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Calculus and Linear Mathematics Problems

1. Sketch the graphs of the lines

$$x+y=3, \quad x-2y=-3, \quad 2x-4y=2,$$

$$5x-2y=2, \quad 10x-4y=4$$

all on the same diagram, labelling each line. [A]

2. Give the derivatives of the functions
(a) $x\ln x - x$ (defined for $x>0$) (b) $\sqrt{\frac{1}{x}-1}$ (defined for $0<x\le 1$)
(c) $e^{-x^2/2}$ (defined for $x\in\mathbb{R}$) (d) $\arctan x$ (defined for $-1<x<1$)
(e) $\frac{1}{2}\ln(\frac{1+x}{1-x})$ (defined for $-1<x<1$) (f) 7^x (defined for $x\in\mathbb{R}$)
(g) $|x|$ (defined for $x\in\mathbb{R}$) (h) $\ln(|x|)$ (defined for $x\neq 0$)
(i) $-\ln|\cos x|$ (defined for $\cos x\neq 0$).
[AA]

3. Find the stationary points of
(a) $x\ln x$ (defined for $x>0$) (b) $\frac{1}{x}\ln x$ (defined for $x>0$) (c) $e^{-|x|}$ (defined for $x\in\mathbb{R}$)
(d) $x^2e^{-|x|}$ (defined for $x\in\mathbb{R}$) (e) $|x|e^{-x}$ (defined for $x\in\mathbb{R}$).
[AB]

4. Find the local maxima of
(a) $\ln\cos x$ (defined when $\cos x>0$) (b) $x\ln x$ (defined for $x>0$) (c) x^3-x (defined for $x\in\mathbb{R}$)
(d) $\frac{1}{x}\ln x$ (defined for $x>0$) (e) $e^{-|x|}$ (defined for $x\in\mathbb{R}$) (f) $x^2e^{-|x|}$ (defined for $x\in\mathbb{R}$).
[AC]

5. Find the global maxima of
(a) $\ln\cos x$ (defined when $\cos x>0$) (b) $x/|x|$ (defined for $x\neq 0$) (c) x^3-x (defined for $x\in\mathbb{R}$)
(d) $\frac{1}{x}\sin x$ (defined for $x\neq 0$) (e) $\frac{1}{x}\ln x$ (defined for $x>0$) (f) xe^{-x} (defined for $x\in\mathbb{R}$).
[AD]

6. Find the local minima of
(a) $\sqrt{x^2}$ (defined for $x\in\mathbb{R}$) (b) x^3-x (defined for $x\in\mathbb{R}$)
(c) $x\ln x$ (defined for $x>0$) (d) $\sqrt{\frac{1}{x}-1}$ (defined for $0<x\le 1$)
(e) $\frac{1}{\sqrt{1-x^2}}$ (defined for $-1<x<1$) (f) $x^2e^{-|x|}$ (defined for $x\in\mathbb{R}$).
[AE]

7. Find the global minima of
(a) $\exp\ln x$ (defined for $x>0$) (b) $\sqrt{x}$ (defined for $x\ge 0$)
(c) $x\ln x$ (defined for $x>0$) (d) $|x|e^{-x}$ (defined for $x\in\mathbb{R}$).
[AF]

8. Find the points of inflection of
(a) x^3-x (defined for $x\in\mathbb{R}$) (b) $x\ln x$ (defined for $x>0$)
(c) $\sqrt{\frac{1}{x}-1}$ (defined for $0<x\le 1$) (d) $\frac{1}{x}\ln x$ (defined for $x>0$).
[AG]

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9. On the same piece of graph paper, plot the functions

$$x^1,\ x^0,\ x^{-1},\ x^{1/2},\ x^{-1/2},\ x^2,\ x^{-2},\ x^{\sqrt{2}},\ x^{1/\sqrt{2}}$$

all for $0 < x \leq 2$. [AH]

10. Consider the formula $8\sqrt{x} - 6x + x^2$.
(a) What is the natural domain to choose when using this formula to define a real function f?
(b) Find its point of inflection.
(c) Sketch the function.
(d) Find the stationary points and local and global maxima and minima (if any) of f.
[AI]

11. Plot (on the same piece of graph paper) the graphs of $\exp(x)$, $1 + x + \frac{x^2}{2!}$ and $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$, for $-4 \leq x \leq 4$. [AJ]

