Calculus and Linear Mathematics Answers

1.

(Diagram not available.)

2. (a) $\ln x$
(b) $-\frac{1}{2x^2\sqrt{\frac{1}{x}-1}}$ or $-\frac{1}{2x\sqrt{x-x^2}}$.
(c) $-xe^{-x^2/2}$.
(d) $\frac{1}{1+x^2}$.
(e) $\frac{1}{1-x^2}$.
(f) $7^x \ln 7$.
(g) $\frac{x}{|x|}$ (defined for $x \neq 0$).
(h) $\frac{1}{x}$.
(i) $\tan x$.

3. (a) $x = \frac{1}{e}$, $x \ln x = -\frac{1}{e}$.
$f'(x) = \ln x + x.\frac{1}{x} = \ln x + 1 = 0$ iff $\ln x = -1$, i.e., $x = \frac{1}{e}$.
(b) $x = e$, $\frac{\ln x}{x} = \frac{1}{e}$.
$f'(x) = (-\frac{1}{x^2}) \ln x + \frac{1}{x}.\frac{1}{x} = (1 - \ln x)/x^2 = 0$ iff $\ln x = 1$, i.e., $x = e$.
(c) None .
$f'(x) = -e^{-x}$ for $x > 0$, e^x for $x < 0$, neither of which is ever 0. At $x = 0$ the function has a corner, not a stationary point.
(d) $x = 0$, $x^2e^{-|x|} = 0$, $x = 2$, $x^2e^{-|x|} = \frac{4}{e^2}$, $x = -2$, $x^2e^{-|x|} = \frac{4}{e^2}$.
$f'(x) = 2xe^{-x} - x^2e^{-x} = (2 - x)xe^{-x}$ for $x > 0$, zero when $x = 2$; $f'(x) = 2xe^x + x^2e^x = (2 + x)xe^x$ for $x < 0$, zero when $x = -2$; at $x = 0$, the x^2 factor squeezes the function flat.
(e) $x = 1$, $|x|e^{-x} = \frac{1}{e}$.
$f'(x) = e^{-x} - xe^{-x} = (1 - x)e^{-x}$ when $x > 0$, zero when $x = 1$; $f'(x) = -e^{-x} + xe^{-x} = (x - 1)e^{-x}$ when $x < 0$, never 0; f turns a sharp corner when $x = 0$.

4. (a) $x = 2k\pi$, any $k \in \mathbb{Z}$, $\ln \cos x = 0$.
(b) None .
(c) $x = -\frac{1}{\sqrt{3}}$, $x^3 - x = \frac{2}{3\sqrt{3}}$.
$f'(x) = 3x^2 - 1 = 0$ iff $x = \pm\frac{1}{\sqrt{3}}$.
(d) $x = e$, $\frac{\ln x}{x} = \frac{1}{e}$.
(e) $x = 0$, $e^{-|x|} = 1$.
(f) $x = \pm 2$, $x^2e^{-|x|} = \frac{4}{e^2}$.

5. (a) $x = 2k\pi$, for any $k \in \mathbb{Z}$, $\ln\cos x = 0$.
(b) Any $x > 0$, $\frac{x}{|x|} = 1$.
(c) None .
(d) None .
The function approaches a value of 1 as x approaches 0, but does not actually achieve it.
(e) $x = e$, $\frac{\ln x}{x} = \frac{1}{e}$.
(f) $x = 1$, $xe^{-x} = \frac{1}{e}$.
$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x} = 0$ iff $x = 1$.

6. (a) $x = 0$, $\sqrt{x^2} = 0$.
(b) $x = \frac{1}{\sqrt{3}}$, $x^3 - x = \frac{2}{3\sqrt{3}}$.
(c) $x = \frac{1}{e}$, $x\ln x = -\frac{1}{e}$.
(d) $x = 1$, $\sqrt{\frac{1}{x} - 1} = 0$.
(e) $x = 0$, $\frac{1}{\sqrt{1-x^2}} = 1$.
$\frac{1}{\sqrt{1-x^2}}$ is at a minimum when $1 - x^2$ is at a maximum, that is, when x^2 is at a minimum.
(f) $x = 0$, $x^2e^{-|x|} = 0$.

7. (a) None .
The function is undefined for $x \leq 0$.
(b) $x = 0$, $\sqrt{x} = 0$.
(c) $x = \frac{1}{e}$, $x\ln x = -\frac{1}{e}$.
(d) $x = 0$, $|x|e^{-x} = 0$.

8. (a) $x = 0$, $x^3 - x = 0$.
$f''(x) = 6x = 0$ iff $x = 0$.
(b) None .
$f''(x) = \frac{1}{x}$ is never 0.
(c) $x = \frac{3}{4}$, $\sqrt{\frac{1}{x} - 1} = \frac{1}{\sqrt{3}}$.
$f'(x) = -\frac{1}{2x^2\sqrt{\frac{1}{x}-1}} = -\frac{1}{2x^2f(x)}$ so

$$\begin{aligned} f''(x) &= -\frac{1}{4x^4f(x)^2}\frac{d}{dx}(2x^2f(x)) \\ &= -\frac{1}{4x^4f(x)^2}(4xf(x) + 2x^2f'(x)) \\ &= -\frac{1}{2x^3f(x)^2}(2f(x) + xf'(x)) \\ &= -\frac{1}{2x^3f(x)^2}(2f(x) - \frac{1}{2xf(x)}) \\ &= -\frac{1}{4x^4f(x)^3}(4xf(x)^2 - 1) \end{aligned}$$

is zero iff $1 = 4xf(x)^2 = 4x(\frac{1}{x} - 1) = 4 - 4x$, that is, iff $x = 3/4$.
(d) $x = e^{3/2}$, $\frac{\ln x}{x} = \frac{3}{2}e^{-3/2}$.
$f''(x) = \frac{2\ln x - 3}{x^3} = 0$ iff $\ln x = \frac{3}{2}$, i.e., iff $x = e^{3/2}$.

9.

(Diagram not available.)

10. (a) $[0, \infty[$.

(b) The second derivative is $2 - \frac{2}{x\sqrt{x}}$, so the sole point of inflection is when $x = 1$, $f(x) = 3$, $f'(x) = 0$.

(c)

(Diagram not available.)

(d) Because $f(0) = 0$ and there are no other points of inflection, we must have $f'(x) > 0$ for $0 < x < 1$, so there are no stationary points there; and because $f(x) \to \infty$ as $x \to \infty$, and there are no points of inflection in $]1, \infty[$, we must have $f'(x) > 0$ for $x > 1$. Thus the only stationary point of f is at $x = 1$, $f(x) = 3$. Now this is neither a local maximum nor a local minimum. The only remaining point to examine is $x = 0$, which is a global (and, of course, local) minimum.

11.

(Diagram not available.)

12.

(Diagram not available.)

13. (a) $\tanh 0 = 0{\cdot}00$, $\tanh 1 \simeq 0{\cdot}76$, $\tanh 2 \simeq 0{\cdot}96$, $\tanh 3 \simeq 1{\cdot}00$, $\tanh' 0 = 1{\cdot}00$.

(b)

(Diagram not available.)

(c)(i)

$$\begin{aligned}
&\frac{z - \frac{1}{z}}{z + \frac{1}{z}} = y \\
&\iff \frac{z^2 - 1}{z^2 + 1} = y \\
&\iff z^2 - 1 = y(z^2 + 1) \\
&\iff z^2 - z^2 y = y + 1 \\
&\iff z^2 = \frac{1 + y}{1 - y} \\
&\iff z = \pm\sqrt{\frac{1 + y}{1 - y}}.
\end{aligned}$$

(ii) $e^x = z$ iff $x = \ln z$.

