player II has to choose $n \geq n_0$. So in order to make him choose an n simultaneously greater than or equal to n_1 and n_2, Player II chooses n_0 so that $n_0 \geq n_1$ and $n_0 \geq n_2$, and the easiest such choice is $n_0 = \max(n_1, n_2)$. But in the case of $\lim_{x\to a}$, Player II has to choose x such that $|x - a| \leq \delta$, and Player I wants to be sure that $|x-a| \leq \delta_1$ and that $|x-a| \leq \delta_2$; so she takes $\delta = \min(\delta_1, \delta_2)$. As a general rule, when choosing moves for Player I, you aim to make things difficult for Player II, by reducing his choices as much as you can. In this case, it's done by making δ close to 0 (but remembering that $\delta = 0$ is cheating).

After this, the difference is mostly that every x_n turns into $f(x)$ and every y_n turns into $g(x)$; but, just as in the theorem on $\lim_{x\to\infty}(f + g)(x)$, we have to preface every statement about $f(x)$, $g(x)$ or $(f \times g)(x)$ with an explanation of why x belongs to the domain of the function. In the lines

there is a $\delta_1 > 0$ such that $x \in \operatorname{dom} f$ and $|f(x) - b| \leq \eta$ whenever $0 < |x - a| \leq \delta_1$, there is a $\delta_2 > 0$ such that $x \in \operatorname{dom} g$ and $|g(x) - c| \leq \eta$ whenever $0 < |x - a| \leq \delta_2$

this came from the definitions of $\lim_{x\to a} f(x) = b$, $\lim_{x\to a} g(x) = c$; in the line

$x \in \operatorname{dom} f$ and $x \in \operatorname{dom} g$ and $x \in \operatorname{dom}(f \times g)$

it came from the definition of $f \times g$.

Three more definitions As well as $\lim_{x\to\infty}$ and $\lim_{x\to a}$, we sometimes want to look at $\lim_{x\to-\infty} f(x)$, $\lim_{x\uparrow a} f(x)$ and $\lim_{x\downarrow a} f(x)$. The definitions are as follows.

$\lim_{x\to-\infty} f(x) = b$ means

for every $\epsilon > 0$ there is an $M \in \mathbb{R}$ such that $x \in \operatorname{dom} f$ and $|f(x) - b| \leq \epsilon$ whenever $x \leq M$.

(Note that the only difference between this and the definition of '$\lim_{x\to\infty} f(x) = b$' is that we have '$x \leq M$' instead of '$x \geq M$'. But of course this makes a big difference to Player I's tactics. When playing from the initial position '$\lim_{x\to\infty} f(x) = b$', Player I will generally take an M far, far to the right, so that Player II will be seriously constrained by the rule '$x \geq M$'. While if playing from initial position '$\lim_{x\to infty} f(x) = b$', Player I will generally take an M correspondingly far to the left, so that Player II will have to go to the edge of the universe to satisfy the requirement '$x \leq M$'.

$\lim_{x\downarrow a} f(x) = b$ means

for every $\epsilon > 0$ there is a $\delta > 0$ such that $x \in \operatorname{dom} f$ and $|f(x)-b| \leq \epsilon$ whenever $a < x \leq a+\delta$.

The condition '$a < x \leq a + \delta$' is an exact way of saying '$x \doteq a$ and $x > a$', or 'x is just to the right of a'.

Finally, $\lim_{x\uparrow a} f(x) = b$ means

for every $\epsilon > 0$ there is a $\delta > 0$ such that $x \in \operatorname{dom} f$ and $|f(x)-b| \leq \epsilon$ whenever $a-\delta \leq x < a$.

Theorem Suppose that $\lim_{n\to\infty} x_n = b \neq 0$. Then $\lim_{n\to\infty} \frac{1}{x_n} \to \frac{1}{b}$.

proof Let $\epsilon > 0$. Set $\eta = \min(\frac{|b|}{2}, \frac{1}{2}b^2\epsilon) > 0$. Then there is an $n_0 \in \mathbb{N}$ such that $|x_n - b| \leq \eta$ for every $n \geq n_0$. If $n \geq n_0$, then

$$|x_n| \geq |b| - |x_n - b|$$

(because $|x_n| + |x_n - b| = |x_n| + |b - x_n| \geq |x_n + b - x_n|$)

$$\geq |b| - \eta \geq |b| - \frac{1}{2}|b| = \frac{1}{2}|b| > 0,$$

so that $x_n \neq 0$ and $\frac{1}{x_n}$ is defined and

$$|\frac{1}{x_n} - \frac{1}{b}| = |\frac{b-x_n}{x_n b}| = \frac{|b-x_n|}{|x_n||b|}$$

$$\leq \frac{\eta}{|x_n||b|}$$

(because $|b - x_n| = |x_n - b| \leq \eta$)

$$\leq \frac{\eta}{\frac{1}{2}|b||b|}$$

(because $|x_n| \geq \frac{1}{2}|b|$)

$$= \frac{2\eta}{b^2} \leq \frac{2}{b^2} \cdot \frac{1}{2} b^2 \epsilon = \epsilon.$$

As ϵ is arbitrary, $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{b}$.

Remarks Some of the ideas here are new, some are taken from earlier proofs. The idea of taking n_0 such that $|x_n - b| \leq \eta$ for $n \geq n_0$, where η is some more or less complicated function of ϵ, is taken from the theorems on products of sequences or functions. This time we have only one sequence so we don't have the step '$n_0 = \max(n_1, n_2)$'. What we have instead is a more complicated string of inequalities at the end. The steps

$$|\frac{1}{x_n} - \frac{1}{b}| = \ldots = \frac{|b-x_n|}{|x_n||b|}$$

are standard; nearly always, in these expressions, if we have the modulus of a product or quotient we express it as the product or quotient of the moduli and see what happens. Now in the expression

$$\frac{|b-x_n|}{|x_n||b|}$$

the $|b - x_n|$ on top is no problem at all; we know that by the time we've reached this stage, we shall have $|b - x_n| \leq \eta$, so we just replace it by η. The new difficulty is in the $|x_n|$ on the bottom. If we are to be sure that that $\frac{\eta}{|x_n||b|}$ is *less* than or equal to ϵ, we shall need to know that $|x_n|$ is *greater* than or equal to something. And that's where the line

$$|x_n| \geq |b| - |x_n - b| \ldots$$

comes in. Provided Player I takes $\eta < |b|$, she can be sure that $|x_n| \geq |b| - \eta$ so that $\frac{1}{|x_n|} \leq \frac{1}{|b|-\eta}$ and $\frac{\eta}{|x_n||b|} \leq \frac{\eta}{(|b|-\eta)|b|}$, and by making η small enough she can ensure that this will be at most ϵ. The actual formula

$$\eta = \min(\frac{|b|}{2}, \frac{1}{2}b^2\epsilon)$$

is just a trick for guaranteeing the three facts

$$\eta > 0, \quad \eta < |b|, \quad \frac{\eta}{(|b|-\eta)|b|} \leq \epsilon$$

without solving any quadratic equations.

Definition Let f be a real function. Then the real function $\frac{1}{f}$ is defined by saying

$$\operatorname{dom}\frac{1}{f} = \{x : x \in \operatorname{dom} f,\ f(x) \neq 0\},$$

$$\frac{1}{f}(x) = \frac{1}{f(x)} \text{ for every } x \in \operatorname{dom}\frac{1}{f}.$$

Theorem Let f be a real function, and suppose that $\lim_{x\to-\infty} f(x) = b \neq 0$. Then $\lim_{x\to-\infty} \frac{1}{f}(x) = \frac{1}{b}$.

proof Let $\epsilon > 0$. Set $\eta = \min(\frac{|b|}{2}, \frac{1}{2}b^2\epsilon) > 0$. Then there is an $M \in \mathbb{R}$ such that $x \in \operatorname{dom} f$ and $|f(x) - b| \leq \eta$ for every $x \leq M$. If $x \leq M$, then $f(x)$ is defined and

$$\begin{aligned}|f(x)| &\geq |b| - |f(x) - b| \\ &\geq |b| - \eta \geq \frac{1}{2}|b| > 0,\end{aligned}$$

so that $f(x) \neq 0$ and $x \in \operatorname{dom}\frac{1}{f}$ is defined and

$$\begin{aligned}\left|\frac{1}{f(x)} - \frac{1}{b}\right| = \frac{|b - f(x)|}{|f(x)||b|} &\leq \frac{\eta}{\frac{1}{2}|b||b|} \\ &= \frac{2\eta}{b^2} \leq \epsilon.\end{aligned}$$

As ϵ is arbitrary, $\lim_{x\to-\infty} \frac{1}{f(x)} = \frac{1}{b}$.

Continuous Functions: Definition Let f be a real function. We say that f **is continuous at** x_0 if $x_0 \in \operatorname{dom} f$ and

for every $\epsilon > 0$ there is a $\delta > 0$ such that

$|f(x) - f(x_0)| \leq \epsilon$ whenever $x \in \operatorname{dom} f$ and $|x - x_0| \leq \delta$.