12. Plot (on the same piece of graph paper) the graphs of $\ln(1+x)$, $x - \frac{x^2}{2} + \frac{x^3}{3}$, $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$, for $-1 < x \leq 1$. [AK]

13. Consider the function $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, defined for all real x.
(a) Calculate $\tanh 0$, $\tanh 1$, $\tanh 2$, $\tanh 3$, $\tanh' 0$, all to two places of decimals.
(b) Sketch the graph of tanh.
(c) Solve the equation $\tanh x = y$ for x in terms of y, by (i) solving the equation $\frac{z - \frac{1}{z}}{z + \frac{1}{z}} = y$ for z in terms of y (ii) solving the equation $e^x = z$ for x in terms of z.
(d) Find the derivative of the inverse function $\tanh^{-1}(x)$. [AL]

14. Find the effective annual rate of interest when interest at 6% p.a. is compounded (a) half-yearly (b) monthly. (Give your answers correct to 3 significant figures.) [AM]

15. A project requires an investment of £1000000, spread evenly over two years, starting now, and is then expected to yield £100000 p.a. for the following 20 years, spread evenly. Find the present value of the expected net return
(a) at a discount rate of 4% p.a.
(b) at a discount rate of 6% p.a.
(Give your answers correct to 2 significant figures.) [AN]

16. Now suppose that the project of the previous question has a close-down cost of £500000, a one-off payment at the end of its 22 years of life. Recompute the present value of the project at the two discount rates. [AO]

17. (a) Give the formula for the sum $a + ax + ax^2 + \ldots + ax^n$ of a geometric series. What happens when $x = 1$?
(b) Find the present value of an annual payment of £1000 p.a. for twenty years, starting one year from now, assuming a discount rate of 5% p.a., giving your answer to 3 significant figures.
(c) Find the present value of an annual payment of £1000 p.a. in perpetuity, starting one year from now, assuming a discount rate of 5% p.a., giving your answer to 3 significant figures. [AP]

18. Consider the function arcsin, defined by saying that if $-1 \leq x \leq 1$ then $\arcsin(x) = y$ iff $\sin y = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
(a) Sketch arcsin.
(b) Find its derivative. [AQ]

19. Give the first-order partial derivatives $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ (and $\dfrac{\partial f}{\partial z}$, if appropriate) of the following functions:
(a) $f(x,y) = x^2y^3$
(b) $f(x,y) = \frac{x}{y}$ for $y \neq 0$
(c) $f(x,y) = \sqrt{x^2+y^2+z^2}$ for $(x,y,z) \neq (0,0,0)$
(d) $f(x,y) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ for $(x,y,z) \neq (0,0,0)$
(e) $f(x,y) = \ln(x^2+y^2)$ for $(x,y) \neq (0,0)$
(f) $f(x,y) = xy + 2x - \ln(x^2y)$ for $x \neq 0$, $y > 0$. [AR]

20. Give the second-order partial derivatives of the following functions. (You may assume that in every case $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.)
(a) $f(x,y) = x^2 + 2xy^2 + 3$
(b) $f(x,y) = y^2 \ln(x+y)$ for $x+y > 0$
(c) $f(x,y) = 3x^2 - 2xy + 3y^2 + 2x - 6y$
(d) $f(x,y) = x^3 - y^3 - 2xy + 6$. [AS]

21. For each of the following functions (i) find its stationary point (x^*, y^*) (ii) obtain a formula for $g(\xi,\eta) = f(\xi + x^*, \eta + y^*)$ (iii) classify the stationary point as local maximum, local minimum or saddle point:
(a) $f(x,y) = 3x^2 - 2xy + 3y^2 + 2x - 6y$
(b) $f(x,y) = 2x^2 - 3y^2 - 2x$
(c) $f(x,y) = 2xy - 5x^2 - 10y^2$. [AT]

22. (a) Find $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$ where $f(x,y) = \ln(x^2+y^2)$ for $(x,y) \neq (0,0)$.
(b) Find $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$ where $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ for $(x,y,z) \neq (0,0,0)$. [AU]

