Real Analysis Answers

1. (a) $|x+y| \le |x|+|y|$ for all real numbers x and y.

(b)(i) $|x-y| = |x+(-y)| \le |x| + |-y| = |x|+|y|$.
(ii) $|x| = |(x-y)+y| \le |x-y|+|y|$; subtracting $|y|$ from both sides, $|x|-|y| \le |x-y|$.
(iii) By (ii), $|x|-|x-y| \le |x-(x-y)| = |y|$.
(iv) By (ii), $|x|-|y| \le |x-y|$; similarly, $|y|-|x| \le |y-x| = |x-y|$; so

$$||x|-|y|| = \max(|x|-|y|, |y|-|x|) \le |x-y|.$$

2. (a)(i) An *upper bound* of A is an $a \in \mathbb{R}$ such that $x \le a$ for every $x \in A$.
(ii) $\sup A$ (when it exists) is the least upper bound of A.

(b)(i) $A = [0,1]$ (ii) $A =]0,1[$ (iii) $A = \emptyset$ *or* $A = \mathbb{R}$.

(There are many alternatives.)

(c)(i) $[-1,3]$, $[1,3[$, $]0,2[$.
(ii) If $z \in A+B$, then $z = x+y$ for some $x \in A$, $y \in B$. Now $x \le a$ and $y \le b$, so

$$z = x+y \le a+y \le a+b.$$

As z is arbitrary, $a+b$ is an upper bound of A.

(iii) Since $x+y \in A+B$, we know that $x+y \le c$; subtracting y from both sides, $x \le c-y$.

(iv) Take any $y \in B$. By (iii), $x \le c-y$ for every $x \in A$, so $c-y$ is an upper bound of A and $a \le c-y$. So $a+y \le c$ and $y \le c-a$. This is true for every $y \in B$, so $c-a$ is an upper bound of B.

(v) By (ii), $a+b$ is an upper bound of $A+B$. On the other hand, if c is any upper bound of $A+B$, then $c-a$ is an upper bound of B, by (iv); but this means that $b \le c-a$, so $a+b \le c$. So $a+b$ is the least upper bound of $A+B$.

3. (i) False; try $a=0$, $A=[0,1]$.
(ii) True.
(iii) False; try $a=2$, $A=[0,1]$.
(iv) False; try $a=1$, $A=[0,1]$.
(v) True.

4. (i) $\exists\, x \in A$, $x > 2$.
(ii) $\forall\, x \in A$, $x \le 0$.
(iii) $\exists\, x \in A\ \exists\, y \in B$, $x > y$.
(iv) $\forall\, x \in A\ \forall\, y \in B$, $x > y$.
(v) $\exists\, x \in A\ \forall\, y \in B$, $x+y \le 0$.
(vi) $\forall\, x \in A\ \exists\, y \in B$, $x+y \le 0$.

5. (i) is false. For instance, if $A = B = \mathbb{R}$, then for every $x \in A$ there is a $y \in B$ such that $x \le y$ (just take $y = x$), but there is no $y \in B$ such that $x \le y$ for every $x \in A$.

(ii) is true. For if $y_0 \in B$ is such that $x \le y_0$ for every $x \in A$, then, for every $x \in A$, there is certainly a $y \in B$ such that $x \le y$: just take $y = y_0$.

6. (a) False.
(b) False.
(c) True.
(d) False.
(e) True.
(f) True.
(g) True.
(h) True.

7. Possible correct moves are marked * .

*	*	O
*	X	*
*	*	*

draw

*	O	*
*	X	*
*		*

X wins

O		X
		*
*		*

X wins

*		X
O		*

X wins

*	X	
O		

X wins

8. (a) Second player plays next, and loses.
(b) Second player plays next, and should win; $\epsilon = 0{\cdot}005$ wins. (Any $\epsilon \in \,]0, 0{\cdot}01[$ will do.)
(c) Second player plays next, and should win; $\epsilon = \frac{1}{2}$ wins. (Any $\epsilon \in \,]0, 1[$ will do.)
(d) *First* player plays next, and should win; he must say '$\lim_{n\to\infty} \frac{n}{n+1} = 1$'.

9. $\lim_{x\downarrow a} f(x) = b$ means:

$$\forall\, \epsilon > 0\; \exists\, \delta > 0,\; f(x) \text{ is defined and } |f(x) - b| \le \epsilon \text{ whenever } 0 < x - a \le \delta.$$

If $\lim_{x\downarrow a} f(x) = b$ and $\lim_{x\downarrow a} g(x) = c$, let $\epsilon > 0$. Then $\frac{1}{2}\epsilon > 0$, so there are δ_1, $\delta_2 > 0$ such that

$$f(x) \text{ is defined and } |f(x) - b| \le \tfrac{1}{2}\epsilon \text{ whenever } 0 < x - a \le \delta_1,$$

$$g(x) \text{ is defined and } |g(x) - c| \le \tfrac{1}{2}\epsilon \text{ whenever } 0 < x - a \le \delta_2.$$

Now set $\delta = \min(\delta_1, \delta_2) > 0$. Then whenever $0 < x - a \le \delta$, $f(x)$ and $g(x)$ are both defined and

$$\begin{aligned} |(f(x) + g(x)) - (b + c)| &= |(f(x) - b) + (g(x) - c)| \le |f(x) - b| + |g(x) - c| \\ &\le \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon. \end{aligned}$$

As ϵ is arbitrary, $\lim_{x\downarrow a} f(x) + g(x) = b + c$.