Remark Note a very important rule change compared with the formula $\lim_{x\to x_0} f(x) = f(x_0)$. If Player I says '$f$ is continuous at x_0', then Player II chooses ϵ, Player I chooses δ and Player II chooses x, just as before. But this time it is Player II's responsibility to ensure that $x \in \operatorname{dom} f$. For '$\lim_{x\to x_0} f(x) = b$', Player II was allowed any x such that $0 < |x - x_0| \leq \delta$, and if he could find one outside the domain of f he would win. But for 'f is continuous at x_0', Player II has to pick $x \in \operatorname{dom} f$. He is now allowed to pick $x = x_0$, but of course that does him no good at all (which is why it's allowed). So Player II has a lot less freedom at his second move, and it's easier for Player I to win.

The reason for this change is that it's useful to be able to say that $\sqrt{}$ is continuous at 0. But there is no way of making sense of the formula $\lim_{x\uparrow 0} \sqrt{x}$ (if we want to stick to real numbers), and if we want to keep the rule

$$\lim_{x\to a} f(x) = b \text{ iff } \lim_{x\downarrow a} f(x) = \lim_{x\uparrow a} f(x) = b,$$

then we are going to have to abandon any attempt to interpret the formula $\lim_{x\to 0} \sqrt{x} = 0$, at least for the 'real' function $\sqrt{}$.

Theorem If f and g are continuous real functions, so are $f + g$, $f \times g$ and $\frac{1}{f}$.

proof (a) Take $x_0 \in \operatorname{dom}(f + g)$ and $\epsilon > 0$. Let δ_1, $\delta_2 > 0$ be such that

$$|f(x)-f(x_0)| \le \frac{1}{2}\epsilon \text{ whenever } x \in \operatorname{dom} f \text{ and } |x-x_0| \le \delta_1,$$

$$|g(x)-g(x_0)| \le \frac{1}{2}\epsilon \text{ whenever } x \in \operatorname{dom} g \text{ and } |x-x_0| \le \delta_2.$$

Set $\delta = \min(\delta_1, \delta_2) > 0$. Then if $x \in \operatorname{dom}(f+g)$ and $|x-x_0| \le \delta$,

$$|(f+g)(x)-(f+g)(x_0)| = |f(x)+g(x)-f(x_0)-g(x_0)| \le |f(x)-f(x_0)|+|g(x)-g(x_0)| \le \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.$$

As x_0 and ϵ are arbitrary, $f+g$ is continuous.

(b) Take $x_0 \in \operatorname{dom}(f \times g)$ and $\epsilon > 0$. Set $\eta = \min(1, \frac{\epsilon}{1+|f(x_0)|+|g(x_0)|}) > 0$. Let δ_1, $\delta_2 > 0$ be such that

$$|f(x)-f(x_0)| \le \eta \text{ whenever } x \in \operatorname{dom} f \text{ and } |x-x_0| \le \delta_1,$$

$$|g(x)-g(x_0)| \le \eta \text{ whenever } x \in \operatorname{dom} g \text{ and } |x-x_0| \le \delta_2.$$

Set $\delta = \min(\delta_1, \delta_2) > 0$. Then if $x \in \operatorname{dom}(f \times g)$ and $|x-x_0| \le \delta$,

$$\begin{aligned}
|(f\times g)(x)-(f\times g)(x_0)| &= |f(x)g(x)-f(x_0)g(x_0)| \\
&\le |f(x)-f(x_0)||g(x)-g(x_0)| + |f(x_0)||g(x)-g(x_0)| + |f(x)-f(x_0)||g(x_0)| \\
&\le \eta(\eta+|f(x_0)|+|g(x_0)| \le \eta(1+|f(x_0)|+|g(x_0)| \le \epsilon.
\end{aligned}$$

As x_0 and ϵ are arbitrary, $f+g$ is continuous.

(c) Take $x_0 \in \operatorname{dom}(\frac{1}{f})$ and $\epsilon > 0$. Set $\eta = \min(\frac{1}{2}|f(x_0)|, \frac{1}{2}\epsilon|f(x_0)|^2) > 0$. Let $\delta > 0$ be such that

$$|f(x)-f(x_0)| \le \eta \text{ whenever } x \in \operatorname{dom} f \text{ and } |x-x_0| \le \delta.$$

Then if $x \in \operatorname{dom}(\frac{1}{f})$ and $|x-x_0| \le \delta$,

$$|f(x)| \ge |f(x_0)| - |f(x)-f(x_0)| \ge |f(x_0)| - \eta \ge \frac{1}{2}|f(x_0)|,$$

so

$$\begin{aligned}
|(\frac{1}{f})(x) - (\frac{1}{f})(x_0)| &= |\frac{1}{f(x)} - \frac{1}{f(x_0)}| = \frac{|f(x_0)-f(x)|}{|f(x)||f(x_0)|} \\
&\le \frac{\eta}{\frac{1}{2}|f(x_0)||f(x_0)|} \le \epsilon.
\end{aligned}$$

As x_0 and ϵ are arbitrary, $\frac{1}{f}$ is continuous.

Definition If f and g are real functions, their **composition** $f \circ g$ is defined by saying

$$\operatorname{dom}(f\circ g) = \{x : x \in \operatorname{dom} g,\ g(x) \in \operatorname{dom} f\}, \quad (f\circ g)(x) = f(g(x)) \text{ for } x \in \operatorname{dom}(f\circ g).$$

Theorem If f and g are continuous real functions, then $f\circ g$ is continuous.

proof Take $x_0 \in \operatorname{dom}(f\circ g)$ and $\epsilon > 0$. Then $g(x_0) \in \operatorname{dom} f$ so there is an $\eta > 0$ such that $|f(y) - f(g(x_0))| \le \epsilon$ whenever $y \in \operatorname{dom} f$ and $|y - g(x_0)| \le \eta$. Next, $x_0 \in \operatorname{dom} g$ so there is a $\delta > 0$ such that $|g(x)-g(x_0)| \le \eta$ whenever $x \in \operatorname{dom} g$ and $|x-x_0| \le \delta$.

If $x \in \operatorname{dom}(f\circ g)$ and $|x-x_0| \le \delta$, then $g(x) \in \operatorname{dom} f$ and $|g(x)-g(x_0)| \le \eta$, so

$$|(f\circ g)(x) - (f\circ g)(x_0)| = |f(g(x)) - f(g(x_0))| \le \epsilon.$$

As x_0 and ϵ are arbitrary, $f\circ g$ is continuous.

Examples (a) Constant functions are continuous. **P** Suppose that $f(x) = c$ for $x \in \operatorname{dom} f$. Take $x_0 \in \operatorname{dom} f$ and $\epsilon > 0$. Set $\delta = 1$. Then if $x \in \operatorname{dom} f$ and $|x-x_0| \le \delta$,

$$|f(x) - f(x_0)| = |c - c| = 0 \leq \epsilon.$$

As x_0 and ϵ are arbitrary, f is continuous. **Q**

Remark Observe that in this (quite exceptional) case, Player I can announce her second move $\delta = 1$ *before* Player II has played his first move; her position is so strong that he has no way of using the advance knowledge.

(b) Identity functions are continuous. **P** Suppose that $f(x) = x$ for $x \in \operatorname{dom} f$. Take $x_0 \in \operatorname{dom} f$ and $\epsilon > 0$. Set $\delta = \epsilon$. Then if $x \in \operatorname{dom} f$ and $|x - x_0| \leq \delta$,

$$|f(x) - f(x_0)| = |x - x_0| \leq \delta = \epsilon.$$

As x_0 and ϵ are arbitrary, f is continuous. **Q**

Remark In *this* case, Player I has to know part of Player II's move; but if she doesn't pay attention properly and misses the announcement of x_0, she can still win by playing $\delta = \epsilon$, and ask what x_0 is afterwards, when they come to check the calculation of $f(x) - f(x_0)$.