23. Consider the function $f(x,y) = xy + 2x - \ln(x^2y)$ for x, $y > 0$. (i) Find its stationary point (x^*, y^*). (ii) Obtain a formula for $g(\xi,\eta) = f(\xi + x^*, \eta + y^*)$. (iii) Determine whether the stationary point is a local maximum, local minimum or saddle point.
(*Hint:* recall that $\ln(1+a) \doteq a - \dfrac{a^2}{2}$ if $a \doteq 0$.) [AV]

24. The following appeared in the *Times*, 2.12.93:

> **Belgrade:** The rate of inflation in Serbia rose 0·7 per cent an hour and amounted to 20,190 per cent in November, Mirjana Rankovic, deputy director of the federal statistics bureau, said. That translates into an annual inflation rate of 286,125,293,792 per cent. *(AP).*

Comment. [AW]

25. For the following quadratic forms, which have maxima at $\mathbf{0}$, which have minima, and which have saddle points?
(a) $f(x,y) = x^2 + 4xy + 3y^2$
(b) $f(x,y) = -x^2 + 4xy - 5y^2$
(c) $f(x,y) = 2x^2 - 4xy + 3y^2$
(d) $f(x,y,z) = x^2 + 5y^2 + 7z^2 + 4xy - 2xz$. [AX]

26. Set $f(x,y) = x^2y + xy^2 - x^2 - y^2 - 3xy + 2x + 2y - 1$.
(a) Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}$.
(b) Find the stationary points of f. (*Hint:* note that $y - 1$ is a factor of $\frac{\partial f}{\partial x}$.)
(c) At each of the stationary points of f compute the matrix $\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$.
(d) Classify each stationary point of f as a local maximum, local minimum or saddle point.
(e) Sketch the contour $f = 0$ (*hint:* compute $f(1,y)$) and on your diagram mark (i) the saddle points of f (ii) the regions in which f is greater than 0. [AY]

27. In each of the following cases, determine whether $(0,0)$ is a local maximum or local minimum (or neither) of the given function f:
(a) $f(x,y) = (x+y)(x^2 - xy + y^2)$
(b) $f(x,y) = \cos x - y^4$
(c) $f(x,y) = e^{-x^2/2} + y^4$
(d) $f(x,y) = x^2 e^{-y^2/2}$. [AZ]

28. Solve the pairs of equations
(a) $x + y = 3$, $x - 2y = -3$
(b) $x - 2y = -3$, $2x - 4y = 2$
(c) $5x - 2y = 2$, $10x - 4y = 4$. [B]

29. Consider the quadratic form $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is the matrix $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.
(a) Find the eigenvalues and eigenvectors of $\mathbf{A}$.
(b) Set $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$. Find $\mathbf{P}^{-1}$, $\mathbf{PD}$, $\mathbf{AP}$, $\mathbf{P}^{-1}\mathbf{AP}$.
(c) Setting $\mathbf{z} = \mathbf{P}^{-1}\mathbf{x}$, show that $f(\mathbf{x}) = 2z_1^2 + 4z_2^2$.
(d) Hence, or otherwise, show that f is positive definite. [BA]

30. Set $f(x,y,z) = (x + 2y + 2z)e^{-\frac{1}{2}(x^2+y^2+z^2)}$. Find the stationary points of f and classify each as a local maximum, local minimum or neither. (*Hint:* do not examine the second partial derivatives of f, but rather consider its behaviour far from the origin.) [BB]

31. Find the present value, at a discount rate of 10% p.a., of an investment of £10,000,000 spread evenly over the next five years and yielding a return of £20,000,000 spread evenly over the subsequent 10 years. [BC]

32. The following appeared in the *Times*, 9.12.93, p. 23:

> Recent figures show the cost to the UK's banks of shifting cash to fill dispensing machines is £4.5 billion a year. Electronic purses would dramatically reduce that figure.