10. $\lim_{x\downarrow a} f(x) = b$ means:

$$\forall\, \epsilon > 0\; \exists\, \delta > 0,\; f(x) \text{ is defined and } |f(x) - b| \le \epsilon \text{ whenever } a < x \le a + \delta.$$

If $\lim_{x\downarrow a} f(x) = b$ and $\lim_{x\downarrow a} g(x) = c$, let $\epsilon > 0$. Set $\eta = \min(1, \frac{\epsilon}{1+|b|+|c|}) > 0$. Then there are δ_1, $\delta_2 > 0$ such that

$$f(x) \text{ is defined and } |f(x) - b| \le \eta \text{ whenever } a < x \le a + \delta_1,$$

$$g(x) \text{ is defined and } |g(x) - c| \le \eta \text{ whenever } a < x \le a + \delta_2.$$

Set $\delta = \min(\delta_1, \delta_2) > 0$. Then for any x such that $a < x \le a + \delta$, $f(x)$ and $g(x)$ are both defined and

$$\begin{aligned} |f(x)g(x) - bc| &\le |f(x) - b||g(x) - c| + |f(x) - b||c| + |b||g(x) - c| \\ &\le \eta^2 + \eta|c| + |b|\eta \le \eta(1 + |b| + |c|) \le \epsilon. \end{aligned}$$

As ϵ is arbitrary, $\lim_{x\downarrow a} f(x)g(x) = bc$.

11. $\lim_{x\to a} f(x) = b$ means:

$$\forall\, \epsilon > 0\; \exists\, \delta > 0,\; f(x) \text{ is defined and } |f(x) - b| \le \epsilon \text{ whenever } 0 < |x - a| \le \delta.$$

If $\lim_{x\downarrow a} f(x) = b$ and $\lim_{x\downarrow a} g(x) = c$, let $\epsilon > 0$. Set $\eta = \min(1, \frac{\epsilon}{1+|b|+|c|}) > 0$. Then there are $\delta_1,\, \delta_2 > 0$ such that

$$f(x) \text{ is defined and } |f(x) - b| \le \eta \text{ whenever } 0 < |x - a| \le \delta_1,$$

$$g(x) \text{ is defined and } |g(x) - c| \le \eta \text{ whenever } 0 < |x - a| \le \delta_2.$$

Set $\delta = \min(\delta_1, \delta_2) > 0$. Then for any x such that $0 < |x - a| \le \delta$, $f(x)$ and $g(x)$ are both defined and

$$\begin{aligned} |f(x)g(x) - bc| &\le |f(x) - b||g(x) - c| + |f(x) - b||c| + |b||g(x) - c| \\ &\le \eta^2 + \eta|c| + |b|\eta \le \eta(1 + |b| + |c|) \le \epsilon. \end{aligned}$$

As ϵ is arbitrary, $\lim_{x\to a} f(x)g(x) = bc$.

12. I have no convincing candidate. I did once write a noughts-and-crosses program, which always played correctly (I think), but in effect I gave it a list of possible positions & told it what to do in each.

13. (a) No. Suppose the game goes like this:

	X	

	O	
	X	

X	O	
	X	

X	O	
	X	
		O

as it very well could, since you played in the centre for your first move and in a corner for your next. Then you ought to be able to win (play in bottom left corner). But the rule allows you to pick a corner at random, and if you pick top right then it's a draw (unless your opponent is a total idiot).

(b) Yes. Play as follows:

X		

X		
	O	

X		
	O	
		X

Your opponent's next move will be in a corner and you can hardly help winning.

14. Set $\epsilon = \frac{1}{3}|b - c|$, so that $\epsilon \ge 0$.
If $\epsilon = 0$ then $|b - c| = 0$ and $b = c$, so we can stop.
If $\epsilon > 0$, then there is an $n_1 \in \mathbb{N}$ such that $|x_n - b| \le \epsilon$ for every $n \ge n_1$.
Also there is an $n_2 \in \mathbb{N}$ such that $|x_n - c| \le \epsilon$ for every $n \ge n_2$.
Set $n_0 = \max(n_1, n_2)$.
Because $n_0 \ge n_1$, $|x_{n_0} - b| \le \epsilon$.
Because $n_0 \ge n_2$, $|x_{n_0} - c| \le \epsilon$.
So $|b - c| \le |x_{n_0} - b| + |x_{n_0} - c| \le 2\epsilon = \dfrac{2}{3}|b - c|$.
Therefore $\dfrac{1}{3}|b - c| \le 0$ and $b = c$ in this case also.
[Of course the point of this proof is that it is *impossible* for ϵ to be non-zero, so that all the deductions after the 'if $\epsilon > 0$' are counter-factual. But we don't know this until we come to the end.]

15. Let $\epsilon > 0$. Then there is an $\eta > 0$ such that $|f(y) - c| \le \epsilon$ whenever $0 < |y - b| \le \eta$; but since we know also that $f(b) = c$, we see that $|f(y) - c| \le \epsilon$ whenever $|y - b| \le \eta$. Now there is a $\delta > 0$ such that $|g(x) - b| \le \eta$ whenever $0 < |x - a| \le \delta$. If we now take any x such that $0 < |x - a| \le \delta$, then $|g(x) - b| \le \eta$ so (because of the way we chose η) $|f(g(x)) - c| \le \epsilon$. As ϵ is arbitrary, $\lim_{x\to a} f(g(x)) = c$, as claimed.

16. Let $\epsilon > 0$. Then there is an $\eta > 0$ such that $|f(y) - c| \leq \epsilon$ whenever $0 < |y - b| \leq \eta$; but since we know also that $f(b) = c$, we see that $|f(y) - c| \leq \epsilon$ whenever $|y - b| \leq \eta$. Now there is an $n_0 \in \mathbb{N}$ such that $|x_n - b| \leq \eta$ whenever $n \geq n_0$. If we now take any n such that $n \geq n_0$, then $|x_n - b| \leq \eta$ so (because of the way we chose η) $|f(x_n)) - c| \leq \epsilon$. As ϵ is arbitrary, $\lim_{n\to\infty} f(x_n) = c$, as claimed.

17. Let $\langle x_n \rangle_{n\in\mathbb{N}}$ be a bounded non-increasing sequence. Then $A = \{x_n : n \in \mathbb{N}\}$ is a non-empty bounded set, so has a greatest lower bound; set $b = \inf A$. For any $\epsilon > 0$, $b + \epsilon$ is not a lower bound of A, so there is an $n_0 \in \mathbb{N}$ such that $x_{n_0} < b + \epsilon$. If now $n \geq n_0$, $x_n \leq x_{n_0} \leq b + \epsilon$, because the sequence is non-increasing; but also $b \leq x_n$, because b is a lower bound for A. Thus $|x_n - b| \leq \epsilon$. As ϵ is arbitrary, $\lim_{n\to\infty} x_n = b$ and $\langle x_n \rangle_{n\in\mathbb{N}}$ is convergent.