(iii) So $\tanh x = y$ iff $x = \ln\sqrt{\frac{1+y}{1-y}} = \frac{1}{2}(\ln(1 + y) - \ln(1 - y))$.

(The point is that the negative solution for z must be rejected, as $e^x = z$ can have a solution only if $z > 0$.)

(d) $(\tanh^{-1})'(x) = \frac{d}{dx}\frac{1}{2}(\ln(1 + x) - \ln(1 - x)) = \frac{1}{2}(\frac{1}{1+x} + \frac{1}{1-x}) = \frac{1}{1-x^2}$ for $|x| < 1$.

14. (a) $1{\cdot}03^2 = 1{\cdot}0609$, answer $6{\cdot}09\%$.

(b) $1{\cdot}005^{12} \simeq 1{\cdot}061678$, answer $6{\cdot}17\%$.

15. Taking $r = \ln(1+i)$, and working in millions of pounds, the present value of the investment is

$$\int_0^2 \frac{1}{2} e^{-rt} dt = \frac{1}{2r}[1 - e^{-2r}],$$

while the present value of the expected yield is

$$\int_2^{22} \frac{1}{10} e^{-rt} dt = \frac{1}{10r}[e^{-2r} - e^{-22r}].$$

(a) If $i = 0{\cdot}04$, $r \simeq 0{\cdot}039221$ then the present value of the project is

$$P \simeq -0{\cdot}962 + 1{\cdot}281 \simeq 0{\cdot}320,$$

and the required answer is £320000.

(b) If $i = 0{\cdot}06$, $r \simeq 0{\cdot}058269$ then the present value of the project is

$$P \simeq -0{\cdot}944 + 1{\cdot}051 \simeq 0{\cdot}107,$$

and the required answer is £110000.

16. The present value of the final payment is (in millions) $\frac{1}{2}/(1+i)^{22}$, so the net value becomes
(a) $P \simeq 0{\cdot}320 - 0{\cdot}211 \simeq 0{\cdot}109$, answer £110000,
(b) $P \simeq 0{\cdot}10722 - 0{\cdot}13875 \simeq -0{\cdot}03154$, answer -£32000.

17. (a) $S = \frac{a - ax^{n+1}}{1-x}$ or $S = \frac{ax^{n+1} - a}{1-x}$ or $S = a\frac{1-x^{n+1}}{1-x}$ or $S = a\frac{x^{n+1}-1}{x-1}$, provided $x \neq 1$.
If $x = 1$ then $S = (n+1)a$.

(b) $P = \sum_{k=1}^{20} 1000/1{\cdot}05^k = \frac{1}{1-\frac{1}{1\cdot 05}}(\frac{1000}{1{\cdot}05} - \frac{1000}{1{\cdot}05^{21}}) \simeq 12462{\cdot}21$, so the required answer is £12500.

(c) £20000.

18. (a)

(Diagram not available.)

(b) $\arcsin'(y) = \frac{1}{\sin'(\arcsin y)} = \frac{1}{\cos(\arcsin y)}$. Now if $x = \arcsin y$ then $y = \sin x$ so $\cos x = \pm\sqrt{1-y^2}$; thus $\arcsin'(y) = \pm\frac{1}{\sqrt{1-y^2}}$. But from the graph it is evident that $\arcsin'(y) > 0$, so $\arcsin'(y) = \frac{1}{\sqrt{1-y^2}}$ for $-1 < y < 1$.

19. (a) $\frac{\partial f}{\partial x} = 2xy^3$, $\frac{\partial f}{\partial y} = 3x^2y^2$

(b) $\frac{\partial f}{\partial x} = \frac{1}{y}$, $\frac{\partial f}{\partial y} = -\frac{x}{y^2}$

(c) $\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}}$, $\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}$, $\frac{\partial f}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$

(d) $\frac{\partial f}{\partial x} = -\frac{x}{(x^2+y^2+z^2)^{3/2}}$, $\frac{\partial f}{\partial y} = -\frac{y}{(x^2+y^2+z^2)^{3/2}}$, $\frac{\partial f}{\partial z} = -\frac{z}{(x^2+y^2+z^2)^{3/2}}$

(e) $\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2}$, $\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2}$

(f) $\frac{\partial f}{\partial x} = y + 2 - \frac{2}{x}$, $\frac{\partial f}{\partial y} = x - \frac{1}{y}$.

20. (a) $\frac{\partial f}{\partial x} = 2x + 2y^2$, $\frac{\partial f}{\partial y} = 4xy$,
$\frac{\partial^2 f}{\partial x^2} = 2$, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 4y$, $\frac{\partial^2 f}{\partial y^2} = 4x$

(b) $\frac{\partial f}{\partial x} = \frac{y^2}{x+y}$, $\frac{\partial f}{\partial y} = 2y \ln(x+y) + \frac{y^2}{x+y}$,
$\frac{\partial^2 f}{\partial x^2} = -\frac{y^2}{(x+y)^2}$,
$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{y(2x+y)}{(x+y)^2}$,
$\frac{\partial^2 f}{\partial y^2} = 2\ln(x+y) + \frac{2y}{x+y} + \frac{y(2x+y)}{(x+y)^2} = 2\ln(x+y) + \frac{y(4x+3y)}{(x+y)^2}$

(c) $\frac{\partial f}{\partial x} = 6x - 2y + 2$, $\frac{\partial f}{\partial y} = -2x + 6y - 6$,
$\frac{\partial^2 f}{\partial x^2} = 6$, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -2$, $\frac{\partial^2 f}{\partial y^2} = 6$

(d) $\frac{\partial f}{\partial x} = 3x^2 - 2y$, $\frac{\partial f}{\partial y} = -3y^2 - 2x$,
$\frac{\partial^2 f}{\partial x^2} = 6x$, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -2$, $\frac{\partial^2 f}{\partial y^2} = -6y$.

21.
(a) $\frac{\partial f}{\partial x} = 6x - 2y + 2$, $\frac{\partial f}{\partial y} = -2x + 6y - 6$ so the stationary point is when $3x - y + 1 = -x + 3y - 3 = 0$, i.e., $x^* = 0$, $y^* = 1$. Now

$$\begin{aligned} g(\xi, \eta) &= 3\xi^2 - 2\xi(\eta + 1) + 3(\eta + 1)^2 + 2\xi - 6(\eta + 1) \\ &= 3\xi^2 - 2\xi\eta - 2\xi + 3\eta^2 + 6\eta + 3 + 2\xi - 6\eta - 6 \\ &= 3\xi^2 - 2\xi\eta + 3\eta^2 - 3 \\ &= 3(\xi - \tfrac{1}{3}\eta)^2 + \frac{8}{3}\eta^2 - 3 \end{aligned}$$

is never less than -3, so the stationary point is a local (and global) minimum.

(b) $\frac{\partial f}{\partial x} = 4x - 2$, $\frac{\partial f}{\partial y} = -6y$, so the stationary point is when $4x - 2 = -6y = 0$, i.e. $x^* = \frac{1}{2}$, $y^* = 0$. Now

$$\begin{aligned} g(\xi + x^*, \eta + y^*) &= 2(\xi + \frac{1}{2})^2 - 3\eta^2 - 2(\xi + \frac{1}{2}) \\ &= 2\xi^2 + \frac{1}{2} - 3\eta^2 \end{aligned}$$

takes values both above and below $\frac{1}{2}$ for ξ, η near 0, so the stationary point is a saddle point.