(c) The function $x \mapsto |x| : \mathbb{R} \to \mathbb{R}$ is continuous. **P** Take $x_0 \in \mathbb{R}$ and $\epsilon > 0$. Set $\delta = \epsilon$. Then if $x \in \operatorname{dom} f$ and $|x - x_0| \leq \delta$,

$$||x| - |x_0|| \leq |x - x_0| \leq \delta = \epsilon.$$

As x_0 and ϵ are arbitrary, $|\ |$ is continuous. **Q**

Corollary For any continuous real function f, $x \mapsto |f(x)| : \operatorname{dom} f \to \mathbb{R}$ is continuous. **P** This is just the composition $|\ ||\circ f$ of two continuous functions. **Q**

Remark We now know that $f + g$, $f \times g$, $\frac{1}{f}$ and $f\circ g$ are continuous whenever f and g are, and that $x \mapsto c$, $x \mapsto x$ and $x \mapsto |x|$ are continuous functions from $\mathbb{R}$ to itself for any $c \in \mathbb{R}$. Putting these together, we see (for instance) that

$x \mapsto x^2 : \mathbb{R} \to \mathbb{R}$ is continuous

(because this is just $f \times f$, where $f(x) = x$ for every $x \in \mathbb{R}$),

$x \mapsto 2x^2 : \mathbb{R} \to \mathbb{R}$ is continuous

(being the product of the continuous functions $x \mapsto x^2$ and $x \mapsto 2$),

$x \mapsto 2x^3 : \mathbb{R} \to \mathbb{R}$ is continuous

(being the product of the continuous functions $x \mapsto 2x^2$ and $x \mapsto x$),

$x \mapsto \frac{1}{x^2} : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is continuous

(being the reciprocal of a continuous function),

$x \mapsto x + \frac{1}{x^2} : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is continuous

(being the sum of two continuous functions),

$x \mapsto 2(x + \frac{1}{x^2})^3 : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is continuous

(being the composition of the functions $x \mapsto 2x^3$ and $x \mapsto x + \frac{1}{x^2}$). Generally, most of the functions we have names for are continuous; and we can prove that a function with a formula like

$$h(x) = \tfrac{\sin(3x+1)}{1 - \ln\cos x}$$

is continuous as soon as we know that sin, cos, ln are continuous (which I am afraid will be just outside the scope of this course).

Dedekind completeness I come now to the final basic property of the real number system which I left out of the initial list of properties of addition, multiplication and the order relation. For this we need some terminology. If $A \subseteq \mathbb{R}$ is any set, an **upper bound** of A is an $x \in \mathbb{R}$ such that $a \leq x$ for every $a \in A$, and a **lower bound** of A is an $x \in \mathbb{R}$ such that $x \leq a$ for every $a \in A$.

Examples If $A = [0, 1]$ then 2, π, 1 are upper bounds of A and -1, 0 are lower bounds.

If $A = \mathbb{N}$ then -1, $-\frac{1}{4}$, 0 are lower bounds of A, but A has no upper bounds.
If $A = \mathbb{Z}$ then A has no upper bounds and no lower bounds.
If $A =]0,1[$ then 1 is an upper bound of A but $0{\cdot}999999$ is not, because $0{\cdot}9999991 \in A$.
If $A = \emptyset$ then every real number is both an upper bound and a lower bound for A.

If A has a least upper bound, I will call it the **supremum** of A, $\sup A$; if it has a greatest lower bound, I call it the **infimum** of A, $\inf A$.

Examples If $A = [0,1]$ then $\sup A = 1$ and $\inf A = 0$.
If $A = \mathbb{N}$ then $\inf A = 0$ but A has no supremum (because it has no upper bounds at all).
If $A = \mathbb{Z}$ then A has no supremum and no infimum.
If $A =]0,1[$ then $\sup A = 1$ and $\inf A = 0$.
If $A = \emptyset$ then A has no supremum and no infimum.

Note that $]0,1[$ and $[0,1]$ have exactly the same upper bounds; $x \in \mathbb{R}$ is an upper bound for either iff $x \geq 1$, and 1 is the least upper bound of both. This shows that $\sup A$ (when defined) may or may not belong to the set A.

FUNDAMENTAL FACT If $A \subseteq \mathbb{R}$ is non-empty and has an upper bound, it has a least upper bound. Similarly, if $A \subseteq \mathbb{R}$ is non-empty and has a lower bound, it has a greatest lower bound.

Remarks What these principles are saying is that a subset of $\mathbb{R}$ has a supremum and an infimum unless it plainly can't, either because it doesn't have any bounds on the appropriate side, or because it is empty and has altogether too many upper and lower bounds.

The idea goes back to classical times; in a geometric form it was proposed by Eudoxus. It re-surfaced in the nineteenth century as part of the general programme of putting calculus on a sound logical footing, and is now generally called 'Dedekind's axiom' or 'the principle of Dedekind completeness of $\mathbb{R}$'.

Convergent sequences A large part of the rest of the course will involve Dedekind completeness in one way or another. For the first application I will give two of the most important theorems on convergence of sequences. We need some definitions.

Definitions A subset A of $\mathbb{R}$ is **bounded** if it has both upper and lower bounds. A real sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ is **bounded** if $\{x_n : n \in \mathbb{N}\}$ is bounded, that is, there are a, $b \in \mathbb{R}$ such that $a \leq x_n \leq b$ for every $n \in \mathbb{N}$.

A real sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ is **non-decreasing** if $x_n \leq x_{n+1}$ for every $n \in \mathbb{N}$, and **non-increasing** if $x_{n+1} \leq x_n$ for every n. Finally, a sequence is **monotonic** if it is either non-decreasing or non-increasing (or both).

Examples (i) If $x_n = 2^n$ for $n \in \mathbb{N}$, then $\langle x_n\rangle_{n\in\mathbb{N}}$ is unbounded, monotonic (non-decreasing), not convergent.

(ii) If $x_n = \frac{1}{n}$ for $n \geq 1$, then $\langle x_n\rangle_{n\geq 1}$ is bounded, monotonic (non-increasing), convergent (to 0).

(iii) If $x_n = (-1)^n$ for $n \in \mathbb{N}$, then $\langle x_n\rangle_{n\in\mathbb{N}}$ is bounded, not monotonic, not convergent.

(iv) If $x_n = (-2)^n$ for $n \in \mathbb{N}$, then $\langle x_n\rangle_{n\in\mathbb{N}}$ is unbounded, not monotonic, not convergent.

(v) If $x_n = 3$ for $n \in \mathbb{N}$, then $\langle x_n\rangle_{n\in\mathbb{N}}$ is bounded, monotonic (simultaneously non-decreasing and non-increasing), convergent (to 3).

(vi) If $x_n = \frac{(-1)^n}{n}$ for $n \geq 1$, then $\langle x_n\rangle_{n\geq 1}$ is bounded, not monotonic, convergent (to 0).

Theorem A bounded monotonic sequence is convergent.

proof Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be a bounded monotonic sequence.

case 1 Suppose that $\langle x_n\rangle_{n\in\mathbb{N}}$ is non-decreasing. Set $A = \{x_n : n \in \mathbb{N}\}$. Because $\langle x_n\rangle_{n\in\mathbb{N}}$ is bounded, A has an upper bound; because $x_0 \in A$, A is not empty; so $\sup A$ is defined in $\mathbb{R}$; call it b.

Let $\epsilon > 0$. Then $b - \epsilon < b$, so $b - \epsilon$ is not an upper bound of A; let $n_0 \in \mathbb{N}$ be such that $x_{n_0} > b - \epsilon$. If $n \geq n_0$, then

$$b - \epsilon \leq x_{n_0} \leq x_n$$

(because $\langle x_i\rangle_{i\in\mathbb{N}}$ is non-decreasing and $n_0 \leq n$)

$$\leq b$$

(because $x_n \in A$ and b is an upper bound of A)

$$\leq b + \epsilon.$$

So $x_n \in [b - \epsilon, b + \epsilon]$ and $|x_n - b| \leq \epsilon$. As ϵ is arbitrary, $\lim_{n\to\infty} x_n = b$ and $\langle x_n \rangle_{n\in\mathbb{N}}$ is convergent.

case 2 Suppose that $\langle x_n \rangle_{n\in\mathbb{N}}$ is non-increasing.

first method Set $A = \{x_n : n \in \mathbb{N}\}$. Because $\langle x_n \rangle_{n\in\mathbb{N}}$ is bounded, A has a lower bound; because $x_0 \in A$, A is not empty; so $\inf A$ is defined in $\mathbb{R}$; call it b.

Let $\epsilon > 0$. Then $b + \epsilon > b$, so $b + \epsilon$ is not a lower bound of A; let $n_0 \in \mathbb{N}$ be such that $x_{n_0} < b + \epsilon$. If $n \geq n_0$, then

$$b + \epsilon \geq x_{n_0} \geq x_n$$

(because $\langle x_i \rangle_{i\in\mathbb{N}}$ is non-increasing and $n_0 \leq n$)

$$\geq b$$

(because $x_n \in A$ and b is a lower bound of A)

$$\geq b - \epsilon.$$

So $x_n \in [b - \epsilon, b + \epsilon]$ and $|x_n - b| \leq \epsilon$. As ϵ is arbitrary, $\lim_{n\to\infty} x_n = b$ and $\langle x_n \rangle_{n\in\mathbb{N}}$ is convergent.

second method $\langle -x_n \rangle_{n\in\mathbb{N}}$ is non-decreasing and bounded, so by case 1 there is a b such that $\lim_{n\to\infty}(-x_n) = b$. Now $\lim_{n\to\infty} x_n = -b$ so $\langle x_n \rangle_{n\in\mathbb{N}}$ is convergent.

Remarks If we think of this theorem in terms of the Analysis Game, the moves are as follows:

Player I says 'bounded monotonic sequences are convergent'.

This is of the form 'every bounded monotonic sequence ...', so

Player II chooses a bounded monotonic sequence $\langle x_n \rangle_{n\in\mathbb{N}}$.