18. Base step: if $n = 0$ then LHS is $(1 + \eta)^0 = 1$ and RHS is $1 + 0 \cdot \eta = 1$, so the induction starts.

Inductive step: if $(1 + \eta)^n \geq 1 + n\eta$, then

$$(1 + \eta)^{n+1} = (1 + \eta)(1 + \eta)^n \geq (1 + \eta)(1 + n\eta)$$

(because $(1 + \eta)^n \geq 1 + n\eta$, by the inductive hypothesis, and $1 + \eta \geq 0$)

$$= 1 + (n + 1)\eta + n\eta^2 \geq 1 + (n + 1)\eta,$$

so the induction proceeds.

If $x = 0$ then $x^n = 0$ for every n so $\lim_{n\to\infty} x^n = 0$.

If $0 < |x| < 1$, then, given $\epsilon > 0$, take $n_0 \in \mathbb{N}$ such that $n_0 \geq \frac{1}{\epsilon(1-|x|)}$. If $n \geq n_0$, then

$$\frac{1}{|x^n|} = \left(1 + (\frac{1}{|x|} - 1)\right)^n \geq 1 + n(\frac{1}{|x|} - 1)$$

(by (a))

$$\geq n_0(\frac{1}{|x|} - 1) \geq \frac{\frac{1}{|x|} - 1}{\epsilon(1 - |x|)}$$
$$= \frac{1-|x|}{\epsilon|x|(1-|x|)} = \frac{1}{\epsilon|x|} \geq \frac{1}{\epsilon},$$

so $|x^n - 0| = |x^n| \leq \epsilon$. As ϵ is arbitrary, $\lim_{n\to\infty} x^n = 0$.

$$(1 - x)\sum_{k=0}^{n} x^k = \sum_{k=0}^{n} x^k - x^{k+1} = \sum_{k=0}^{n} x^k - \sum_{k=0}^{n} x^{k+1}$$
$$= \sum_{k=0}^{n} x^k - \sum_{k=1}^{n+1} x^k = x^0 - x^{n+1} = 1 - x^{n+1};$$

dividing both sides by $1 - x$, we have the result.

Alternatively: Induce on n. If $n = 0$ the LHS is 1 and the RHS is $\frac{1-x}{1-x} = 1$, so the induction starts. For the inductive step, given that $\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$, then

$$\sum_{k=0}^{n+1} x^k = \sum_{k=0}^{n} x^k + x^{n+1} = \frac{1-x^{n+1}}{1-x} + x^{n+1}$$

(by the inductive hypothesis)

$$= \frac{1-x^{n+1}+x^{n+1}-x^{n+2}}{1-x} = \frac{1-x^{(n+1)+1}}{1-x}$$

and the induction proceeds.

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} \sum_{k=0}^{n} x^k$$

(by the definition of $\sum_0^\infty$)

$$= \lim_{n\to\infty} \tfrac{1-x^{n+1}}{1-x}$$

(by (c))

$$= \lim_{n\to\infty} \tfrac{1}{1-x} - \tfrac{x}{1-x}x^n = \tfrac{1}{1-x} - \tfrac{x}{1-x} \lim_{n\to\infty} x^n = \tfrac{1}{1-x}.$$

19. (a) Let M, $M' \in \mathbb{R}$ be such that $|x_n| \le M$ and $|y_n| \le M'$ for every $n \in \mathbb{N}$. Then $|x_n + y_n| \le |x_n| + |y_n| \le M + M'$ and $|x_n y_n| = |x_n||y_n| \le MM'$ for every n, so $\langle x_n + y_n\rangle_{n\in\mathbb{N}}$ and $\langle x_n y_n\rangle_{n\in\mathbb{N}}$ are bounded.

(b) Try $x_n = \frac{1}{n+1}$ for every n.

(c) For each $n \in \mathbb{N}$,

$$x_n + y_n \le x_{n+1} + y_n$$

(because $x_n \le x_{n+1}$)

$$\le x_{n+1} + y_{n+1}$$

because $y_n \le y_{n+1}$.

(d) For each $n \in \mathbb{N}$,

$$x_n y_n \le x_{n+1} y_n$$

(because $x_n \le x_{n+1}$ and $y_n \ge 0$)

$$\le x_{n+1} y_{n+1}$$

because $y_n \le y_{n+1}$ and $x_{n+1} \ge 0$.

20. (a) $x_n = 0$ for every n.

(b) No. Try $x_0 = 0$, $x_n = 1$ for $n \ge 1$ (so that $\langle x_n\rangle_{n\in\mathbb{N}}$ is non-decreasing) and $y_0 = y_1 = 1$, $y_n = 0$ for $n \ge 2$ (so that $\langle y_n\rangle_{n\in\mathbb{N}}$ is non-increasing) while $\langle x_n + y_n\rangle_{n\in\mathbb{N}} = (1, 2, 1, \dots)$ is not monotonic.

21. (a) A sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ of real numbers is a Cauchy sequence if for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $|x_m - x_n| \le \epsilon$ for all m, $n \ge n_0$.

The General Principle of Convergence states: a real sequence is convergent iff it is Cauchy.

(b)(i) If $n < m$ then

$$|s_m - s_n| = |\sum_{k=0}^{m} x_k - \sum_{k=0}^{n} x_k| = |\sum_{k=n+1}^{m} x_k| = \sum_{k=n+1}^{m} x_k$$

(because $x_k \ge 0$ for every k)

$$\le \sum_{k=n+1}^{m} y_k$$

(because $x_k \le y_k$ for every k)

$$= |\sum_{k=n+1}^{m} y_k| = |t_m - t_n|.$$

If $m = n$ then $|s_m - s_n| = 0 = |t_m - t_n|$.
If $m < n$ then

$$|s_m - s_n| = |s_n - s_m| = \sum_{k=m+1}^{n} x_k \le \sum_{k=m+1}^{n} y_k = |t_m - t_n|.$$

Now suppose that $\langle t_n\rangle_{n\in\mathbb{N}}$ is Cauchy. In this case, given $\epsilon > 0$, there is an $n_0 \in \mathbb{N}$ such that $|t_m - t_n| \le \epsilon$ for all m, $n \ge n_0$. But this means that $|s_m - s_n| \le |t_m - t_n| \le \epsilon$ for all m, $n \ge n_0$; as m and n are arbitrary, $\langle s_n\rangle_{n\in\mathbb{N}}$ is Cauchy.