(c) $\frac{\partial f}{\partial x} = 2y - 10x$, $\frac{\partial f}{\partial y} = 2x - 20y$ so the saddle point is when $2y - 10x = 2x - 20y = 0$, i.e., $x^* = y^* = 0$. Now

$$\begin{aligned} g(\xi, \eta) &= -5\xi^2 + 2\xi\eta - 10\eta^2 \\ &= -5(\xi - \frac{1}{5}\eta)^2 - \frac{49}{5}\eta^2 \end{aligned}$$

is never greater than 0, so the saddle point is a local maximum.

22. (a) $\dfrac{\partial f}{\partial x} = \frac{2x}{x^2+y^2}$,

$$\frac{\partial^2 f}{\partial x^2} = \frac{2(x^2+y^2)-4x^2}{(x^2+y^2)^2} = 2\frac{y^2-x^2}{(x^2+y^2)^2},$$

so $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$.

(b) $\dfrac{\partial f}{\partial x} = -\frac{x}{(x^2+y^2+z^2)^{3/2}}$,

$$\frac{\partial^2 f}{\partial x^2} = -\frac{(x^2+y^2+z^2)^{3/2}-3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} = -\frac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^{5/2}} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}},$$

so $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 0$.

23. (i) $\frac{\partial f}{\partial x} = y + 2 - \frac{2}{x}$, $\frac{\partial f}{\partial y} = x - \frac{1}{y}$ so the stationary point is when $xy + 2x - 2 = xy - 1 = 0$, i.e., $x^* = \frac{1}{2}$, $y^* = 2$. Now

$$\begin{aligned} g(\xi,\eta) &= (\xi + \frac{1}{2})(\eta + 2) + 2(\xi + \frac{1}{2}) - 2\ln(\frac{1}{2} + \xi) - \ln(2 + \eta) \\ &= \xi\eta + \frac{1}{2}\eta + 4\xi + 2 - 2\ln(1 + 2\xi) + 2\ln 2 - \ln 2 - \ln(1 + \frac{1}{2}\eta) \\ &\simeq \xi\eta + \frac{1}{2}\eta + 4\xi + 2 + \ln 2 - 2(2\xi - \frac{4\xi^2}{2}) - \frac{1}{2}\eta + \frac{1}{2}(\frac{1}{2}\eta)^2 \\ &= \xi\eta + 2 + \ln 2 + 4\xi^2 + \frac{1}{8}\eta^2 \\ &= \frac{1}{8}(\eta + 4\xi)^2 + 2\xi^2 + 2 + \ln 2 \end{aligned}$$

which is always greater than $2 + \ln 2$ if $(\xi, \eta) \neq (0, 0)$, so that $(\frac{1}{2}, 2)$ is a local minimum.

24. The first point which struck me was the phrase 'the rate of inflation in Serbia rose ...'. As usual, this means that the rate of inflation rose *to* the figure given, not *by* the figure given.

A monthly rate of 20190%, that is, a multiplier of 202·9, corresponds to a daily multiplier (compounded) of $202{\cdot}9^{1/30}$ (November having 30 days) and an hourly multiplier of $(202{\cdot}9^{1/30})^{1/24} \simeq 1{\cdot}0074$, that is, an hourly inflation rate of about 0·74%, so the figure of 0·7% is consistent with the figure of 20190%.

On the other hand, a monthly multiplier of 202·9 would seem to correspond to an annual multiplier of $202{\cdot}9^{12} \simeq 4.9 \times 10^{27}$, that is, an annual inflation rate of 4.9×10^{29} per cent. So the rate of $2{\cdot}9 \times 10^{11}$ per cent given is not consistent with the other figures.

The twelve-significant-figure precision of the annual rate given suggests that it is the result of using a calculator with a twelve-digit display, rather than any attempt to represent reality.

25. (a) Saddle point. (Setting $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, we have $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 4 & 6 \end{pmatrix}$. Now $a_{11} = 2 > 0$ and $\det \mathbf{A} = -4$ so $\mathbf{0}$ is a saddle point.)

(b) Local maximum. (Here $\mathbf{A} = \begin{pmatrix} -2 & 4 \\ 4 & -10 \end{pmatrix}$ so $-\mathbf{A} = \begin{pmatrix} 2 & -4 \\ -4 & 10 \end{pmatrix}$ has $a_{11} = 2 > 0$, $\det(-\mathbf{A}) = 4 > 0$ and is positive definite, that is, $\mathbf{A}$ is negative definite and $\mathbf{0}$ is a local maximum.)

(c) Local minimum. (Here $\mathbf{A} = \begin{pmatrix} 4 & -4 \\ -4 & 6 \end{pmatrix}$ has $a_{11} = 4 > 0$, $\det \mathbf{A} = 8 > 0$ so $\mathbf{A}$ is positive definite and $\mathbf{0}$ is a local minimum.)

(d) Local minimum. (Here $\mathbf{A} = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 10 & 0 \\ -2 & 0 & 14 \end{pmatrix}$ has

$$a_{11} = 2 > 0, \qquad \det \begin{pmatrix} 2 & 4 \\ 4 & 10 \end{pmatrix} = 4 > 0, \qquad \det \mathbf{A} = 16 > 0$$

so is positive definite, and $\mathbf{0}$ is a local minimum.)

26. (a) $\dfrac{\partial f}{\partial x} = 2xy + y^2 - 2x - 3y + 2$, $\qquad \dfrac{\partial f}{\partial y} = x^2 + 2xy - 2y - 3x + 2$,

$\dfrac{\partial^2 f}{\partial x^2} = 2y - 2$, $\qquad \dfrac{\partial^2 f}{\partial x \partial y} = 2x + 2y - 3$, $\qquad \dfrac{\partial^2 f}{\partial y^2} = 2x - 2$.

(b) $\dfrac{\partial f}{\partial x} = (y-1)(2x+y-2)$, $\dfrac{\partial f}{\partial y} = (x-1)(2y+x-2)$ so the four stationary points are

$(1,1)$ (when $y - 1 = x - 1 = 0$)
$(0,1)$ (when $y - 1 = 2y + x - 2 = 0$)
$(1,0)$ (when $2x + y - 2 = x - 1 = 0$)
$(\frac{2}{3}, \frac{2}{3})$ (when $2x + y - 2 = 2y + x - 2 = 0$).

(c) (at $(1,1)$) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(at $(0,1)$) $\begin{pmatrix} 0 & -1 \\ -1 & -2 \end{pmatrix}$

(at $(1,0)$) $\begin{pmatrix} -2 & -1 \\ -1 & 0 \end{pmatrix}$

(at $(\frac{2}{3}, \frac{2}{3})$) $\begin{pmatrix} -\frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} \end{pmatrix}$.

(d) $(1,1)$, $(0,1)$ and $(1,0)$ are all saddle points; $(\frac{2}{3}, \frac{2}{3})$ is a local maximum.

(e)

The point is that $f(x,y) = (x-1)(y-1)(x+y-1)$.

27. (a) Neither. ($f(x,y) = x^3 + y^3$ takes both positive and negative values near $\mathbf{0}$.)

(b) Local maximum. ($\cos x \le 1$ and $y^4 \ge 0$, so $\cos x - y^4 \le 1$ for all x, y.)

(c) Neither. ($e^{-x^2/2} + y^4$ takes values greater than 1 for $x = 0$, less than 1 for $y = 0$.)

(d) Local minimum. ($f \ge 0$ everywhere.)

28. (a) $x = 1$, $y = 2$.
(b) There are no numbers x, y satisfying these two equations simultaneously.
(c) There are infinitely many solutions . Possible descriptions of the set of solutions are
$y = \frac{5}{2}x - 2$ for any x
or $x = \frac{2}{5}y + \frac{2}{5}$ for any y
or $x = 2\alpha$, $y = 5\alpha - 2$ for any α
or 'the whole line $5x - 2y = 2$'
or a sketch of the line
or as approved by the class teacher.