Now Player I is claiming, of this particular bounded monotonic sequence, that it has a limit. So she has to tell us what that limit is; that is,

Player I chooses b.

At this point, Player I is claiming that $\lim_{n\to\infty} x_n = b$, so they turn to the very first game:

Player II chooses $\epsilon > 0$
Player I chooses $n_0 \in \mathbb{N}$
Player II chooses $n \geq n_0$
and they look to see whether $|x_n - b| \leq \epsilon$, or not.

The proof of the theorem consists, as usual, of a description of a strategy for Player I. Because there are two separate moves (the choice of b and the choice of n_0) to manage, we either have to be very good at looking ahead or remember at least one of them. The rule for b is

$$\begin{aligned} b &= \sup\{x_n : n \in \mathbb{N}\} \text{ if } \langle x_n \rangle_{n\in\mathbb{N}} \text{ is non-decreasing,} \\ &= \inf\{x_n : n \in \mathbb{N}\} \text{ if } \langle x_n \rangle_{n\in\mathbb{N}} \text{ is non-increasing.} \end{aligned}$$

The rule for n_0 is

$$\begin{aligned} \text{choose } n_0 \text{ such that } x_{n_0} &\geq b - \epsilon \text{ if } \langle x_n \rangle_{n\in\mathbb{N}} \text{ is non-decreasing,} \\ &\leq b + \epsilon \text{ if } \langle x_n \rangle_{n\in\mathbb{N}} \text{ is non-increasing.} \end{aligned}$$

Most of the proof amounts, in fact, to making sure that these rules can be applied; if we are going to set $b = \sup\{x_n : n \in \mathbb{N}\}$, we must first check that $\{x_n : n \in \mathbb{N}\}$ is non-empty and bounded above.

Note that the theorem depends on two hypotheses: the sequence must be simultaneously bounded and monotonic; if it's not bounded (like $x_n = 2^n$), or not monotonic (like $x_n = (-1)^n$) it may fail to be convergent. In the proof, therefore, we must use both these facts. The assumption that $\langle x_n \rangle_{n\in\mathbb{N}}$ is bounded is used when Player I chooses b, and the assumption that it is monotonic is used at the checking stage, to see that x_n is on the correct side of x_{n_0} and is therefore at least as close to b as x_{n_0} is.

There is a similar exact economy in our use of the properties of b. Looking at case 1 in the proof above, in which $b = \sup A$, we use the fact that b is an upper bound of A at the end, where we say that $x_n \le b \le b + \epsilon$; we have already used the fact that A has no upper bound less than b, when we said that $b - \epsilon$ is not an upper bound of A, so there is an $n_0 \in \mathbb{N}$ such that $x_{n_0} > b - \epsilon$. When you are writing out a proof of this kind, you should look for these things; they are a check that you aren't leaving anything out.

Cauchy sequences: Definition A real sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ is a **Cauchy sequence** if for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $|x_m - x_n| \le \epsilon$ for all m, $n \ge n_0$.

Proposition Every Cauchy sequence is bounded.

proof Let $\langle x_n \rangle_{n\in\mathbb{N}}$ be a Cauchy sequence. Then there is an $n_0 \in \mathbb{N}$ such that $|x_m - x_n| \le 1$ for every m, $n \ge n_0$; in particular, $|x_n - x_{n_0}| \le 1$ for every $n \ge n_0$. Set

$$M = \max(|x_0|, |x_1|, \dots, |x_{n_0}|, |x_{n_0}| + 1);$$

then M is the maximum of a finite string of real numbers, so is finite. If $n \le n_0$, then $|x_n| \le M$ because $|x_n|$ is listed in the string; if $n \ge n_0$, then

$$|x_n| \le |x_{n_0}| + |x_n - x_{n_0}| \le |x_{n_0}| + 1 \le M.$$

So $|x_n| \le M$, that is, $-M \le x_n \le M$, for every $n \in \mathbb{N}$, and $\{x_n : n \in \mathbb{N}\}$ is bounded.

Theorem Every Cauchy sequence is convergent.

proof Let $\langle x_n \rangle_{n\in\mathbb{N}}$ be a Cauchy sequence. Then it is bounded, by the proposition just above; say a, $b \in \mathbb{R}$ are such that $a \le x_n \le b$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ set $A_n = \{x_i : i \ge n\}$. Then a is a lower bound for A_n and $x_n \in A_n$, so A_n is a non-empty set with a lower bound and has an infimum $y_n = \inf A_n$, with $a \le y_n \le x_n$. Next, $A_{n+1} \subseteq A_n$ (in fact, $A_n = A_{n+1} \cup \{x_n\}$), so y_n is also a lower bound for A_{n+1}, and $y_n \le y_{n+1}$.

This is true for every $n \in \mathbb{N}$, so $\langle y_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence. Moreover, $a \le y_n \le x_n \le b$ for every n, so $\langle y_n \rangle_{n\in\mathbb{N}}$ is bounded. It is therefore convergent, by the last theorem; let c be its limit.

Now $\lim_{n\to\infty} x_n - y_n = 0$. **P** Let $\epsilon > 0$. Then there is an $n_0 \in \mathbb{N}$ such that $|x_i - x_n| \le \epsilon$ whenever n, $i \ge n_0$. Let $n \ge n_0$. Then, for any $i \ge n$, $|x_i - x_n| \le \epsilon$, so $x_i \ge x_n - \epsilon$. This means that $x_n - \epsilon$ is a lower bound for A_n; since y_n is the greatest lower bound, $x_n - \epsilon \le y_n$. But we already know that $y_n \le x_n$, so $|x_n - y_n| \le \epsilon$, and this is true for every $n \ge n_0$. As ϵ is arbitrary, $\lim_{n\to\infty} x_n - y_n = 0$. **Q**

Since $\lim_{n\to\infty} x_n - y_n = 0$ and $\lim_{n\to\infty} y_n = c$,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_n - y_n + y_n = 0 + c = c.$$

Thus $\langle x_n \rangle_{n\in\mathbb{N}}$ is convergent; as $\langle x_n \rangle_{n\in\mathbb{N}}$ was arbitrary, the theorem is proved.

***Remark** Observe that we have a formula for the limit of $\langle x_n \rangle_{n\in\mathbb{N}}$: it is

$$c = \lim_{n\to\infty} y_n = \sup\{y_n : n \in \mathbb{N}\}$$

(see the proof of the theorem that bounded monotonic sequences are convergent)

$$= \sup\{\inf\{x_i : i \ge n\} : n \in \mathbb{N}\}.$$

This last formula makes sense for any *bounded* sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ (note that in the proof above we used the fact that $\langle x_n \rangle_{n\in\mathbb{N}}$ is bounded right at the beginning, but didn't need to know any more until we came to look at $\lim_{n\to\infty} x_n - y_n$); and it has a name; it's called '$\liminf_{n\to\infty} x_n$'.

Theorem Every convergent sequence is Cauchy.

proof Let $\langle x_n \rangle_{n\in\mathbb{N}}$ be a convergent sequence, with limit b. Let $\epsilon > 0$. Then there is an $n_0 \in \mathbb{N}$ such that $|x_n - b| \le \frac{\epsilon}{2}$ for every $n \ge n_0$. If m, $n \ge n_0$,

$$|x_m - x_n| \le |x_m - b| + |x_n - b| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

As ϵ is arbitrary, $\langle x_n\rangle_{n\in\mathbb{N}}$ is Cauchy.

Cauchy's General Principle of Convergence Putting the last two theorems together, we have the following fundamental principle:

A real sequence is convergent iff it is Cauchy.

This is one of the most basic properties of the real numbers.

Summation: Definitions

A 'series' is a sequence we mean to try to add up.

If $\langle x_k\rangle_{k\in\mathbb{N}}$ is a series, its **sequence of partial sums** is the sequence $\langle s_n\rangle_{n\in\mathbb{N}}$ defined by

$$s_n = \sum_{k=0}^{n} x_k \text{ for every } n\in\mathbb{N}.$$

The **sum** of the series is

$$\sum_{k=0}^{\infty} x_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\sum_{k=0}^{n} x_k$$

if this is defined.

The sum of the series is the limit of the sequence of partial sums.

A series $\langle x_k\rangle_{k\in\mathbb{N}}$ is **summable** if its sequence of partial sums is convergent, that is, $\sum_{k=0}^{\infty} x_k$ is defined as a real number (not allowing $\pm\infty$).

A series $\langle x_k\rangle_{k\in\mathbb{N}}$ is **absolutely summable** if the series $\langle |x_k|\rangle_{k\in\mathbb{N}}$ of its absolute values is summable, that is, $\sum_{k=0}^{\infty}|x_k| = \lim_{n\to\infty}\sum_{k=0}^{n}|x_k|$ is defined as a real number (not allowing ∞).