(ii) If $\langle y_n\rangle_{n\in\mathbb{N}}$ is summable, that is, $\langle t_n\rangle_{n\in\mathbb{N}}$ is convergent, then $\langle t_n\rangle_{n\in\mathbb{N}}$ is Cauchy, so $\langle s_n\rangle_{n\in\mathbb{N}}$ is Cauchy, therefore convergent, and $\langle x_n\rangle_{n\in\mathbb{N}}$ is summable.

22. (i) $f(x) = \frac{1}{x}$ for $x \in]0,1[$. ($x \mapsto x$ is continuous, and the reciprocal of a continuous function is continuous.)

(ii) $f(x) = \frac{x}{1+x}$ for $x \in]0,1[$. ($x \mapsto x$ and $x \mapsto 1$ are continuous, and the sum of two continuous functions is continuous, so $x \mapsto 1+x$ is continuous; the reciprocal of a continuous function is continuous, so $x \mapsto \frac{1}{1+x}$ is continuous; the product of two continuous functions is continuous, so $x \mapsto x \times \frac{1}{1+x}$ is continuous.)

The supremum is $\frac{1}{2}$ and the infimum is 0.

(iii) $f(x) = x$ for $x \ge 0$.

(iv) $f(x) = \frac{1}{1+x}$ for $x \ge 0$; supremum is 1, infimum is 0.

(Of course there are many alternative answers.)

23. Let $\epsilon > 0$. Because f is uniformly continuous, there is a $\delta_1 > 0$ such that $|f(x) - f(y)| \le \frac{\epsilon}{2}$ whenever x, $y \in \mathrm{dom}(f)$ and $|x - y| \le \delta_1$. Because g is uniformly continuous, there is a $\delta_2 > 0$ such that $|g(x) - g(y)| \le \frac{\epsilon}{2}$ whenever x, $y \in \mathrm{dom}(g)$ and $|x - y| \le \delta_2$. Set $\delta = \min(\delta_1, \delta_2) > 0$. If x, $y \in \mathrm{dom}(f+g)$ and $|x - y| \le \delta$, then

$$|(f+g)(x) - (f+g)(y)| \le |f(x) - f(y)| + |g(x) - g(y)| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

As ϵ is arbitrary, $f + g$ is uniformly continuous.

24. Let $\epsilon > 0$. Let $\eta > 0$ be such that $|f(x) - f(y)| \leq \epsilon$ whenever x, $y \in \operatorname{dom} f$ and $|x - y| \leq \eta$. Let $\delta > 0$ be such that $|g(x) - g(y)| \leq \eta$ whenever x, $y \in \operatorname{dom} g$ and $|x - y| \leq \delta$. Now if x, $y \in \operatorname{dom}(f \circ g)$ and $|x - y| \leq \delta$, x and y belong to $\operatorname{dom} g$ so $|g(x) - g(y)| \leq \eta$, while also $g(x)$ and $g(y)$ belong to $\operatorname{dom} f$ so

$$|(f \circ g)(x) - (f \circ g)(y)| = |f(g(x)) - f(g(y))| \leq \epsilon.$$

As ϵ is arbitrary, $f \circ g$ is uniformly continuous.

25. (a) Suppose first that $\lim_{x \to a} f(x) = b$.

(i) For any $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - b| \leq \epsilon$ whenever $0 < |x - a| \leq \delta$. In particular, $|f(x) - b| \leq \epsilon$ whenever $a < x \leq a + \delta$. As ϵ is arbitrary, $\lim_{x \downarrow a} f(x) = b$.

(ii) For any $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - b| \leq \epsilon$ whenever $0 < |x - a| \leq \delta$. In particular, $|f(x) - b| \leq \epsilon$ whenever $a - \delta \leq x < a$. As ϵ is arbitrary, $\lim_{x \uparrow a} f(x) = b$.

(b) Now suppose that $\lim_{x \downarrow a} f(x) = \lim_{x \uparrow a} f(x) = b$. Let $\epsilon > 0$. Then there are δ_1, $\delta_2 > 0$ such that

$|f(x) - b| \leq \epsilon$ whenever $a < x \leq a + \delta_1$

$|f(x) - b| \leq \epsilon$ whenever $a - \delta_2 \leq x < a$.

Set $\delta = \min(\delta_1, \delta_2) > 0$. If $0 < |x - a| \leq \delta$, then

either $x > a$ so $a < x \leq a + \delta \leq a + \delta_1$ and $|f(x) - b| \leq \epsilon$

or $x < a$ so $a - \delta_2 \leq a - \delta \leq x < a$ and $|f(x) - b| \leq \epsilon$.

Thus $|f(x) - b| \leq \epsilon$ whenever $0 < |x - a| \leq \delta$; as ϵ is arbitrary, $\lim_{x \to a} f(x) = b$.

26. (a) **?** If $b > c$, set $\epsilon = \frac{1}{3}(b - c) > 0$. Then there are an $n_1 \in \mathbb{N}$ such that $|x_n - b| \leq \epsilon$ whenever $n \geq n_1$, and an $n_2 \in \mathbb{N}$ such that $|y_n - c| \leq \epsilon$ whenever $n \geq n_2$. Set $n = \max(n_1, n_2)$; then $x_n \geq b - |x_n - b| \geq b - \epsilon$ and $y_n \leq c + |y_n - c| \leq c + \epsilon$, so

$$x_n - y_n \geq (b - \epsilon) - (c + \epsilon) = b - c - 2\epsilon = \epsilon.$$

But $x_n - y_n \leq 0$, so this is impossible. **X**

We must therefore have $b \leq c$, as claimed.