29. (a) $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 6\lambda + 8 = (\lambda - 4)(\lambda - 2)$, so the eigenvalues are 2, 4.
(For $\lambda = 2$) $\mathbf{A} - 2\mathbf{I} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ so the corresponding eigenvectors are $\alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for $\alpha \ne 0$.
(For $\lambda = 4$) $\mathbf{A} - 4\lambda\mathbf{I} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ so the corresponding eigenvectors are $\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $\alpha \ne 0$.

(b) $\mathbf{P}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, $\mathbf{PD} = \begin{pmatrix} 2 & 4 \\ -2 & 4 \end{pmatrix}$, $\mathbf{AP} = \begin{pmatrix} 2 & 4 \\ -2 & 4 \end{pmatrix}$, $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$.

(c)

$$\begin{aligned} f(\mathbf{x}) &= \frac{1}{2}(\mathbf{Pz})^T\mathbf{APz} \\ &= \frac{1}{2}\mathbf{z}^T\mathbf{P}^T\mathbf{APz} \\ &= \mathbf{z}^T\mathbf{P}^{-1}\mathbf{APz} \\ &= \mathbf{z}^T\mathbf{Dz} \\ &= 2z_1^2 + 4z_2^2. \end{aligned}$$

(d) It follows at once that $f(\mathbf{x}) > 0$ whenever $\mathbf{z} \ne \mathbf{0}$, that is, whenever $\mathbf{x} \ne \mathbf{0}$; which is just what is meant by saying that f is positive definite.

30. Setting $E = e^{-\frac{1}{2}(x^2+y^2+z^2)}$, $S = x + 2y + 2z$ we have
$\frac{\partial f}{\partial x} = S\frac{\partial E}{\partial x} + E\frac{\partial S}{\partial x} = -xSE + E = (1 - xS)E$,
$\frac{\partial f}{\partial y} = S\frac{\partial E}{\partial y} + E\frac{\partial S}{\partial y} = -ySE + 2E = (2 - yS)E$,
$\frac{\partial f}{\partial z} = S\frac{\partial E}{\partial z} + E\frac{\partial S}{\partial z} = -zSE + 2E = (2 - zS)E$,
so the stationary points are when $xS = 1$, $yS = zS = 2$. Setting $\alpha = 1/S$ we must have $x = \alpha$, $y = z = 2\alpha$ so $S = 9\alpha$ and $1 = xS = 9\alpha^2$. Accordingly $\alpha = \pm\frac{1}{3}$ and the two stationary points are $\mathbf{x}^* = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ and $-\mathbf{x}^*$.

Now $\mathbf{x}^*$ is a (global) maximum and $-\mathbf{x}^*$ is a (global) minimum. To see this, observe that $f = SE \doteq 0$ far from $\mathbf{0}$. (Setting $r = \sqrt{x^2 + y^2 + z^2}$, the distance from $\mathbf{x}$ to $\mathbf{0}$, we have $|S| \le 5r$ so $|f| \le 5re^{-r^2/2}$, which is practically 0 if r is large.) Since f definitely takes positive values (for instance, $f(\mathbf{x}^*) > 0$), it must have a highest point. This must be a stationary point, and the only possible candidate is $\mathbf{x}^*$. Accordingly $\mathbf{x}^*$ is a global maximum. Similarly, $-\mathbf{x}^*$ is a global minuimum.

31. Set $r = \ln 1{\cdot}1 \simeq 0{\cdot}0953102$. Working in millions, the present value of the investment is

$$\int_0^5 2e^{-rt}dt = [-\frac{2}{r}e^{-rt}]_0^5 = \frac{2}{r}(1 - e^{-5r}) \simeq 7.955,$$

while the present value of the return is

$$\int_5^{15} 2e^{-rt}dt = [-\frac{2}{r}e^{-rt}]_5^{15} = \frac{2}{r}(e^{-5r} - e^{-15r}) \simeq 8.006,$$

so the net present value is approximately

$$£8{,}006{,}000\text{-}£7{,}955{,}000 = £51{,}000.$$

32. According to the *Guardian*, the figure of £4.5 billion is an estimate of the total annual cost to the banks of shifting cash – not just the filling of dispensing machines.

33. (i) $2x - 4\ln(|2+x|)$. $(\frac{2x}{2+x} = 2 - \frac{4}{2+x}.)$

(ii) $-\frac{1}{2}e^{-2x}$.

(iii) $-\frac{1}{7}\cos 7x$.

(iv) $-\ln(|2-x|)$.

(v) $\frac{1}{a}\arctan\frac{x}{a}$.

34. (i) $\ln(|\sin x|)$. $(\cot x = \frac{\cos x}{\sin x} = \frac{\frac{d}{dx}\sin x}{\sin x}.)$

(ii) $\frac{1}{2}\ln(x^2+2x+2)$. $(x^2+2x+2 = (x+1)^2+1$ is always positive, so you don't need to write $\ln(|x^2+2x+2|)$, though this is of course correct.)

(iii) $\ln(1+e^x)$.

(iv) $\ln(|\ln x|)$. ($\ln|\ln|x||$ acceptable but not required, because the question had $\ln x$, so gives you the right to suppose that $x > 0$.)

35. (i) $e^{-1/x}$.

(ii) $\sin(e^x)$.

(iii) $\frac{2}{3}(x^2+3)^{3/2}$.

(iv) $-2\sqrt{1-x} + 4\ln(2+\sqrt{1-x})$. (Substitute $u = \sqrt{1-x}$ to turn the problem into $\int \frac{1}{2+u}(-2u)du$.)

(v) $2\sqrt{x+2} - 4\arctan(\frac{1}{2}\sqrt{x+2})$. (Substitute $u = \sqrt{x+2}$ to turn the problem into $\int \frac{2u^2}{u^2+4}du$.)

(vi) $-\frac{1}{2}\cos x^2$.

36. (i) $x \sin x + \cos x$.

$$\begin{aligned}(\int x \cos x dx &=_{\text{utc}} x \sin x - \int 1 \cdot \sin x dx \\ &=_{\text{utc}} x \sin x + \cos x.)\end{aligned}$$

(ii) $\frac{1}{2} e^x (\sin x - \cos x)$.

$$\begin{aligned}(\int e^x \sin x\, dx &=_{\text{utc}} e^x \sin x - \int e^x \cos x\, dx \\ &=_{\text{utc}} e^x \sin x - (e^x \cos x - \int e^x(-\sin x) dx) \\ &= e^x(\sin x - \cos x) - \int e^x \sin x\, dx.)\end{aligned}$$

(iii) $2 \cos x + 2x \sin x - x^2 \cos x$.

$$\begin{aligned}(\int x^2 \sin x\, dx &=_{\text{utc}} x^2(-\cos x) - \int 2x(-\cos x) dx \\ &= -x^2 \cos x + 2 \int x \cos x\, dx \\ &= -x^2 \cos x + 2(x \sin x - \int \sin x\, dx) \\ &=_{\text{utc}} -x^2 \cos x + 2x \sin x + 2 \cos x.)\end{aligned}$$

37. $F_n(x) = [t^n(-e^{-t})]_0^x - \int_0^1 nt^{n-1}(-e^{-t})dt = -x^n e^{-x} + nF_{n-1}(x)$.