Remark Some of our favourite series don't begin at x_0; e.g., $x_k = \frac{1}{k}$. For such a series $\langle x_k\rangle_{k\ge1}$, the sequence of partial sums has to start at the same point, and we have $\langle s_n\rangle_{n\ge1}$, where $s_n = \sum_{k=1}^{n} x_k$, and

$$\sum_{k=1}^{\infty} x_k = \lim_{n\to\infty}\sum_{k=1}^{n} x_k$$

if this is defined.

Theorem An absolutely summable series is summable.

proof Let $\langle x_k\rangle_{k\in\mathbb{N}}$ be an absolutely summable series. For each $n\in\mathbb{N}$, set

$$s_n = \sum_{k=0}^{n} x_k, \quad t_n = \sum_{k=0}^{n}|x_k|.$$

Then $|s_m - s_n| \le |t_m - t_n|$ for all $m, n\in\mathbb{N}$. **P** (i) If $n < m$, then

$$|s_m - s_n| = |x_{n+1} + x_{n+2} + \ldots + x_m| \le |x_{n+1}| + |x_{n+2}| + \ldots + |x_m| = t_m - t_n = |t_m - t_n|.$$

(ii) If $m = n$ then $|s_m - s_n| = 0 = |t_m - t_n|$. (iii) If $n < m$ then $|s_m - s_n| = |s_n - s_m| \le t_n - t_m = |t_m - t_n|$. **Q**

Now we are supposing that $\langle x_k\rangle_{k\in\mathbb{N}}$ is absolutely summable, that is, that $\langle t_n\rangle_{n\in\mathbb{N}}$ is convergent. We know that convergent sequences are Cauchy, so $\langle t_n\rangle_{n\in\mathbb{N}}$ is Cauchy. It follows that $\langle s_n\rangle_{n\in\mathbb{N}}$ is Cauchy. **P** For any $\epsilon > 0$, there is an $n_0\in\mathbb{N}$ such that $|t_m - t_n| \le \epsilon$ for all $m, n \ge n_0$. So $|s_m - s_n| \le |t_m - t_n| \le \epsilon$ for all m, $n \ge n_0$. As ϵ is arbitrary, $\langle s_n\rangle_{n\in\mathbb{N}}$ is Cauchy. **Q**

But now remember that all Cauchy sequences are convergent. So $\langle s_n\rangle_{n\in\mathbb{N}}$ is convergent, that is, $\langle x_k\rangle_{k\in\mathbb{N}}$ is summable.

The same ideas can be used to prove the Comparison Test. I give a simple form.

Theorem Suppose that $0 \le x_k \le y_k$ for every $k\in\mathbb{N}$ and that $\langle y_k\rangle_{k\in\mathbb{N}}$ is summable. Then $\langle x_k\rangle_{k\in\mathbb{N}}$ is summable.

proof For each $n\in\mathbb{N}$, set

$$s_n = \sum_{k=0}^{n} x_k, \quad t_n = \sum_{k=0}^{n} y_k.$$

Then $|s_m - s_n| \le |t_m - t_n|$ for all $m, n\in\mathbb{N}$. **P** (i) If $n < m$, then

$$|s_m - s_n| = x_{n+1} + x_{n+2} + \ldots + x_m \le y_{n+1} + y_{n+2} + \ldots + y_m = |t_m - t_n|.$$

(ii) If $m = n$ then $|s_m - s_n| = 0 = |t_m - t_n|$. (iii) If $n < m$ then $|s_m - s_n| = s_n - s_m \le t_n - t_m = |t_m - t_n|$. **Q**

Now we are supposing that $\langle y_k\rangle_{k\in\mathbb{N}}$ is summable, that is, that $\langle t_n\rangle_{n\in\mathbb{N}}$ is convergent. We know that convergent sequences are Cauchy, so $\langle t_n\rangle_{n\in\mathbb{N}}$ is Cauchy. It follows that $\langle s_n\rangle_{n\in\mathbb{N}}$ is Cauchy, as in the last theorem. So $\langle s_n\rangle_{n\in\mathbb{N}}$ is convergent, that is, $\langle x_k\rangle_{k\in\mathbb{N}}$ is summable.

Series of positive terms Note that if $\langle x_k\rangle_{k\in\mathbb{N}}$ is a series with every $x_k \ge 0$, and $s_n = \sum_{k=0}^{n} x_k$ for each n, then $s_{n+1} = s_n + x_{n+1} \ge s_n$ for each n, and $\langle s_n\rangle_{n\in\mathbb{N}}$ is non-decreasing. It follows that $\langle x_k\rangle_{k\in\mathbb{N}}$ is summable iff $\langle s_n\rangle_{n\in\mathbb{N}}$ is bounded. (I gave as a theorem the fact that a bounded monotonic sequence is convergent. But note also that every convergent sequence is bounded, because it is Cauchy, or otherwise.)

Continuous functions on closed bounded intervals Continuous functions on closed bounded intervals have several special properties of the greatest importance. One of the ones which we use most often is the following.

Theorem Let f be a real function which is defined and continuous at every point of a closed interval $[a,b]$, where $a \le b$. Then f is bounded on $[a,b]$, that is, $\{f(x) : x \in [a,b]\}$ is a bounded set.

proof Set $A = \{x : x \in [a,b],\ f$ is bounded on $[a,x]\}$. Then A is bounded above by b and $a \in A$ (because $[a,a] = \{a\}$, so $\{f(y) : y \in [a,a]\} = \{f(a)\}$ is bounded). So A has a least upper bound c say, and $a \le c \le b$. Now f is continuous at c, so there is a $\delta > 0$ such that $|f(x) - f(c)| \le 1$ whenever $x \in [c-\delta, c+\delta] \cap \operatorname{dom} f$. Set $z = \min(b, c+\delta)$. Then $a \le c \le z \le b$ so $z \in [a,b]$. Now $c - \delta < c$, so there is an $x \in A$ such that $c - \delta \le x$. We know that $\{f(y) : y \in [a,x]\}$ is bounded; let M_0, $M_1 \in \mathbb{R}$ be such that $M_0 \le f(y) \le M_1$ whenever $y \in [a,x]$. Set $M_0' = \min(M_0, f(x)-1)$, $M_1' = \max(M_1, f(c)+1)$. If $y \in [a,z]$, then either $y \le x$ and $y \in [a,x]$ and

$$M_0' \le M_0 \le f(y) \le M_1 \le M_1',$$

or $x \le y$ and $c - \delta \le x \le y \le z \le c+\delta$ and $y \in [c-\delta, c+\delta] \cap [a,b] \subseteq [c-\delta, c+\delta] \cap \operatorname{dom} f$, so $|f(y) - c| \le 1$ and

$$M_0' \le f(c) - 1 \le f(y) \le f(c) + 1 \le M_1'.$$

This shows that M_0', M_1' are bounds for $\{f(y) : y \in [a,z]\}$, so $z \in A$. But this means that $z \le c$; since $z = \min(b, c+\delta)$, we must have $z = b$ so $b \in A$. But this means that f is bounded on $[a,b]$, which is what we set out to prove.

In fact rather more is true.

Theorem Let f be a real function defined, and continuous, at each point of a closed interval $[a,b]$, where $a \le b$. Then f attains its bounds on $[a,b]$, that is, there are points z_1, $z_2 \in [a,b]$ such that $f(z_1) \le f(x) \le f(z_2)$ for every $x \in [a,b]$.

proof Set $B = \{f(x) : x \in [a,b]$. Then B is non-empty (since it contains $f(a)$) and bounded (by the last theorem); so $c_1 = \inf B$ and $c_2 = \sup B$ are defined in $\mathbb{R}$.

? Suppose, if possible, that $f(x) \ne c_1$ for every $x \in [a,b]$, that is, that $c_1 \notin B$. Then $f(x) > c_1$ for every $x \in [a,b]$. Set $g(x) = \frac{1}{f(x)-c_1}$ whenever this is defined, that is, whenever $x \in \operatorname{dom} f$ and $f(x) \ne c_1$. Then g is continuous at any point where f is continuous and $f(x) \ne c_1$ (because the function $y \mapsto \frac{1}{y-c_1}$ is continuous); in particular, g is continuous at every point of $[a,b]$. There is therefore some K such that $g(x) \le K$ for every $x \in [a,b]$, because continuous functions on closed bounded intervals are bounded. But this means that $f(x) - c_1 \ge \frac{1}{K}$ for every $x \in [a,b]$, that is, $f(x) \ge c_1 + \frac{1}{K}$ for every $x \in [a,b]$, that is, $c_1 + \frac{1}{K}$ is a lower bound for B, and c_1 is not the greatest lower bound of B. **X**

So we have to conclude that there is some $z_1 \in [a,b]$ such that $f(z_1) = c_1$, and now we have $f(z_1) \le f(x)$ for every $x \in [a,b]$.