(b) No; consider $x_n = 0$, $y_n = \frac{1}{n+1}$ for every n, $b = c = 0$.

27. (a) If $|x| \geq 1$, $|x|^4 \geq |x|^3 \geq 1$ so

$$|f(x) - x^5| \leq 2|x|^4 + |\lambda||x|^3 + 4 \leq 2|x|^4 + |\lambda||x|^4 + 4|x|^4 = (6 + |\lambda|)x^4.$$

Consequently

$$f(x) \geq x^5 - (6 + |\lambda|)x^4 = (x - 6 - |\lambda|)x^4$$

whenever $x \geq 6 + |\lambda|$. Setting $b = |c| + 6 + |\lambda|$ we have $f(b) \geq |c| \geq c$. Similarly,

$$f(x) \leq x^5 + (6 + |\lambda|)x^4 = (x + 6 + |\lambda|)x^4 \leq x + 6 + |\lambda|$$

whenever $x \leq -6 - |\lambda|$. Setting $a = -|c| - 6 - |\lambda|$ we have $f(a) \leq -|c| \leq c$.

(b) Take $\max(1, \sum_{k=0}^{4} |\lambda_k|)$ instead of $6 + |\lambda|$.

(c) Express $g(x)$ as $\sum_{k=0}^{n} \lambda_k x^k$ where n is odd and $\lambda_n \neq 0$. Set $f(x) = \frac{1}{\lambda_n} g(x)$. Then by the arguments of (a)-(b), $|f(x) - x^n| \leq \sum_{k=0}^{n-1} |\frac{\lambda_k}{\lambda_n}||x|^{n-1}$ whenever $|x| \geq 1$. So if $b = \max(1, \sum_{k=0}^{n-1} |\frac{\lambda_k}{\lambda_n}|)$ and $a = -b$, $f(a) \leq 0 \leq f(b)$. By the Intermediate Value Theorem, there is an $x \in [a, b]$ such that $f(x) = 0$ and $g(x) = 0$.

28. (a) Let $\epsilon > 0$. Let $\eta > 0$ be such that $|\frac{f(y)-f(g(x_0))}{y-g(x_0)} - f'(g(x_0))| \le \epsilon$ whenever $0 < |y - g(x_0)| \le \eta$. Let $\delta > 0$ be such that $|g(x) - g(x_0)| \le \eta$ whenever $0 < |x - x_0| \le \delta$. Now if $0 < |x - x_0| \le \delta$, $g(x)$ is defined and $|g(x) - g(x_0)| \le \eta$. If $g(x) = g(x_0)$ then $h(x) = f'(g(x_0))$, so $|h(x) - f'(g(x_0))| = 0 \le \epsilon$. Otherwise, $f(g(x))$ is defined and

$$|h(x) - f'(g(x_0))| = \frac{f(g(x))-f(g(x_0))}{g(x)-g(x_0)} - f'(g(x_0))| \le \epsilon.$$

Thus in either case $|h(x) - f'(g(x_0))| \le \epsilon$. As ϵ is arbitrary, $\lim_{x\to x_0} h(x) = f'(g(x_0))$.

(b) Now suppose that g is differentiable at x_0, so that $\lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} = g'(x_0)$. Then

$$h(x) \cdot \frac{g(x)-g(x_0)}{x-x_0} = \frac{f(g(x))-f(g(x_0))}{x-x_0}$$

whenever $x \in \mathrm{dom}(f\circ g)$, because if $g(x) = g(x_0)$ then both sides are zero, and otherwise the left-hand-side is

$$\frac{f(g(x))-f(g(x_0))}{g(x)-g(x_0)} \cdot \frac{g(x)-g(x_0)}{x-x_0} = \frac{f(g(x))-f(g(x_0))}{x-x_0}.$$

Since $\lim_{x\to a} h_1(x)h_2(x) = \lim_{x\to a} h_1(x) \cdot \lim_{x\to a} h_2(x)$ whenever both limits on the right are defined, we have

$$\begin{aligned}\lim_{x\to x_0} \frac{(f\circ g)(x)-(f\circ g)(x_0)}{x-x_0} &= \lim_{x\to x_0} h(x) \cdot \frac{g(x)-g(x_0)}{x-x_0} \\ &= \lim_{x\to x_0} h(x) \cdot \lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} \\ &= f'(g(x_0))g'(x_0),\end{aligned}$$

and $(f\circ g)'(x_0)$ is defined (and equal to $f'(g(x_0))g'(x_0)$).

29. (a) Because continuous functions on closed bounded intervals are bounded and attain their bounds, there are z_1, $z_2 \in [a, b]$ such that $f(z_1) \le f(x) \le f(z_2)$ whenever $x \in [a, b]$. Setting $c = f(z_1)$ and $d = f(z_2)$, we see that $[c, d]$ includes the set of values of f. On the other hand, if y is any point of $[c, d]$, it lies between $f(z_1)$ and $f(z_2)$, so is of the form $f(x)$ for some x lying between z_1 and z_2, by the Intermediate Value Theorem; and the set of values of f is the whole of $[c, d]$.

(b) Suppose $f(a) \le f(b)$. **?** If there are x, $y \in [a, b]$ such that $x \le y$ and $f(x) > f(y)$, then we must have $x < y$. Now either $f(a) < f(x)$ or $f(y) < f(b)$. If $f(a) < f(x)$ set $w = \max(f(a), f(y))$, so that $f(a) \le w < f(x)$ and $f(x) > w \ge f(y)$. By the IVT, applied to the intervals $[a, x]$ and $[x, y]$, there are $z \in [a, x]$ and $z' \in [x, y]$ such that $f(z) = f(z') = w$; but as neither z nor z' can be equal to x, $z < z'$ and f is not injective.

Similarly, of $f(y) < f(b)$, set $w = \min(f(x), f(b))$; then $f(y) < w \le f(x)$ and $f(y) < w \le f(b)$, so there are $z \in [x, y]$ and $z' \in [y, b]$ such that $f(z) = f(z') = w$, and again f is not injective. **X**

So $f(x) \le f(y)$ whenever $a \le x \le y \le b$.