Comment. [BD]

33. Find indefinite integrals of
(i) $\frac{2x}{2+x}$ (ii) e^{-2x} (iii) $\sin 7x$ (iv) $\frac{1}{2-x}$ (v) $\frac{1}{a^2+x^2}$ where $a > 0$. [BE]

34. Find indefinite integrals of
(i) $\cot x$ (ii) $\frac{x+1}{x^2+2x+2}$ (iii) $\frac{e^x}{1+e^x}$ (iv) $\frac{1}{x \ln x}$. [BF]

35. Find indefinite integrals of
(i) $\frac{1}{x^2}e^{-1/x}$ (ii) $e^x \cos(e^x)$ (iii) $2x\sqrt{x^2+3}$ (iv) $\frac{1}{2+\sqrt{1-x}}$ (v) $\frac{\sqrt{x+2}}{x+6}$ (vi) $x \sin x^2$. [BG]

36. Find indefinite integrals of
(i) $x \cos x$ (ii) $e^x \sin x$ (iii) $x^2 \sin x$.
[BH]

37. Set $F_n(x) = \int_0^x t^n e^{-t} dt$. Show that

$$F_n(x) = nF_{n-1}(x) - x^n e^{-x}$$

for every integer $n \geq 1$. Find $F_4(1)$. [BI]

38. (a) Find $\frac{d}{dx} \frac{x}{(1+x)^2}$.
(b) Give indefinite integrals of
(i) $\frac{1}{1+x^2}$ (ii) $\frac{2x}{1+x^2}$ (iii) $\frac{2x}{(1+x^2)^2}$.
(c) Find an indefinite integral of $\frac{1}{(1+x^2)^2}$. (*Hint:* express $\frac{1}{(1+x^2)^2}$ as $A\frac{1}{1+x^2} + B\frac{2x}{1+x^2} + C\frac{2x}{(1+x^2)^2} + D\frac{1-x^2}{(1+x^2)^2}$.)
[BJ]

39. Find $\mathbf{A}^3$ where $\mathbf{A} = \begin{pmatrix} -2 & -1 & 3 \\ 1 & 0 & -1 \\ 1 & 2 & -2 \end{pmatrix}$. [BK]

40. Use row operations ('Gaussian elimination') to solve the following sets of equations:
(i)

$$\begin{aligned} y + 2z &= 1 \\ 2x + 3y - z &= 3 \\ -x - y + 3z &= 2 \end{aligned}$$

(ii)

$$\begin{aligned} 2x + 3y - 2z &= 5 \\ x - 2y + 3z &= 2 \\ 4x - y + 4z &= 1 \end{aligned}$$

(iii)

$$\begin{aligned} x + 2y - 3y + 2z &= 2 \\ 2x + 5y - 8z + 6w &= 5 \\ 3x + 4y - 5z + 2w &= 4 \end{aligned}$$

[BL]

41. Find the inverse of the matrix $\begin{pmatrix} 1 & 2 & 0 \\ -2 & 5 & -5 \\ 5 & 7 & 2 \end{pmatrix}$. [BM]

42. Find the determinant of the matrix $\begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 5 \\ -1 & 10 & -1 \end{pmatrix}$. [BN]

43. Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 4 & 5 & 0 \\ 3 & 6 & 0 \\ 1 & 2 & 1 \end{pmatrix}$. [BO]

44. Find the local maxima and minima, if any, of the functions
(i) $\frac{1}{x^2-x-2}$ (ii) $\frac{x}{x^2+2}$ (iii) $\frac{e^x-1}{e^x+1}$ (iv) $\sqrt{1+2x}$ (v) x^x
in each case stating precisely the domain you are attaching to the formula given. [BP]

45. Solve the equations

$$x + 2y = 3,\ y + 2z = 3,\ z + 2x = 3.$$

[C]

46. Arrange the following numbers in order, smallest first, noting any which happen to be equal:

$$\pi,\ 3,\ 3\tfrac{1}{7},\ \tfrac{355}{113},\ \sqrt{10},\ e,\ \sqrt{8},\ 2(\arccos\tfrac{8}{17} + \arccos\tfrac{15}{17}).$$

('arccos' is very likely called $\cos^{-1}$ on your calculator. Make sure that you calculate the two angles in *radians*, not degrees.) [D]

47. Factorise into linear factors

(a) $x^2 + 5x + 6$	(e) $x^2 + 5xy + 6y^2$	(i) $x^4 + 4x^3 + 6x^2 + 4x + 1$
(b) $x^2 - 5x + 6$	(f) $-2x^2 + 3x + 2$	(j) $x^4 - 4x^3 + 6x^2 - 4x + 1$
(c) $6x^2 + 5x + 1$	(g) $x^3 + 6x^2 + 11x + 6$	(k) $x^2 - 3$
(d) $2x^2 - 3x - 2$	(h) $6x^3 + 11x^2 + 6x + 1$	(l) $x^2 + x - 1$