I have still to find z_2. To do this, *either* repeat the argument just above, upside down, as follows:

> **?** Suppose, if possible, that $f(x) \ne c_2$ for every $x \in [a,b]$, that is, that $c_2 \notin B$. Then $f(x) < c_2$ for every $x \in [a,b]$. Set $g(x) = \frac{1}{c_2 - f(x)}$ whenever this is defined, that is, whenever $x \in \operatorname{dom} f$ and $f(x) \ne c_2$. Then g is continuous at any point where f is continuous and $f(x) \ne c_2$ (because the

function $y \mapsto \frac{1}{c_2 - y}$ is continuous); in particular, g is continuous at every point of $[a, b]$. There is therefore some K such that $g(x) \leq K$ for every $x \in [a, b]$, because continuous functions on closed bounded intervals are bounded. But this means that $c_2 - f(x) \geq \frac{1}{K}$ for every $x \in [a, b]$, that is, $f(x) \leq c_2 - \frac{1}{K}$ for every $x \in [a, b]$, that is, $c_2 - \frac{1}{K}$ is an upper bound for B, and c_2 is not the least upper bound of B. **✗**

So we have to conclude that there is some $z_2 \in [a, b]$ such that $f(z_2) = c_2$, and now we have $f(x) \leq f(z_2)$ for every $x \in [a, b]$

or do the same inversion by introducing a new function h say, defined by saying that $h(x) = -f(x)$ for every $x \in \operatorname{dom} f$. Then h is defined and continuous at any point where f is (because the function $y \mapsto -y$ is continuous), in particular, at every point of $[a, b]$. By the previous argument, there is a point $z_2 \in [a, b]$ such that $-f(z_2) = h(z_2) \leq h(x) = -f(x)$ for every $x \in [a, b]$, that is, $f(x) \leq f(z_2)$ for every $x \in [a, b]$.

Theorem Let f be a real function defined, and continuous, at each point of a closed interval $[a, b]$, where $a \leq b$. If c lies between $f(a)$ and $f(b)$, then there is a $z \in [a, b]$ such that $f(z) = c$.

proof (a) Suppose that $f(a) \leq c \leq f(b)$. Set

$$A = \{x : x \in [a, b],\ f(x) \leq c\}.$$

Then A is bounded above by b, and $a \in A$, so $z = \sup A$ is defined and $a \leq z \leq b$. As $z \in [a, b]$, f is continuous at z.

? If $f(z) < c$, then $c - f(z) > 0$, so there is a $\delta > 0$ such that $|f(x) - f(c)| \leq c - f(z)$ whenever $x \in \operatorname{dom} f$ and $|x - z| \leq \delta$. Consider $x = \min(b, z + \delta)$. Then $a \leq z \leq x \leq b$ so $f(x)$ is defined, and $z \leq x \leq z + \delta$ so $f(x) \leq f(z) + (c - f(z)) = c$. But this means that $x \in A$. On the other hand, since $f(z) < c \leq f(b)$, z cannot be equal to b, so $z < b$ and $z < x$ and z is not an upper bound for A. **✗**

? If $f(z) > c$, then $\frac{1}{2}(f(z) - c) > 0$, so there is a $\delta > 0$ such that $|f(x) - f(c)| \leq \frac{1}{2}(f(z) - c)$ whenever $x \in \operatorname{dom} f$ and $|x - z| \leq \delta$. Now there must be an $x \in A$ such that $z - \delta \leq x \leq z$, in which case $f(x) \leq c$ and $|x - z| \leq \delta$, so

$$f(z) - c \leq f(z) - f(x) \leq |f(x) - f(z)| \leq \frac{1}{2}(f(z) - c),$$

which is impossible. **✗**

We are forced to conclude that $f(z) = c$; which is what we were looking for.

(b) We still have to deal with the case in which $f(b) \leq c \leq f(a)$. Just as in the last theorem, we have a choice: *either* repeat the argument above with half the signs exchanged, as follows:

Set

$$A = \{x : x \in [a, b],\ f(x) \geq c\}.$$

Then A is bounded above by b, and $a \in A$, so $z = \sup A$ is defined and $a \leq z \leq b$. As $z \in [a, b]$, f is continuous at z.

? If $f(z) > c$, then $f(z) - c > 0$, so there is a $\delta > 0$ such that $|f(x) - f(c)| \leq f(z) - c$ whenever $x \in \operatorname{dom} f$ and $|x - z| \leq \delta$. Consider $x = \min(b, z + \delta)$. Then $a \leq z \leq x \leq b$ so $f(x)$ is defined, and $z \leq x \leq z + \delta$ so $f(x) \geq f(z) - (f(z) - c) = c$. But this means that $x \in A$. On the other hand, since $f(z) > c \geq f(b)$, z cannot be equal to b, so $z < b$ and $z < x$ and z is not an upper bound for A. **✗**

? If $f(z) < c$, then $\frac{1}{2}(c - f(z)) > 0$, so there is a $\delta > 0$ such that $|f(x) - f(c)| \leq \frac{1}{2}(c - f(z))$ whenever $x \in \operatorname{dom} f$ and $|x - z| \leq \delta$. Now there must be an $x \in A$ such that $z - \delta \leq x \leq z$, in which case $f(x) \geq c$ and $|x - z| \leq \delta$, so

$$c - f(z) \leq f(x) - f(z) \leq |f(x) - f(z)| \leq \frac{1}{2}(c - f(z)),$$

which is impossible. **✗**

We are forced to conclude that $f(z) = c$

or define a new function h by saying that

$$h(x) = -f(x) \text{ for every } x \in \operatorname{dom} f,$$

so that h is defined and continuous wherever f is, in particular, at every point of $[a, b]$. Now

$$h(a) = -f(a) \leq -c \leq -f(b) = h(b),$$

so by part (a) we know that there is a $z \in [a, b]$ such that $h(z) = -c$, that is, $f(z) = c$.

So the theorem is true in this case also.

Differentiable Functions

Definition Let f be a real function. We say that f is **differentiable** at $a \in \mathbb{R}$, with **derivative** $b = f'(a)$, if $a \in \operatorname{dom} f$ and $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = b$.

Remark Note that if the limit is to be defined, then there must be some $\delta > 0$ such that $f(x)$ is defined whenever $0 < |x - a| \leq \delta$; since we must also be able to calculate $f(a)$, we see that the whole interval $[x - \delta, x + \delta]$ must be included in $\operatorname{dom} f$.

Lemma If f is a real function, $a \in \operatorname{dom} f$ and $\lim_{x\to a} f(x) = f(a)$, then f is continuous at a.

proof For every $\epsilon > 0$ there is a $\delta > 0$ such that $x \in \operatorname{dom} f$ and $|f(x) - f(a)| \leq \epsilon$ whenever $0 < |x - a| \leq \delta$. Of course $|f(a) - f(a)| \leq \epsilon$, so we see that if $|x - a| \leq \delta$ and $x \in \operatorname{dom} f$ then $|f(x) - f(a)| \leq \epsilon$. As ϵ is arbitrary, f is continuous at a.

Theorem If a real function f is differentiable at $a \in \mathbb{R}$, then f is continuous at a.

proof Of course $a \in \operatorname{dom} f$. I seek to show that $f(a) = \lim_{x\to a} f(x)$. We know that $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$ and that $\lim_{x\to a} x - a = 0$. Since the limit of a product is the product of the limits whenever the latter is defined, we have

$$\begin{aligned} \lim_{x\to a} f(x) - f(a) &= \lim_{x\to a} \tfrac{f(x)-f(a)}{x-a} \cdot (x - a) \\ &= \lim_{x\to a} \tfrac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a} x - a = f'(a) \cdot 0 = 0. \end{aligned}$$

Next, the limit of a sum is the sum of the limits whenever the latter is defined, so

$$\begin{aligned} \lim_{x\to a} f(x) &= \lim_{x\to a} f(x) - f(a) + f(a) \\ &= \lim_{x\to a} f(x) - f(a) + \lim_{x\to a} f(a) = 0 + f(a) = f(a). \end{aligned}$$

By the last lemma, f is continuous at a.

Proposition Let f and g be real functions, both differentiable at $a \in \mathbb{R}$. Then $f + g$ is differentiable at a and $(f + g)'(a) = f'(a) + g'(a)$.

proof We know that

$$\lim_{x\to a} \tfrac{f(x)-f(a)}{x-a} = f'(a), \quad \lim_{x\to a} \tfrac{g(x)-g(a)}{x-a} = g'(a).$$

Now the limit of a sum is the sum of the limits whenever the latter is defined, so

$$\begin{aligned} \lim_{x\to a} \tfrac{(f+g)(x)-(f+g)(a)}{x-a} &= \lim_{x\to a} \tfrac{f(x)+g(x)-f(a)-g(a)}{x-a} \\ &= \lim_{x\to a} \tfrac{f(x)-f(a)}{x-a} + \tfrac{g(x)-g(a)}{x-a} \\ &= \lim_{x\to a} \tfrac{f(x)-f(a)}{x-a} + \lim_{x\to a} \tfrac{g(x)-g(a)}{x-a} = f'(a) + g'(a), \end{aligned}$$

as required.