30. We know that dom f^{-1} is an interval $[c,d]$ where $c \leq d$, and that $f(x) \leq f(y)$ whenever $a \leq x \leq y \leq b$, so that $c = f(a)$ and $d = f(b)$ and in fact $f(x) < f(y)$ whenever $a \leq x < y \leq b$ (since f is injective). Let $v \in [c,d]$ and $\epsilon > 0$. Set $x = f^{-1}(v) \in [a,b]$. Define δ_1, $\delta_2 > 0$ by setting

$$\begin{aligned}\delta_1 &= v - f(x-\epsilon) \text{ if } x-\epsilon \geq a,\\ &= 1+d-c \text{ otherwise,}\\ \delta_2 &= f(x+\epsilon) - v \text{ if } x+\epsilon \leq b,\\ &= 1+d-c \text{ otherwise.}\end{aligned}$$

Because δ_1 and δ_2 must both be greater than 0. Set $\delta = \min(\delta_1,\delta_2) > 0$. If $w \in [c,d]$ and $|v-w| \leq \delta$, then

$$f(\max(a, x-\epsilon)) = \max(c, v-\delta_1)$$

(because if $a \leq x-\epsilon$ then $f(x-\epsilon) = v-\delta_1$, while if $x-\epsilon < a$ then $v-\delta_1 < c$)

$$\leq w \leq \min(d, v+\delta_2) = f(\min(b, x+\epsilon))$$

because if $x+\epsilon \leq b$ then $f(x+\epsilon) = v+\delta_1$, while if $b < x+\epsilon < a$ then $d < v+\delta_1$. But this means that $f^{-1}(w)$ must lie in the interval $[\max(a,x-\epsilon), \min(b,x+\epsilon)]$ and $|f^{-1}(w)-x| \leq \epsilon$. As ϵ is arbitrary, f^{-1} is continuous at v; as v is arbitrary, f^{-1} is continuous.

31. (a) Let $\epsilon > 0$. Then (writing $f(x) = x$ for $x \in \mathbb{R}$) $|f(x) - f(y)| \leq \epsilon$ whenever $|x-y| \leq \epsilon$, so f is uniformly continuous.

(b) Write $f(x) = x^2$ for $x \in \mathbb{R}$. Set $\epsilon = 1$. If $\delta > 0$ and $x = \frac{1}{\delta}$, $y = x+\delta$ then $f(y) - f(x) = \delta^2 + 2 > \epsilon$. So there is no $\delta > 0$ such that $|f(x) - f(y)| \leq \epsilon$ whenever $|x-y| \leq \delta$, and f is not uniformly continuous.

(c) Set $f(x) = (\sin x)^2$. Then $f'(x) = \sin x \sin' x + \sin' x \sin x = 2 \sin x \sin' x$, so $|f'(x)| \leq 2$ for every x. If $a < b$ there is an $x \in \,]a,b[$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$, so that $\sin^2 b - \sin^2 a = (b-a)f'(x)$ and $|\sin^2 b - \sin^2 a| \leq 2|b-a|$. Similarly, if $a > b$, $|\sin^2 b - \sin^2 a| = |\sin^2 a - \sin^2 b| \leq 2|a-b| = 2|b-a|$, while if $a = b$ we have $|\sin^2 b - \sin^2 a| = 0 = 2|b-a|$.

Now let $\epsilon > 0$. Set $\delta = \frac{1}{2}\epsilon > 0$. Then if x, $y \in \mathbb{R}$ and $|x-y| \leq \delta$,

$$|\sin^2 x - \sin^2 y| \leq 2|x-y| \leq 2\delta = \epsilon.$$

As ϵ is arbitrary, $\sin^2$ is uniformly continuous.

(d) Let $\epsilon > 0$. Let M_1 be such that $|f(x)| \leq M_1$ for every $x \in \operatorname{dom} f$. Let M_2 be such that $|g(x)| \leq M_2$ for every $x \in \operatorname{dom} g$. Set $\eta = \min(1, \frac{\epsilon}{1+M_1+M_2})$. Let $\delta_1 > 0$ be such that $|f(x) - f(y)| \leq \eta$ whenever x, $y \in \operatorname{dom} f$ and $|x-y| \leq \delta_1$. Let $\delta_2 > 0$ be such that $|g(x)-g(y)| \leq \eta$ whenever x, $y \in \operatorname{dom} g$ and $|x-y| \leq \delta_2$. Set $\delta = \min(\delta_1, \delta_2) > 0$. Now if x, $y \in \operatorname{dom}(f \times g)$ and $|x-y| \leq \delta$, then

$$\begin{aligned}|f(x)g(x) - f(y)g(y)| &\leq |f(x) - g(x)||f(y) - g(y)|\\ &\qquad + |f(x)||g(x) - g(y)| + |f(x) - f(y)||g(x)|\\ &\leq \eta^2 + M_1\eta + \eta M_2 \leq \eta(1+M_1+M_2) \leq \epsilon.\end{aligned}$$

As ϵ is arbitrary, $f \times g$ is uniformly continuous.

32. (a) Setting $f(y) = y^2$ for $y \in \mathbb{R}$, we know that f is continuous and that $f(0) \le x$, while $f(\max(1,x)) = \max(1,x^2) \ge x$. By the Intermediate Value Theorem, there is a $y \in [0, \max(1,x)]$ such that $y^2 = x$.

If now $y_1^2 = y_2^2 = x$ we must have $(y_1 + y_2)(y_1 - y_2) = 0$, that is, at least one of $y_1 + y_2$, $y_1 - y_2$ is zero. If y_1 and y_2 are both greater than or equal to 0, this can happen only when $y_1 = y_2$.