$F_0(1) = \int_0^1 e{-}tdt = [-e^{-t}]_0^1 = 1 - \frac{1}{e}$,

$F_1(1) = (1 - \frac{1}{e}) - \frac{1}{e} = 1 - \frac{2}{e}$,

$F_2(1) = 2(1 - \frac{2}{e}) - \frac{1}{e} = 2 - \frac{5}{e}$,

$F_3(1) = 3(2 - \frac{5}{e}) - \frac{1}{e} = 6 - \frac{16}{e}$,

$F_4(1) = 4(6 - \frac{16}{e}) - \frac{1}{e} = 24 - \frac{65}{e}$.

38. (a) $\frac{1-x^2}{(1+x^2)^2}$.

(b)(i) $\arctan x$ (ii) $\ln(1 + x^2)$ (iii) $-\frac{1}{1+x^2}$.

(c) $\frac{1}{(1+x^2)^2} = \frac{1}{2}(\frac{1}{1+x^2} + \frac{1-x^2}{(1+x^2)^2})$, so

$$\int \frac{1}{(1+x^2)^2} dx =_{\text{utc}} \frac{1}{2}(\arctan x + \frac{x}{1+x^2}).$$

39. $\begin{pmatrix} -15 & -28 & 32 \\ 8 & 13 & -16 \\ 4 & 12 & -11 \end{pmatrix}$. $(\mathbf{A}^2 = \begin{pmatrix} 6 & 8 & -11 \\ -3 & -3 & 5 \\ -2 & -5 & 5 \end{pmatrix}$.)

40. (i) $x = 7$, $y = -3$, $z = 2$.

(ii) No solution.

(iii) $x = -\alpha + 2\beta$, $y = 2\alpha - 2\beta + 1$, $z = \alpha$, $w = \beta$ for any α, β.

41. $\begin{pmatrix} 15 & -\frac{4}{3} & -\frac{10}{3} \\ -7 & \frac{2}{3} & \frac{5}{3} \\ -13 & 1 & 3 \end{pmatrix}$.

42. 0.

43. Eigenvalues are 1, 9; eigenvectors for eigenvalue 1 are $\alpha \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ for $\alpha \neq 0$, eigenvectors for eigenvalue 9 are $\alpha \begin{pmatrix} 8 \\ 8 \\ 3 \end{pmatrix}$.

44. (i) Domain $\mathbb{R} \setminus \{-1, 2\}$, local maximum at $x = \frac{1}{2}$, $f(x) = -\frac{4}{9}$.

(ii) Domain $\mathbb{R}$, local (and global) maximum at $x = \sqrt{2}$, $f(x) = \frac{1}{2\sqrt{2}}$, local (and global) minimum at $x = -\sqrt{2}$, $f(x) = -\frac{1}{2\sqrt{2}}$.

(iii) Domain $\mathbb{R}$, no local maxima or minima.

(iv) Domain $\left[-\frac{1}{2}, \infty\right[$, local (and global) minimum at $x = -\frac{1}{2}$, $f(x) = 0$.

(v) Domain *either* $]0, \infty[$ *or* $[0, \infty[$, if you interpret 0^0 as 1.
In either case there is a local (and global) minimum at $x = \frac{1}{e} \simeq 0{\cdot}37$, $f(x) = e^{-1/e} \simeq 0{\cdot}69$; in the latter case there is a local maximum at $x = 0$, $f(x) = 1$.
(The point is that, for $x > 0$, $x^x = e^{x \ln x}$ has its minimum when $x \ln x$ is at a minimum.)

45. $x = y = z = 1$.

46. $e < \sqrt{8} < 3 < \pi = 2(\arccos\frac{8}{17} + \arccos\frac{15}{17}) < \frac{355}{113} < 3\frac{1}{7} < \sqrt{10}$.

47. (a) $x^2 + 5x + 6 = (x + 2)(x + 3)$
(b) $x^2 - 5x + 6 = (x - 2)(x - 3)$
(c) $6x^2 + 5x + 1 = (2x + 1)(3x + 1)$
(d) $2x^2 - 3x - 2 = (2x + 1)(x - 2)$
(e) $x^2 + 5xy + 6y^2 = (x + 2y)(x + 3y)$
(f) $-2x^2 + 3x + 2 = -(2x + 1)(x - 2)$ (*or* $(2 - x)(2x + 1)$, etc.)
(g) $x^3 + 6x^2 + 11x + 6 = (x + 1)(x + 2)(x + 3)$
(h) $6x^3 + 11x^2 + 6x + 1 = (x + 1)(2x + 1)(3x + 1)$
(i) $x^4 + 4x^3 + 6x^2 + 4x + 1 = (x + 1)^4$
(j) $x^4 - 4x^3 + 6x^2 - 4x + 1 = (x - 1)^4$
(k) $x^2 - 3 = (x + \sqrt{3})(x - \sqrt{3})$
(l) $x^2 + x - 1 = (x + \frac{1}{2})^2 - \frac{5}{4} = (x + \frac{1}{2} + \frac{\sqrt{5}}{2})(x + \frac{1}{2} - \frac{\sqrt{5}}{2})$.

48.

$$\mathbf{AB} = \begin{pmatrix} -2 & 4 & -5 \\ 7 & -1 & 7 \\ 9 & 4 & 5 \end{pmatrix}, \quad \mathbf{BA} = \begin{pmatrix} 3 & 13 & 2 \\ -1 & -2 & 2 \\ 3 & 13 & 1 \end{pmatrix},$$

$\mathbf{AC}$ is undefined, because $\mathbf{A}$ has three columns and $\mathbf{C}$ has only two rows,

$$\mathbf{CA} = \begin{pmatrix} 0 & 3 & 5 \\ 1 & 2 & -2 \end{pmatrix}, \quad \mathbf{CD} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{DC} = \begin{pmatrix} -10 & -15 & -5 \\ 6 & 9 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

49.

$$\mathbf{AB} = \begin{pmatrix} 141 & 216 & 78 & 462 \\ 170 & 220 & 60 & 540 \end{pmatrix}$$

represents the raw materials required for the clients' orders; the (i,k)th coefficient being the quantity of material R_k required to fill the order from client C_i.

$$\mathbf{Bc} = \begin{pmatrix} 330 \\ 560 \\ 535 \end{pmatrix}$$

represents the cost of raw materials used in each product; the jth element being the cost of materials for unit quantity of product P_j.

$$\mathbf{ABc} = \begin{pmatrix} 12900 \\ 14500 \end{pmatrix}$$

represents the costs of raw materials for the two clients' orders; the ith element being the cost of materials used to fill the order from client F_i.

$$(1 \quad 1)\,\mathbf{A} = (19 \quad 32 \quad 6)$$

represents the total orders received; the jth element being the total order for product P_j.

50.

$$\mathbf{AB} = \begin{pmatrix} -1 & 5 & 8 \\ -1 & 2 & 1 \\ 2 & 5 & -6 \end{pmatrix}, \quad \mathbf{BA} = \begin{pmatrix} -1 & -1 & 2 \\ 5 & 2 & 5 \\ 8 & 1 & -6 \end{pmatrix}.$$

Using 1G, and the fact that $\mathbf{A}$ and $\mathbf{B}$ are both 'symmetric' (that is, equal to their transposes) we get

$$\mathbf{BA} = \mathbf{B}^T\mathbf{A}^T = (\mathbf{AB})^T,$$

so we can write down one of $\mathbf{AB}$, $\mathbf{BA}$ as soon as we have found the other.