[E]

48. Consider the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & -2 \\ -1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 3 & 0 & 4 \\ 1 & 2 & -1 \\ 4 & 1 & 3 \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & -2 & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} -5 & 5 \\ 3 & -3 \\ 1 & -1 \end{pmatrix}.$$

Evaluate those of the following that are defined, and explain why any others are not:
(a) **AB** (b) **BA** (c) **AC** (d) **CA** (e) **CD** (f) **DC**. [F]

49. A business produces three products P_1, P_2, P_3 for clients F_1, F_2 using raw materials $R_1, \ldots, R_4$. Let matrix **A** represent the purchase orders, so that a_{ij} is the quantity of product P_j ordered by client F_i, and matrix **B** the quantities of raw materials used, so that b_{jk} is the quantity of material R_k needed for production of a unit quantity of product P_j; and let the column vector **c** represent the cost of materials, so that c_k is the price of unit quantity of material R_k.

$$\mathbf{A} = \begin{pmatrix} 9 & 12 & 6 \\ 10 & 20 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 3 & 4 & 0 & 14 \\ 7 & 9 & 3 & 20 \\ 5 & 12 & 7 & 16 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 10 \\ 5 \\ 15 \\ 20 \end{pmatrix}.$$

Find and interpret **AB**, **Bc**, **ABc** and $(1 \quad 1)\,\mathbf{A}$, where $(1 \quad 1)$ is a row vector. [G]

50. Set

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & -1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}.$$

Find **AB**, **BA**. Which theorem covered in lectures could (or did) enable you to save trouble in this calculation? [H]

51. Set

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 3 \\ -2 & 0 & 1 \end{pmatrix}.$$

Find $\mathbf{B} = \mathbf{AA}$, $\mathbf{AB}$, $\mathbf{BA}$. Which theorem covered in lectures could (or did) enable you to save trouble in this calculation? [I]

52. Use row operations ('Gaussian elimination') to solve the following sets of equations:

(a)

$$\begin{aligned} x + 4y + 11z &= 7 \\ x + 4y + 8z &= 4 \\ x + 6y + 17z &= 9, \end{aligned}$$

(b)

$$\begin{aligned} 2x + 3y + 4z &= 1 \\ 5x + 6y + 7z &= 2 \\ 8x + 9y + 10z &= 4, \end{aligned}$$

(c)

$$\begin{aligned} x - y + 2z &= 5 \\ -x + 2y - z &= 0 \\ x + 3z &= 10. \end{aligned}$$

[J]

53. Solve the following sets of equations by row operations:

(a)

$$\begin{aligned} x + 5y - 2z + 4w &= 1 \\ 2x + 8y - 4z + 6w &= 2 \\ -3x + 7y + 6z + 10w &= -3. \end{aligned}$$

(b)

$$\begin{aligned} 2x - y + 5z + 6w &= -8 \\ 2x + y + 3z + 3w &= -1 \\ x - y + 3z + 4w &= -6. \end{aligned}$$

54. Consider the following set of equations for x, y and z:

$$\begin{aligned} x + y - z &= 1 \\ 2x + 3y + kz &= 3 \\ x + ky + 3z &= 2. \end{aligned}$$

Determine the values of k for which these equations have

(i) no solution

(ii) just one solution

(iii) infinitely many solutions.

55. Recall from lectures (2E) that if

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 & -1 & 2 \\ -1 & 1 & -1 & -1 & 1 \\ 1 & -3 & 1 & 0 & -1 \\ 1 & -4 & 1 & -2 & 2 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & -1 & -1 & 0 \\ \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

then the equation $\mathbf{Ax} = \mathbf{b}$ has a solution $\mathbf{x} = \mathbf{Cb}$ whenever $b_1 + b_2 + b_3 - b_4 = 0$. Find $\mathbf{AC}$ and $\mathbf{CA} - \mathbf{I}$, and explain (i) why the bottom row of $\mathbf{AC}$ is what it is (ii) why the columns of $\mathbf{CA} - \mathbf{I}$ are what they are.
[M]