(b) We have $(\sqrt{x} + \sqrt{\delta})^2 = x + \delta + 2\sqrt{x}\sqrt{\delta} \ge x + \delta$, so $(\sqrt{x})^2 \le y \le (\sqrt{x} + \sqrt{\delta})^2$ and there is a $z \in [\sqrt{x}, \sqrt{x} + \sqrt{\delta}]$ such that $z^2 = y$; since positive square roots are unique, $z = \sqrt{y}$ and $\sqrt{x} \le \sqrt{y} \le \sqrt{x} + \sqrt{\delta}$, so that $|\sqrt{y} - \sqrt{x}| \le \sqrt{\delta}$.

Now let $\epsilon > 0$. Set $\delta = \epsilon^2$. If $x, y \ge 0$ and $|x - y| \le \delta$, either $x \le y \le x + \delta$ or $y \le x \le y + \delta$. In either case, $|\sqrt{y} - \sqrt{x}| = |\sqrt{x} - \sqrt{y}| \le \sqrt{\delta} = \epsilon$. As ϵ is arbitrary, $\sqrt{\ }$ is uniformly continuous.

33. Let $\epsilon > 0$. Let $\delta_0 > 0$ be such that $\frac{f'(x)}{g'(x)}$ is defined and $|\frac{f'(x)}{g'(x)} - b| \le \frac{1}{2}\epsilon$ whenever $x_0 < x \le x_0 + \delta_0$. Note that this implies that $f'(x)$ and $g'(x)$ are defined, and that $g'(x) \ne 0$, for $x_0 < x \le x_0 + \delta_0$; so that $f(x)$ and $g(x)$ are defined for $x_0 < x \le x_0 + \delta_0$, and g is continuous at every point of $]x_0, x_0 + \delta_0]$. By Rolle's theorem, $g(x)$ cannot be zero for more than one value of x in $]x_0, x_0 + \delta_0]$. So there is a $\delta \in]0, \delta_0]$ such that $g(x) \ne 0$ for $x_0 < x \le x_0 + \delta$.

Take x such that $x_0 < x \le x_0 + \delta$. Then $\lim_{y \downarrow x_0} \frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f(x)-0}{g(x)-0} = \frac{f(x)}{g(x)}$. Let $\delta_1 > 0$ be such that $\frac{f(x)-f(y)}{g(x)-g(y)}$ is defined and $|\frac{f(x)-f(y)}{g(x)-g(y)} - \frac{f(x)}{g(x)}| \le \frac{1}{2}\epsilon$ whenever $x_0 < y \le x_0 + \delta_1$. Since we cannot have $g(y) = g(x)$ for $x_0 < y \le x_0 + \delta_1$, $x_0 + \delta_1 < x$. Now examine $y = x_0 + \delta_1$. Because $y < x$ and f and g are differentiable (therefore continuous) at every point of $[y, x]$, **Cauchy's Mean Value Theorem** tells us that there is a $z \in]y, x[$ such that $\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(z)}{g'(z)}$. So

$$|\frac{f(x)}{g(x)} - b| \le |\frac{f(x)}{g(x)} - \frac{f(x)-f(y)}{g(x)-g(y)}| + |\frac{f'(z)}{g'(z)} - b| \le \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.$$

As ϵ is arbitrary, $\lim_{x \downarrow 0} \frac{f(x)}{g(x)} = b$.

34. (a) Set $t_n = \sum_{k=0}^{n} |y_k|$ for $n \in \mathbb{N}$. Then $\langle t_n \rangle_{n\in\mathbb{N}}$ is convergent, therefore Cauchy. Now $|s_m - s_n| \le |t_m - t_n|$ for all m, $n \in \mathbb{N}$. **P** If $n < m$, then

$$|s_m - s_n| = |\sum_{k=n+1}^{m} x_k| \le \sum_{k=n+1}^{m} |x_k| \le \sum_{k=n+1}^{m} |y_k| = t_m - t_n = |t_m - t_n|.$$

If $m < n$, then

$$|s_m - s_n| = |s_n - s_m| \le |t_n - t_m| = |t_m - t_n|.$$

If $m = n$ then $|s_m - s_n| = 0 = |t_m - t_n|$. **Q**

Let $\epsilon > 0$. Then there is an $n_0 \in \mathbb{N}$ such that $|t_m - t_n| \le \epsilon$ whenever m, $n \ge n_0$. So $|s_m - s_n| \le \epsilon$ whenever m, $n \ge n_0$. As ϵ is arbitrary, $\langle s_n \rangle_{n\in\mathbb{N}}$ is Cauchy. By Cauchy's General Principle of Convergence, $\langle s_n \rangle_{n\in\mathbb{N}}$ is convergent, that is, $\langle x_n \rangle_{n\in\mathbb{N}}$ is summable.

(b) Set $t_n = \sum_{k=0}^{n} |y_k|$ for $n \in \mathbb{N}$. Then $\langle t_n \rangle_{n\in\mathbb{N}}$ is convergent, therefore Cauchy. Now $|s_m - s_n| \le M|t_m - t_n|$ for all m, $n \ge n_1$. **P** If $n < m$, then

$$|s_m - s_n| = |\sum_{k=n+1}^{m} x_k| \le \sum_{k=n+1}^{m} |x_k| \le \sum_{k=n+1}^{m} M|y_k| = M(t_m - t_n) = M|t_m - t_n|.$$

If $m < n$, then

$$|s_m - s_n| = |s_n - s_m| \le M|t_n - t_m| = M|t_m - t_n|.$$

If $m = n$ then $|s_m - s_n| = 0 = M|t_m - t_n|$. **Q**

Let $\epsilon > 0$. Then there is an $n_0 \in \mathbb{N}$ such that $|t_m - t_n| \le \frac{\epsilon}{M+1}$ whenever m, $n \ge n_0$. So $|s_m - s_n| \le \epsilon$ whenever m, $n \ge \max(n_0, n_1)$. As ϵ is arbitrary, $\langle s_n \rangle_{n\in\mathbb{N}}$ is Cauchy. By Cauchy's General Principle of Convergence, $\langle s_n \rangle_{n\in\mathbb{N}}$ is convergent, that is, $\langle x_n \rangle_{n\in\mathbb{N}}$ is summable.