51.

$$\mathbf{B} = \begin{pmatrix} 3 & 2 & 2 \\ -4 & 5 & 8 \\ -2 & -2 & 3 \end{pmatrix}, \quad \mathbf{AB} = \begin{pmatrix} -2 & 7 & 5 \\ -11 & 6 & 27 \\ -8 & -6 & -1 \end{pmatrix}, \quad \mathbf{BA} = \begin{pmatrix} -2 & 7 & 5 \\ -11 & 6 & 27 \\ -8 & -6 & -1 \end{pmatrix}.$$

Use 1F ('associative law of multiplication') to see that

$$\mathbf{AB} = \mathbf{A}(\mathbf{AA}) = (\mathbf{AA})\mathbf{A} = \mathbf{BA}.$$

52. (a)

$$\begin{pmatrix} 1 & 4 & 11 & | & 7 \\ 1 & 4 & 8 & | & 4 \\ 1 & 6 & 17 & | & 9 \end{pmatrix}$$

$$\begin{array}{l} \rho_2' = \rho_2 - \rho_1 \\ \rho_3' = \rho_3 - \rho_1 \end{array} \begin{pmatrix} 1 & 4 & 11 & | & 7 \\ 0 & 0 & -3 & | & -3 \\ 0 & 2 & 6 & | & 2 \end{pmatrix}$$

$$\begin{array}{l} \rho_2' = \rho_3 \\ \rho_3' = \rho_2 \end{array} \begin{pmatrix} 1 & 4 & 11 & | & 7 \\ 0 & 2 & 6 & | & 2 \\ 0 & 0 & -3 & | & -3 \end{pmatrix}$$

$$\rho_2' = \tfrac{1}{2}\rho_2 \begin{pmatrix} 1 & 4 & 11 & | & 7 \\ 0 & 1 & 3 & | & 1 \\ 0 & 0 & -3 & | & -3 \end{pmatrix}$$

$$\rho_3' = -\tfrac{1}{3}\rho_3 \begin{pmatrix} 1 & 4 & 11 & | & 7 \\ 0 & 1 & 3 & | & 1 \\ 0 & 0 & 1 & | & 1 \end{pmatrix}$$

yields the equations

$$z = 1,$$

$$y + 3z = 1 \text{ so } y = -2,$$

$$x + 4y + 11z = 7 \text{ so } x = 7 - 4(-2) - 11 = 4;$$

final answer $x = 4$, $y = -2$, $z = 1$.

(b)

$$\begin{pmatrix} 2 & 3 & 4 & | & 1 \\ 5 & 6 & 7 & | & 2 \\ 8 & 9 & 10 & | & 4 \end{pmatrix}$$

$$\rho_1' = \tfrac{1}{2}\rho_1 \begin{pmatrix} 1 & \frac{3}{2} & 2 & | & \frac{1}{2} \\ 5 & 6 & 7 & | & 2 \\ 8 & 9 & 10 & | & 4 \end{pmatrix}$$

$$\begin{matrix} \\ \rho_2' = \rho_2 - 5\rho_1 \\ \rho_3' = \rho_3 - 8\rho_1 \end{matrix} \begin{pmatrix} 1 & \frac{3}{2} & 1 & | & \frac{1}{2} \\ 0 & -\frac{3}{2} & -3 & | & -\frac{1}{2} \\ 0 & -3 & -6 & | & 0 \end{pmatrix}$$

$$\rho_2' = -\tfrac{2}{3}\rho_2 \begin{pmatrix} 1 & \frac{3}{2} & 1 & | & \frac{1}{2} \\ 0 & 1 & 2 & | & \frac{1}{3} \\ 0 & -3 & -6 & | & 0 \end{pmatrix}$$

$$\begin{matrix} \\ \\ \rho_3' = \rho_3 + 3\rho_2 \end{matrix} \begin{pmatrix} 1 & \frac{3}{2} & 1 & | & \frac{1}{2} \\ 0 & 1 & -\frac{4}{3} & | & \frac{1}{3} \\ 0 & 0 & 0 & | & 1 \end{pmatrix}$$

yielding the equation

$$0x + 0y + 0z = 1,$$

which is insoluble; so there were **no** solutions to the original equations.

(c)

$$\begin{pmatrix} 1 & -1 & 2 & | & 5 \\ -1 & 2 & -1 & | & 0 \\ 1 & 0 & 3 & | & 10 \end{pmatrix}$$

$$\begin{matrix} \\ \rho_2' = \rho_2 + \rho_1 \\ \rho_3' = \rho_3 - \rho_1 \end{matrix} \begin{pmatrix} 1 & -1 & 2 & | & 5 \\ 0 & 1 & 1 & | & 5 \\ 0 & 1 & 1 & | & 5 \end{pmatrix}$$

$$\begin{matrix} \\ \\ \rho_3' = \rho_3 - \rho_2 \end{matrix} \begin{pmatrix} 1 & -1 & 2 & | & 5 \\ 0 & 1 & 1 & | & 5 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 2 & | & 5 \\ 0 & 1 & 1 & | & 5 \end{pmatrix}$$

yielding the equations

$$y + z = 5 \text{ so } y = 5 - z \text{ (}z\text{ being arbitrary)}$$

$$x - y + 2z = 5 \text{ so } x = 5 - 2z + y = 5 - 2z + (5 - z) = 10 - 3z,$$

and the general solution is

$$z \text{ arbitrary, } x = 10 - 3z,\ y = 5 - z.$$

Alternative solutions:

$$y \text{ arbitrary, } z = 5 - y,\ x = 3y - 5;$$

$$x \text{ arbitrary, } y = \tfrac{5}{3} + \tfrac{5}{3}x,\ z = \tfrac{10}{3} - \tfrac{10}{3}x.$$

53. (a)

$$\begin{pmatrix} 1 & 5 & -2 & 4 & | & 1 \\ 2 & 8 & -4 & 6 & | & 2 \\ -3 & 7 & 6 & 10 & | & -3 \end{pmatrix}$$

$$\begin{array}{l} \\ \rho_2' = \rho_2 - 2\rho_1 \\ \rho_3' = \rho_3 + 3\rho_1 \end{array} \begin{pmatrix} 1 & 5 & -2 & 4 & | & 1 \\ 0 & -2 & 0 & -2 & | & 0 \\ 0 & 22 & 0 & 22 & | & 0 \end{pmatrix}$$

$$\rho_2' = -\tfrac{1}{2}\rho_2 \begin{pmatrix} 1 & 5 & -2 & 4 & | & 1 \\ 0 & 1 & 0 & 1 & | & 0 \\ 0 & 22 & 0 & 22 & | & 0 \end{pmatrix}$$

$$\begin{array}{l} \\ \\ \rho_3' = \rho_3 - 22\rho_2 \end{array} \begin{pmatrix} 1 & 5 & -2 & 4 & | & 1 \\ 0 & 1 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 5 & -2 & 4 & | & 1 \\ 0 & 1 & 0 & 1 & | & 0 \end{pmatrix}$$

yielding

$$w \text{ arbitrary, } z \text{ arbitrary, } y = -w,\ x = 1 - 5y + 2z - 4w = 1 + 2z + w.$$

(b)

$$\begin{pmatrix} 2 & -1 & 5 & 6 & | & -8 \\ 2 & 1 & 3 & 3 & | & -1 \\ 1 & -1 & 3 & 4 & | & -6 \end{pmatrix}$$

$$\begin{array}{l} \rho_1' = \rho 3 \\ \\ \rho_3' = \rho_1 \end{array} \begin{pmatrix} 1 & -1 & 3 & 4 & | & -6 \\ 2 & 1 & 3 & 3 & | & -1 \\ 2 & -1 & 5 & 6 & | & -8 \end{pmatrix}$$

$$\begin{array}{l} \\ \rho_2' = \rho_2 - 2\rho_1 \\ \rho_3' = \rho_3 - 2\rho_1 \end{array} \begin{pmatrix} 1 & -1 & 3 & 4 & | & 6 \\ 0 & 3 & -3 & -5 & | & 11 \\ 0 & 1 & -1 & -2 & | & 4 \end{pmatrix}$$

$$\begin{array}{l} \\ \rho_2' = \rho_3 \\ \rho_3' = \rho_2 \end{array} \begin{pmatrix} 1 & -1 & 3 & 4 & | & 6 \\ 0 & 1 & -1 & -2 & | & 4 \\ 0 & 3 & -3 & -5 & | & 11 \end{pmatrix}$$

$$\begin{array}{l} \\ \\ \rho_3' = \rho_3 - 3\rho_2 \end{array} \begin{pmatrix} 1 & -1 & 3 & 4 & | & 6 \\ 0 & 1 & -1 & -2 & | & 4 \\ 0 & 0 & 0 & 1 & | & -1 \end{pmatrix}$$

yielding $w = -1$, z arbitrary, $y = 4 + z + 2w = 2 + z$, $x = 6 + y - 3z + 4w = -2z$.