56. Set

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}.$$

Find $\mathbf{A}^{-1}$, $(\mathbf{A}^T)^{-1}$. Which theorem from lectures could (or did) enable you to save time in this calculation?
[N]

57. Find $\mathbf{A}^{-1}$ where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ -2 & 3 & 8 \end{pmatrix},$$

and hence find the matrix $\mathbf{X}$ such that

$$\mathbf{AX} = \begin{pmatrix} 3 & 7 & -1 & 0 \\ 4 & 10 & -3 & 1 \\ 6 & 10 & 5 & 1 \end{pmatrix}.$$

[O]

58. Which of the following matrices have inverses, and what are they?

$$\begin{pmatrix} 2 & 4 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 5 \\ -1 & 10 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 0 \\ 1 & 3 & 2 \end{pmatrix}.$$

[P]

59. Which of the following lists of vectors are lineally dependent, and which are lineally independent? For the lineally dependent lists, give non-zero $x_1, \ldots, x_n$ such that $x_1\mathbf{a}^{(1)} + \ldots + x_n\mathbf{a}^{(n)} = \mathbf{0}$.

(a) $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 16 \end{pmatrix}$ (b) $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 9 \\ 27 \end{pmatrix}$ (c) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ (d) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

(e) $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$ (f) $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 16 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ 25 \end{pmatrix}$

[Q]

60. Find the determinants of the four matrices

$$\begin{pmatrix} 2 & 4 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 5 \\ -1 & 10 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 0 \\ 1 & 3 & 2 \end{pmatrix}.$$

(If you can, use the calculations from the last problem sheet.) [R]

61. Find the determinants of the matrices

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 9 & 16 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 4 & 9 & 16 & 25 \\ 8 & 27 & 64 & 125 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

[S]

62. Find the determinants of $\mathbf{A} - \lambda\mathbf{I}$ as quadratics or cubics in λ, where
(a) $\mathbf{A} = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$, (b) $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, (c) $\mathbf{A} = \begin{pmatrix} -1 & 6 & -12 \\ 0 & -13 & 30 \\ 0 & -9 & 20 \end{pmatrix}$, (d) $\mathbf{A} = \begin{pmatrix} 8 & 4 & 10 \\ -3 & 0 & -5 \\ -3 & -2 & -3 \end{pmatrix}$.
[T]

63. Compute $\mathbf{A}^2$, $\mathbf{B}^2$, $\mathbf{AB}$, $\mathbf{BA}$, $\mathbf{A}^2 - \mathbf{B}^2$, $(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B})$, $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B})$ when
(a) $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, (b) $\mathbf{A} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$.
[U]

64. Find the determinants of
(a) $\begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & k \\ 1 & k & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{pmatrix}$.
[V]

65. Find the eigenvalues and eigenvectors of the following matrices:
(a) $\begin{pmatrix} -8 & 5 & -4 \\ -22 & 13 & -10 \\ -8 & 4 & -3 \end{pmatrix}$ (b) $\begin{pmatrix} 7 & 8 & 8 \\ -9 & -9 & -7 \\ 4 & 4 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & -6 & 3 \\ 0 & 1 & 0 \\ -6 & 12 & -5 \end{pmatrix}$. [W]

66. Describe the sets which it is natural to take as the domains of the following real functions, and sketch the functions:
(a) $\ln(\cos x)$ (c) $\exp(\ln x)$ (e) $(\sqrt{x})^2$ (g) $x \ln x$
(b) $\sqrt{x^2}$ (d) $\ln(\exp x)$ (f) $x^3 - x$ (h) $(\sin x)/x$.
[X]

67. Find the eigenvalues and eigenvectors of the following matrices:
(a) $\begin{pmatrix} -1 & 6 & -12 \\ 0 & -13 & 30 \\ 0 & -9 & 20 \end{pmatrix}$ (b) $\begin{pmatrix} 8 & 4 & 10 \\ -3 & 0 & -5 \\ -3 & -2 & -3 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$. [Y]

68. Describe the sets which it is natural to take as the domains of the following real functions, and sketch the functions:
(a) $x/|x|$ (c) $\sqrt{\frac{1}{x} - 1}$ (e) $\frac{1}{\sqrt{1-x^2}}$
(b) $\ln(-x)$ (d) $\frac{1}{x}\ln x$ (f) $x^2 e^{-|x|}$
[Z]