(c) Set $b = \lim_{n\to\infty} \frac{x_n}{y_n}$. There is an $n_1 \in \mathbb{N}$ such that $|\frac{x_n}{y_n} - b| \le 1$ for every $n \ge n_1$, so that $|\frac{x_n}{y_n}| \le |b| + 1$ and $|x_n| \le (|b| + 1)|y_n|$ for every $n \ge n_1$. By (b), $\langle x_n \rangle_{n\in\mathbb{N}}$ is summable.

35. (a) Because $\frac{1}{2}(1 - b) > 0$, there is an $n_0 \in \mathbb{N}$ such that $|\frac{x_{n+1}}{x_n} - b| \le \frac{1}{2}(1 - b)$ for every $n \ge n_0$, so that $|\frac{x_{n+1}}{x_n}| \le b + \frac{1}{2}(1 - b) = c$ for every $n \ge n_0$. (Note that b, being the limit of a sequence of non-negative terms, is non-negative.) This means that $|x_{n+1}| \le c|x_n|$ for $n \ge n_0$. Set $M = |x_{n_0}|c^{-n_0}$. Then $|x_n| \le Mc^n$ for every $n \ge n_0$. **P** Induce on n. For $n = n_0$ we have $Mc^n = |x_{n_0}|c^{-n_0}c^{n_0} = |x_{n_0}|$, so the induction starts. For the inductive step, given that $n \ge n_0$ and $|x_n| \le Mc^n$, then

$$|x_{n+1}| \le c|x_n| \le cMc^n$$

(by the inductive hypothesis)

$$= Mc^{n+1},$$

so the induction continues. **Q**

(c) We know that $\sum_{n=0}^{\infty} c^n = \frac{1}{1-c}$, so $\langle c_n \rangle_{n\in\mathbb{N}}$ is absolutely summable; by the comparison test, so is $\langle x_n \rangle_{n\in\mathbb{N}}$.

36. (a)(i) We have $g'(x) = f'(a+x) - f'(a)$ defined and continuous on $[0,h]$ while $g''(x) = f''(a+x)$ is defined on $]0,h[$. By Cauchy's Mean Value Theorem, there is an $h_1 \in]0,h[$ such that $\frac{g(h)-g(0)}{h^2-0^2} = \frac{g'(h_1)}{2h_1}$, that is, $\frac{f(a+h)-f(a)-xf'(a)}{h^2} = \frac{f'(a+h_1)-f'(a)}{2h_1}$.

(ii) Now g' is continuous on $[0,h_1]$ and differentiable on $]0,h_1[$, so by Cauchy's Mean Value Theorem again there is an $h_2 \in]0,h_1[$ such that

$$\frac{g''(h_2)}{2} = \frac{g'(h_1)-g'(0)}{2h_1-2\cdot 0} = \frac{g'(h_1)}{2h_1},$$

that is,

$$\frac{1}{2}f''(a+h_2) = \frac{f(a+h)-f(a)-hf'(a)}{h^2}.$$

(b) Set $g(x) = f(a+x) - f(a) - xf'(a) - \frac{1}{2}x^2 f''(a)$. Then g is defined and continuous on $[0,h]$ and differentiable on $]0,h[$. So there is an $h_1 \in]0,h[$ such that

$$\frac{f(a+h)-f(a)-hf'(a)-\frac{1}{2}h^2f''(a)}{h^3} = \frac{g(h)-g(0)}{h^3-0^3} = \frac{g'(h_1)}{3h_1^2}.$$

Now apply (a) to the function g'; there is an $h_3 \in]0,h_1[$ such that

$$g'(h_1) = g'(0) + h_1 g''(0) + \frac{1}{2}h_1^2 g'''(h_3) = \frac{1}{2}h_1^2 g'''(h_3),$$

so that

$$\frac{f(a+h)-f(a)-hf'(a)-\frac{1}{2}h^2f''(a)}{h^3} = \frac{1}{6}g'''(h_3) = \frac{1}{6}f'''(a+h_3).$$

37. (no solution written out)

38. (no solution written out)

39. (no solution written out)

40. (no solution written out)

41. Let $\epsilon > 0$.

Set $A = \{x : a \le x \le b, \exists\, \delta > 0, |f(y) - f(y')| \le \epsilon$ whenever $y, y' \in [a,x]$ and $|y - y'| \le \delta\}$.

$a \in A$ because if $y, y' \in [a,a]$ then $|f(y) - f(y') = 0$.

A is bounded above by b.

So $z = \sup A$ is defined and $a \le z \le b$.

Because $z \in [a,b]$, f is continuous at z; let $\delta_0 > 0$ be such that $|f(y) - f(z)| \le \epsilon$ whenever $y \in [a,b] \cap [z-\delta_0, z+\delta_0]$.

Because $z = \sup A$, there is an $x_1 \in A \cap [z - \frac{1}{2}\delta_0, z]$.

Let $\delta_1 > 0$ be such that $|f(y) - f(y')| \le \frac{1}{2}\epsilon$ whenever $y, y' \in [a, x_1]$ and $|y - y_1| \le \delta_1$.

Set $\delta = \min(\frac{1}{2}\delta_0, \delta_1) > 0$.

Suppose that $y, y' \in [a, \min(b, z+\delta_0)]$ and $|y - y'| \le \delta$.

If $\min(y,y') \ge z - \delta_0$, then $|f(y) - f(y')| \le |f(y) - f(z)| + |f(y') - f(z)| \le \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$.

If $\min(y,y') \le z - \delta_0$ then $\min(y,y') \le x - \frac{1}{2}\delta_0$; since $|y - y'| \le \frac{1}{2}\delta_0$, both y and y' belong to $[a,x]$; since $|y - y'| \le \delta_1$, $|f(y) - f(y')| \le \epsilon$.

Since y and y' are arbitrary, $\min(b, z+\delta_0) \in A$.

Since z is an upper bound for A, $\min(b, z+\delta_0) \le z$, so in fact $z = b$ and $b \in A$.

This means that there is a $\delta > 0$ such that $|f(y) - f(y')| \le \epsilon$ whenever $y, y' \in [a,b]$ and $|y - y'| \le \delta$. As ϵ is arbitrary, f is uniformly continuous.