54. We use the tableaux

$$\begin{pmatrix} 1 & 1 & -1 & | & 1 \\ 2 & 3 & k & | & 3 \\ 1 & k & 3 & | & 2 \end{pmatrix}$$

$$\begin{matrix} \rho_2' = \rho_2 - 2\rho_1 \\ \rho_3' = \rho_3 - \rho_1 \end{matrix} \begin{pmatrix} 1 & 1 & -1 & | & 1 \\ 0 & 1 & k+2 & | & 1 \\ 0 & k-1 & 4 & | & 1 \end{pmatrix}$$

$$\rho_3' = \rho_3 - (k-1)\rho_2 \begin{pmatrix} 1 & 1 & -1 & | & 1 \\ 0 & 1 & k+2 & | & 1 \\ 0 & 0 & 6-k-k^2 & | & 2-k \end{pmatrix}$$

If $6-k-k^2 \neq 0$, that is, $k \neq 2$ and $k \neq -3$, then we can divide the bottom line by $6-k-k^2$ and get a unique solution. If $k=2$, the whole bottom line disappears and we get infinitely many solutions. If $k=-3$ the bottom line becomes $(0 \quad 0 \quad 0 \quad | \quad 5)$, representing $0x+0y+0z=5$, and there is no solution.

55.

$$\mathbf{AC} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad \mathbf{CA} - \mathbf{I} = \begin{pmatrix} 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

(i) We have

$$\mathbf{ACb} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_1+b_2+b_3 \end{pmatrix} = \mathbf{b}$$

just when $b_4 = b_1 + b_2 + b_3$, that is, just when there is a solution to the equation $\mathbf{Ax} = \mathbf{b}$. (ii) For any vector $\mathbf{z}$, the equation $\mathbf{Ax} = \mathbf{Az}$ surely has a solution, so one such solution will be $\mathbf{CAz}$, while another is of course $\mathbf{z}$. This means that $(\mathbf{CA} - \mathbf{I})\mathbf{z} = \mathbf{CAz} - \mathbf{z}$ must be a solution of the homogeneous equation $\mathbf{Ax} = \mathbf{0}$. But the two non-zero vectors

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ 0 \\ -2 \\ -1 \end{pmatrix}$$

appearing as columns of $\mathbf{CA} - \mathbf{I}$ are such solutions, and indeed any such solution is expressible as a sum of multiples of these two vectors, that is, in the form $(\mathbf{CA} - \mathbf{I})\mathbf{z}$.

56.

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{3}{2} & \frac{5}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{3}{2} & \frac{1}{2} \end{pmatrix}, \quad (\mathbf{A}^T)^{-1} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{5}{2} & -\frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

The point here is that (by 1G)

$$\mathbf{A}^T(\mathbf{A}^{-1})^T = (\mathbf{A}^{-1}\mathbf{A})^T = \mathbf{I}^T = \mathbf{I} = (\mathbf{AA}^{-1})^T = (\mathbf{A}^{-1})^T\mathbf{A}^T.$$

So $(\mathbf{A}^{-1})^T$ must be the inverse of $\mathbf{A}^T$.

57.

$$\mathbf{A}^{-1} = \begin{pmatrix} -4 & \frac{5}{2} & \frac{1}{2} \\ 16 & -9 & -2 \\ -7 & 4 & 1 \end{pmatrix}.$$

Consequently

$$\begin{aligned} \mathbf{X} &= \mathbf{IX} \\ &= (\mathbf{A}^{-1}\mathbf{A})\mathbf{X} \\ &= \mathbf{A}^{-1}(\mathbf{AX}) \\ &= \begin{pmatrix} -4 & \frac{5}{2} & \frac{1}{2} \\ 16 & -9 & -2 \\ -7 & 4 & 1 \end{pmatrix} \begin{pmatrix} 3 & 7 & -1 & 0 \\ 4 & 10 & -3 & 1 \\ 6 & 10 & 5 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 2 & 1 & -11 \\ 1 & 1 & 0 & 5 \end{pmatrix}. \end{aligned}$$

58.

$$\begin{pmatrix} 2 & 4 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{pmatrix} \text{ has inverse } \begin{pmatrix} -2 & -1 & 3 \\ 1 & 0 & -1 \\ 1 & 2 & -2 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 5 \\ -1 & 10 & -1 \end{pmatrix} \text{ has no inverse },$$

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \text{ has no inverse },$$

$$\begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 0 \\ 1 & 3 & 2 \end{pmatrix} \text{ has inverse } \begin{pmatrix} 2 & 1 & -3 \\ -4 & -1 & 6 \\ 5 & 1 & -7 \end{pmatrix}.$$

59. (a) independent (b) dependent – try $x_1 = 1$, $x_2 = x_3 = 0$ (c) dependent – try $x_1 = 2$, $x_2 = -1$ (d) independent (e) dependent – try $x_1 = 1$, $x_2 = -1$, $x_3 = -1$, $x_4 = 1$ (f) dependent – try $x_1 = -1$, $x_2 = 3$, $x_3 = -3$, $x_4 = 1$.

(There are many alternative answers to part (e).)

60. (a) 1 (b) 0 (c) 0 (d) 1.

(When setting out to find the inverses of these matrices, (b) and (c) were not invertible, so must have determinant 0; in each of the others, the reduction to echelon form produced 1s on the diagonal; for (a), there were two exchanges of rows, giving a factor of $(-1)^2 = 1$; for (d), no exchanges of rows were needed.)

61. (a) 2 (b) 1 (c) 12 (d) 1.

62. (a) $(6-\lambda)^2 - 9 = \lambda^2 - 12\lambda + 27$.

(b) $(a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = \lambda^2 - \lambda(a_{11}+a_{22}) + (a_{11}a_{22} - a_{12}a_{21})$.

(c) $(-1-\lambda)((-13-\lambda)(20-\lambda)+30.9) = -(\lambda+1)(\lambda^2-7\lambda+10) = -\lambda^3+6\lambda^2-3\lambda-10$.

(d)

$$\begin{aligned}(8-\lambda)(-\lambda(-3-\lambda)-10) - 4(-3(-3-\lambda)-15) + 10(6-(-\lambda)(-3))\\ &= (8-\lambda)(\lambda^2+3\lambda-10) - 4(3\lambda-6) + 10(6-3\lambda)\\ &= -\lambda^3+5\lambda^2+34\lambda-80-12\lambda+24-30\lambda+60\\ &= -\lambda^3+5\lambda^2-8\lambda+4.\end{aligned}$$

63. (a) $\begin{pmatrix}1&2\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\2&1\end{pmatrix}, \begin{pmatrix}2&1\\1&1\end{pmatrix}, \begin{pmatrix}1&1\\1&2\end{pmatrix}, \begin{pmatrix}0&2\\-2&0\end{pmatrix}, \begin{pmatrix}1&2\\-2&-1\end{pmatrix}, \begin{pmatrix}-1&2\\-2&1\end{pmatrix}$.