Proposition Let f and g be real functions, both differentiable at $a \in \mathbb{R}$. Then $f \times g$ is differentiable at a and $(f \times g)'(a) = f'(a)g(a) + f(a)g'(a)$.

proof We know that

$$\lim_{x\to a} \tfrac{f(x)-f(a)}{x-a} = f'(a),\ \lim_{x\to a} g(a) = g(a);$$

because the limit of a product is the product of the limits when the latter exists,

$$\lim_{x\to a} \tfrac{f(x)-f(a)}{x-a} g(a) = f'(a)g(a).$$

Similarly,

$$\lim_{x\to a} \tfrac{g(x)-g(a)}{x-a} = g'(a),\ \lim_{x\to a} f(a) = f(a),$$

so $\lim_{x\to a} f(a)\tfrac{g(x)-g(a)}{x-a} = f(a)g'(a)$. Moreover, $\lim_{x\to a} x - a = 0$, so

$$\lim_{x\to a} \tfrac{f(x)-f(a)}{x-a} \cdot \tfrac{g(x)-g(a)}{x-a} \cdot (x-a) = f'(a)g'(a) \cdot 0 = 0.$$

Now we know also that the limit of a sum is the sum of the limits when the latter exists, so

$$\begin{aligned}
\lim_{x\to a} \tfrac{(f\times g)(x)-(f\times g)(a)}{x-a} &= \lim_{x\to a} \tfrac{f(x)g(x)-f(a)g(a)}{x-a} \\
&= \lim_{x\to a} \tfrac{(f(x)-f(a))(g(x)-g(a))+(f(x)-f(a))g(a)+f(a)(g(x)-g(a))}{x-a} \\
&= \lim_{x\to a} \tfrac{(f(x)-f(a))(g(x)-g(a))}{x-a} + \tfrac{f(x)-f(a)}{x-a}g(a) + f(a)\tfrac{g(x)-g(a)}{x-a} \\
&= \lim_{x\to a} \tfrac{f(x)-f(a)}{x-a}\tfrac{g(x)-g(a)}{x-a}(x-a) + \tfrac{f(x)-f(a)}{x-a}g(a) + f(a)\tfrac{g(x)-g(a)}{x-a} \\
&= 0 + f'(a)g(a) + f(a)g'(a) = f'(a)g(a) + f(a)g'(a).
\end{aligned}$$

But this is just what it means to say that $(f \times g)'(a)$ is defined and equal to $f'(a)g(a) + f(a)g'(a)$.

Three basic functions (a) If $f(x) = c$ for x near a, then

$$f'(a) = \lim_{x\to a} \tfrac{f(x)-f(a)}{x-a} = \lim_{x\to a} \tfrac{c-c}{x-a} = \lim_{x\to a} 0 = 0.$$

(Constant functions are differentiable, with derivative zero.)

(b) If $f(x) = x$ for x near a, then

$$f'(a) = \lim_{x\to a} \tfrac{f(x)-f(a)}{x-a} = \lim_{x\to a} \tfrac{x-a}{x-a} = \lim_{x\to a} 1 = 1.$$

(The identity function is differentiable, with derivative 1.)

(c) If $f(x) = \frac{1}{x}$ for x near a, then

$$\begin{aligned}
f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} &= \lim_{x\to a} \frac{\frac{1}{x}-\frac{1}{a}}{x-a} \\
= \lim_{x\to a} -\frac{1}{xa} &= -\frac{1}{a}\lim_{x\to a} 1x = -\frac{1}{a}\cdot\frac{1}{a}
\end{aligned}$$

(because the function $x \mapsto \dfrac{1}{x}$ is continuous and is defined near a)

$$= -\frac{1}{a^2}.$$

(The reciprocal function is differentiable.)

Derivatives without division It is very useful to know the following fact.

Lemma Let f be a real function, and $a \in \operatorname{dom} f$. Then $f'(a)$ is defined and equal to b iff

for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(a) - b(x-a)|$ is defined and less than or equal to $\epsilon|x-a|$ whenever $|x-a| \le \delta$.

proof We have

$$f'(a) = b \iff \forall\, \epsilon > 0\, \exists\, \delta > 0,\ |\tfrac{f(x)-f(a)}{x-a} - b| \text{ exists } \le \epsilon \text{ whenever } 0 < |x-a| \le \delta$$

$$\iff \forall\, \epsilon > 0\, \exists\, \delta > 0,\ |f(x) - f(a) - b(x-a)| \text{ exists } \le \epsilon|x-a| \text{ whenever } 0 < |x-a| \le \delta$$

(multiplying or dividing both sides of the inequality by the strictly positive number $|x-a|$, and remembering that $|y||z| = |yz|$ for all y, $z \in \mathbb{R}$)

$$\iff \forall\, \epsilon > 0\, \exists\, \delta > 0,\ |f(x) - f(a) - b(x-a)| \text{ exists } \le \epsilon|x-a| \text{ whenever } |x-a| \le \delta$$

because if $x = a$ then $|f(x) - f(a) - b(x-a)| = 0 = \epsilon|x-a|$.

Theorem (Chain Rule for differentiable functions) Let f and g be real functions, and suppose that g is differentiable at a and that f is differentiable at $g(a)$. Then $f \circ g$ is differentiable at a, with $(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$.

proof Write $c = f'(g(a))$, $b = g'(a)$. Let $\epsilon > 0$. Set $\eta = \min(1, \frac{1}{1+|b|+|c|})$. Let $\delta_1 > 0$ be such that $g(x)$ is defined and $|g(x) - g(a) - b(x-a)| \le \eta|x-a|$ whenever $|x-a| \le \delta_1$. Then

$$|g(x) - g(a)| \le |b(x-a)| + \eta|x-a| = (|b| + \eta)|x-a| \qquad \text{[key step}$$

whenever $|x-a| \le \delta_1$. Let $\delta_2 > 0$ be such that $f(y)$ is defined and $|f(y) - f(g(a)) - c(y - g(a))| \le \eta|y - g(a)|$ whenever $|y - g(a)| \le \delta_2$.

Set $\delta = \min(\delta_1, \frac{\delta_2}{|b|+\eta}) > 0$. If $|x-a| \le \delta$, then $|x-a| \le \delta_1$ so $g(x)$ is defined and $|g(x) - g(a) - b(x-a)| \le \eta|x-a|$ and

$$|g(x) - g(a)| \le (|b| + \eta)|x-a| \le (|b| + \eta)\delta \le \delta_2.$$

This means that $f(g(x))$ is defined and

$$\begin{aligned} |(f \circ g)(x) - (f \circ g)(a) - cb(x-a)| &= |f(g(x)) - f(g(a)) - cb(x-a)| \\ &\le |f(g(x)) - f(g(a)) - c(g(x) - g(a))| + |c(g(x) - g(a)) - cb(x-a)| \qquad \text{[key step} \\ &\le \eta|g(x) - g(a)| + |c||g(x) - g(a) - b(x-a)| \end{aligned}$$

(because $|g(x) - g(a)| \le \delta_2$)

$$\begin{aligned} &\le \eta(|b| + \eta)|x-a| + |c|\eta|x-a| \\ &= \eta(|b| + \eta + |c|)|x-a| \le \eta(|b| + 1 + |c|)|x-a| \le \epsilon|x-a|. \end{aligned}$$

As ϵ is arbitrary, $(f \circ g)'(a)$ is defined and equal to cb, as claimed.

Rolle's Theorem and the Mean Value Theorem

The last section of the course introduces one of the fundamental theorems of analysis. Before embarking on the main results we need to tidy things up a little.

Proposition (a) Suppose that f is a real function, that $d < a$ in $\mathbb{R}$, and that $f(x) \ge c$ whenever $d \le x < a$. If $\lim_{x \uparrow a} f(x)$ is defined, then $\lim_{x \uparrow a} f(x) \ge c$.

(b) Suppose that f is a real function, that $a < d$ in $\mathbb{R}$, and that $f(x) \ge c$ whenever $a < x \le d$. If $\lim_{x \downarrow a} f(x)$ is defined, then $\lim_{x \downarrow a} f(x) \ge c$.

proof (a) Set $b = \lim_{x \uparrow a} f(x)$. **?** Suppose, if possible, that $b < c$. Then $c - b > 0$; set $\epsilon = \frac{1}{2}(c-b) > 0$. There is a $\delta > 0$ such that $f(x)$ is defined and $|f(x) - b| \le \epsilon$ whenever $a - \delta \le x < a$. Set $x = \max(a - \delta, d)$; then $f(x) \ge c$, and also $|f(x) - b| \le \epsilon$, so

$$b \geq f(x) - |f(x) - b| \geq c - \epsilon = b + 2\epsilon - \epsilon = b + \epsilon > b,$$

which is impossible. **X**

So b must be greater than or equal to c, as claimed.