(b) $\begin{pmatrix}1&2a\\0&1\end{pmatrix}, \begin{pmatrix}1&2b\\0&1\end{pmatrix}, \begin{pmatrix}1&a+b\\0&1\end{pmatrix}, \begin{pmatrix}1&b+a\\0&1\end{pmatrix}, \begin{pmatrix}0&2(a-b)\\0&0\end{pmatrix}, \begin{pmatrix}0&2(a-b)\\0&0\end{pmatrix}, \begin{pmatrix}0&2(a-b)\\0&0\end{pmatrix}$.

64. (a) $6+k-k^2$
(b) $(b-a)(c-a)(c-b)$
(c) $(b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$.

(The point about the last one is that if any pair of a, b, c, d are equal then two columns are identical so the determinant is zero. By the Remainder Theorem it follows that all the six terms $b-a, \dots, d-c$ must be factors of the determinant. Also the determinant is surely a polynomial of degree 6 so that we have only to check the constant term. Looking at the coefficient of bc^2d^3 we see that this is 1, as claimed.)

65. (a)

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I}) &= (-8-\lambda)(13-\lambda)(-3-\lambda)+5(-10)(-8)+(-4)(-22)4\\ &\qquad -(-8-\lambda)(-10)4-5(-22)(-3-\lambda)-(-4)(13-\lambda)(-8)\\ &= -\lambda^3+2\lambda^2+\lambda-2\\ &= -(\lambda-1)(\lambda-2)(\lambda+1).\end{aligned}$$

So the eigenvalues are 1, 2 and -1.

eigenvectors for eigenvalue 1: $\alpha\begin{pmatrix}1\\1\\-1\end{pmatrix}$ for $\alpha\neq 0$.

eigenvectors for eigenvalue 2: $\alpha\begin{pmatrix}1\\2\\0\end{pmatrix}$ for $\alpha\neq 0$.

eigenvectors for eigenvalue -1: $\alpha\begin{pmatrix}1\\3\\2\end{pmatrix}$ for $\alpha\neq 0$.

(b)

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I}) &= (7-\lambda)(-9-\lambda)(3-\lambda)+8(-7)4+8(-9)4\\ &\qquad -(7-\lambda)(-7)4-8(-9)(3-\lambda)-8(-9-\lambda)4\\ &= -\lambda^3+\lambda^2+\lambda-1\\ &= -(\lambda-1)^2(\lambda+1),\end{aligned}$$

so the eigenvalues are 1, -1.

(Note that 1 is a double eigenvalue.)

eigenvectors for eigenvalue 1: $\alpha\begin{pmatrix}4\\-5\\2\end{pmatrix}$ for $\alpha\neq 0$.

eigenvectors for eigenvalue -1: $\alpha\begin{pmatrix}1\\-2\\1\end{pmatrix}$ for $\alpha\neq 0$.

(c)

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I}) &= (4-\lambda)(1-\lambda)(-5-\lambda)-3(1-\lambda)(-6)\\ &= (1-\lambda)((\lambda-4)(\lambda+5)+18)\\ &= (1-\lambda)(\lambda^2+\lambda-2)\\ &= -(\lambda-1)^2(\lambda+2),\end{aligned}$$

so the eigenvalues are 1 and -2.

(Note that 1 is a double eigenvalue.)

eigenvectors for eigenvalue 1: $\alpha\begin{pmatrix}-1\\0\\1\end{pmatrix}+\beta\begin{pmatrix}2\\1\\0\end{pmatrix}$ for α, β not both 0. (There are many alternative descriptions: check with your class teacher.)

eigenvectors for eigenvalue -2: $\alpha\begin{pmatrix}1\\0\\-2\end{pmatrix}$ for $\alpha\neq 0$.

66.
(a) ($\ln\cos x$) Domain $\{x : \cos x > 0\}$ or $\{x : \exists\, k \in \mathbb{Z}, |x - 2k\pi| < \frac{\pi}{2}\}$.

(Diagram not available.)

(b) ($\sqrt{x^2}$) Domain $\mathbb{R}$ or $]-\infty, \infty[$.

(Diagram not available.)

(c) ($\exp\ln x$) Domain $\{x : x > 0\}$ or $]0, \infty[$.

(Diagram not available.)

(d) ($\ln\exp x$) Domain $\mathbb{R}$ or $]-\infty, \infty[$.

(Diagram not available.)

(e) ($(\sqrt{x})^2$) Domain $\{x : x \geq 0\}$ or $[0, \infty[$.

(Diagram not available.)

(f) ($x^3 - x$) Domain $\mathbb{R}$ or $]-\infty, \infty[$.

(Diagram not available.)

(g) ($x\ln x$) Domain $\{x : x > 0\}$ or $]0, \infty[$.

(Diagram not available.)

(h) ($\frac{1}{x}\sin x$) Domain $\{x : x \neq 0\}$.

(Diagram not available.)

67. (a)

$$\det(\mathbf{A}-\lambda\mathbf{I}) = (-1-\lambda)((-13-\lambda)(20-\lambda)-30(-9))$$
$$= -(\lambda+1)(\lambda-5)(\lambda-2),$$

so the eigenvalues are -1, 2 and 5.

eigenvectors for eigenvalue -1: $\alpha\begin{pmatrix}1\\0\\0\end{pmatrix}$ for $\alpha \neq 0$.

eigenvectors for eigenvalue 2: $\alpha\begin{pmatrix}0\\2\\1\end{pmatrix}$ for $\alpha \neq 0$.

eigenvectors for eigenvalue 5: $\alpha\begin{pmatrix}-1\\5\\3\end{pmatrix}$ for $\alpha \neq 0$.

(b)

$$\det(\mathbf{A}-\lambda\mathbf{I}) = -(\lambda-2)^2(\lambda-1),$$

so the eigenvalues are 2 (twice) and 1.

eigenvectors for eigenvalue 1: $\alpha\begin{pmatrix}-2\\1\\1\end{pmatrix}$ for $\alpha \neq 0$.

eigenvectors for eigenvalue 2: $\alpha\begin{pmatrix}-5\\0\\3\end{pmatrix}+\beta\begin{pmatrix}-2\\3\\0\end{pmatrix}$ for α, β not both 0.

(c) $\det(\mathbf{A}-\lambda\mathbf{I}) = (\lambda-1)^2(\lambda-2)^2$, so the eigenvalues are 1 (twice) and 2 (twice).

eigenvectors for eigenvalue 1: $\alpha\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ for $\alpha \neq 0$.

eigenvectors for eigenvalue 2: $\alpha\begin{pmatrix}0\\0\\1\\0\end{pmatrix}+\beta\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ for α, β not both 0.

68.
(a) $(x/|x|)$ Domain $\{x : x \neq 0\}$.

(Diagram not available.)

(b) $(\ln(-x))$ Domain $]-\infty, 0[$.

(Diagram not available.)

(c) $(\sqrt{1-1/x})$ Domain $]0, 1]$.

(Diagram not available.)

(d) $(\ln x/x)$ Domain $]0, \infty[$.

(Diagram not available.)

(e) $(1/\sqrt{1-x^2})$ Domain $]-1,1[$.

(Diagram not available.)

(f) $(x^2 e^{-|x|})$ Domain $\mathbb{R}$.

(Diagram not available.)