(b) Set $b = \lim_{x\downarrow a} f(x)$. **?** Suppose, if possible, that $b < c$. Then $c - b > 0$; set $\epsilon = \frac{1}{2}(c-b) > 0$. There is a $\delta > 0$ such that $f(x)$ is defined and $|f(x) - b| \leq \epsilon$ whenever $a < x \leq a + \delta$. Set $x = \min(a+\delta, d)$; then $f(x) \geq c$, and also $|f(x) - b| \leq \epsilon$, so

$$b \geq f(x) - |f(x) - b| \geq c - \epsilon = b + 2\epsilon - \epsilon = b + \epsilon > b,$$

which is impossible. **X**

So b must be greater than or equal to c, as claimed.

Proposition Let f be a real function and a, $b \in \mathbb{R}$. Then $\lim_{x\to a} f(x) = b$ iff $\lim_{x\uparrow a} f(x) = \lim_{x\downarrow a} f(x) = b$.

proof (a) Suppose that $\lim_{x\to a} f(x) = b$.

(i) Let $\epsilon > 0$. Then there is a $\delta > 0$ such that $f(x)$ is defined and $|f(x)-b| \leq \epsilon$ whenever $0 < |x-a| \leq \delta$. If now $a - \delta \leq x < a$, $|x-a| = a - x \in]0, \delta]$ so $f(x)$ is defined and $|f(x) - b| \leq \epsilon$. As ϵ is arbitrary, $\lim_{x\uparrow a} f(x) = b$.

(ii) Let $\epsilon > 0$. Then there is a $\delta > 0$ such that $f(x)$ is defined and $|f(x)-b| \leq \epsilon$ whenever $0 < |x-a| \leq \delta$. If now $a < x \leq a + \delta$, $|x-a| = x - a \in]0, \delta]$ so $f(x)$ is defined and $|f(x) - b| \leq \epsilon$. As ϵ is arbitrary, $\lim_{x\downarrow a} f(x) = b$.

(b) Now suppose that $\lim_{x\downarrow a} f(x) = \lim_{x\uparrow a} f(x) = b$. Let $\epsilon > 0$. Then

there is a $\delta_1 > 0$ such that $x \in \operatorname{dom} f$ and $|f(x) - b| \leq \epsilon$ whenever $a - \delta_1 \leq x < a$,

there is a $\delta_2 > 0$ such that $x \in \operatorname{dom} f$ and $|f(x) - b| \leq \epsilon$ whenever $a < x \leq a + \delta_2$.

Set $\delta = \min(\delta_1, \delta_2) > 0$. If $0 < |x - a| \leq \delta$, then

either $x < a$, in which case $a - \delta_1 \leq a - \delta \leq x < a$, so $x \in \operatorname{dom} f$ and $|f(x) - b| \leq \epsilon$,

or $x > a$, in which case $a < x \leq a + \delta \leq a + \delta_1$, so $x \in \operatorname{dom} f$ and $|f(x) - b| \leq \epsilon$.

Thus in either case $f(x)$ is defined and $|f(x) - b| \leq \epsilon$; as ϵ is arbitrary, $\lim_{x\to a} f(x) = b$.

Rolle's Theorem Let f be a real function, defined and continuous at each point of a closed interval $[a, b]$, where $a < b$ in $\mathbb{R}$. Suppose that f is differentiable at each point of the open interval $]a, b[$, and that $f(a) = f(b)$. Then there is a $z \in]a, b[$ such that $f'(z) = 0$.

proof Recall that a continuous function on a closed bounded interval is bounded and attains its bounds. So there are z_1, $z_2 \in [a, b]$ such that $f(z_1) \leq f(x) \leq f(z_2)$ for every $x \in [a, b]$.

case 1 Suppose that $z_1 \in]a, b[$. Then $f'(z_1)$ is defined, and

$$f'(z_1) = \lim_{x\to z_1} \frac{f(x)-f(z_1)}{x-z_1} = \lim_{x\downarrow z_1} \frac{f(x)-f(z_1)}{x-z_1} = \lim_{x\uparrow z_1} \frac{f(x)-f(z_1)}{x-z_1}$$

by the last proposition.

If $z_1 < x \leq b$, then $f(x) - f(z_1) \geq 0$ so $\frac{f(x)-f(z_1)}{x-z_1} \geq 0$; but this means that $f'(z_1) = \lim_{x\downarrow z_1} \frac{f(x)-f(z_1)}{z-z_1} \geq 0$, because the limit of a non-negative function (if defined) must be non-negative. If $a \leq x < z_1$, then $f(x) - f(z_1) \geq 0$ so $\frac{f(x)-f(z_1)}{x-z_1} \leq 0$; but this means that $f'(z_1) = \lim_{x\downarrow z_1} \frac{f(x)-f(z_1)}{z-z_1} \leq 0$. Putting these together, we see that $f'(z_1) = 0$ and we can take $z = z_1$.

case 2 Suppose that $z_2 \in]a, b[$. Then $f'(z_2)$ is defined, and

$$f'(z_2) = \lim_{x\to z_2} \frac{f(x)-f(z_2)}{x-z_2} = \lim_{x\downarrow z_2} \frac{f(x)-f(z_2)}{x-z_2} = \lim_{x\uparrow z_2} \frac{f(x)-f(z_2)}{x-z_2}$$

by the last proposition.

If $z_2 < x \le b$, then $f(x) - f(z_2) \le 0$ so $\frac{f(x)-f(z_2)}{x-z_2} \le 0$; but this means that $f'(z_2) = \lim_{x \downarrow z_2} \frac{f(x)-f(z_2)}{z-z_2} \le 0$. If $a \le x < z_2$, then $f(x) - f(z_2) \le 0$ so $\frac{f(x)-f(z_2)}{x-z_2} \ge 0$; but this means that $f'(z_2) = \lim_{x \downarrow z_2} \frac{f(x)-f(z_2)}{z-z_2} \ge 0$. Putting these together, we see that $f'(z_2) = 0$ and we can take $z = z_2$.

case 3 Suppose that neither z_1 nor z_2 belong to $]a, b[$; that is, that both are either a or b. Then $f(z_1)$ must be either $f(a)$ or $f(b)$; but these are equal, so $f(z_1) = f(a)$. Similarly $f(z_2) = f(a)$. This means that $f(a) \le f(x) \le f(a)$, that is, $f(x) = f(a)$, for every $x \in]a, b[$; and if we take any $z \in]a, b[$ (e.g., $z = \frac{1}{2}(a+b)$), we shall have $f'(z) = 0$.

(Note that it is only in case 3 that we need to know that $f(a) = f(b)$.)

Mean Value Theorem Let f be a real function, defined and continuous at each point of a closed interval $[a, b]$, where $a < b$ in $\mathbb{R}$. Suppose that f is differentiable at each point of the open interval $]a, b[$. Then there is a $z \in]a, b[$ such that $f'(z) = \frac{f(b)-f(a)}{b-a}$.

proof Set $h(x) = f(x) - \frac{f(b)-f(a)}{b-a} x$ for $x \in \operatorname{dom} f$. Observe that because the constant function $x \mapsto -\frac{f(b)-f(a)}{b-a}$ and the identity function $x \mapsto x$ are differentiable everywhere, so is their product $x \mapsto -\frac{f(b)-f(a)}{b-a} x$, and h will be differentiable at every point at which f is differentiable and continuous at every point at which f is continuous; in particular, h is continuous everywhere in $[a, b]$ and differentiable everywhere in $]a, b[$. Also

$$h(b) - h(a) = f(b) - f(a) - \frac{f(b)-f(a)}{b-a}(b-a) = 0.$$

By Rolle's theorem, there is a $z \in]a, b[$ such that

$$0 = h'(z) = f'(z) - \frac{f(b)-f(a)}{b-a},$$

and $f'(z) = \frac{f(b)-f(a)}{b-a}$, as required.

Cauchy's Mean Value Theorem Let f and g be real functions, both defined and continuous at each point of a closed interval $[a, b]$, where $a < b$ in $\mathbb{R}$. Suppose that both f and g are differentiable at each point of the open interval $]a, b[$, and that $g'(x) \ne 0$ for every $x \in]a, b[$. Then there is a $z \in]a, b[$ such that $\frac{f'(z)}{g'(z)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.

proof We need to know that $g(a) \ne g(b)$. But Rolle's theorem tells us that if $g(a)$ and $g(b)$ were equal, there would be a point $x \in]a, b[$ such that $g'(x) = 0$; which isn't so.

Set $h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$ for $x \in \operatorname{dom} f$. Then h will be differentiable at every point at which f and g are differentiable and continuous at every point at which f and g are continuous; in particular, h is continuous everywhere in $[a, b]$ and differentiable everywhere in $]a, b[$. Also

$$h(b) - h(a) = (f(b) - f(a))(g(b) - g(a)) - (g(b) - g(a))(f(b) - f(a)) = 0.$$

By Rolle's theorem, there is a $z \in]a, b[$ such that

$$0 = h'(z) = f'(z)(g(b) - g(a)) - g'(z)(f(b) - f(a)).$$

Since $g(b) - g(a)$ and $g'(z)$ are both non-zero, $\frac{f'(z)}{g'(z)} = \frac{f(b)-f(a)}{g(b)-g(a)}$, as required.