Calculus I Questions

1. Which of the following are saying something more or less true? Can any of the others be corrected? Are there any checks you have to make when using them?

(a) $\ln(x+y) = \ln x . \ln y$
(b) $\exp(x+y) = \exp(x) . \exp(y)$
(c) $\cos(x+y) = \cos x \cos y + \sin x \sin y$
(d) $\sin(x-y) = \sin x \cos y - \cos x \sin y$
(e) $1 + x + x^2 + \ldots + x^n = \frac{1-x^{n+1}}{1-x}$ (where $x \in \mathbb{R}$ and $n \geq 0$ is an integer)
(f) $1 + 2 + \ldots + n = \frac{n(n-1)}{2}$ (where $n \geq 1$ is an integer).

[*Notes*: $\ln x = \log_e x$, the 'natural' logarithm of x; for practical purposes, $\exp x$ is the same thing as e^x; in this course, the arguments of trigonometric functions are to be read in radians unless otherwise stated.]
[AA]

2. Which of the following are true?

(a) $\pi = 3.142$	(e) $-\pi > -3.142$
(b) $\pi \geq 3.142$	(f) if $a \leq b$ and $c \leq d$ then $a + c \leq b + d$
(c) $\pi \leq 3.142$	(g) if $a \leq b$ and $c \leq d$ then $a - d \leq b - c$
(d) $-\pi < -3.142$	(h) if $a < b$ and $c < d$ then $a + c \leq b + d$.

[AB]

3. One of the following makes no proper sense. Which is it? Evaluate each of the others in decimal form as accurately as seems to you reasonable, giving an estimate of the error (if any) in the answer you give.

(a) $\ln 3 + \ln \frac{1}{3}$	(d) $\ln \sin \frac{\pi}{4}$	(g) $(\exp 2)^3$
(b) $\sin(-7)$	(e) $\ln \tan 2$	(h) $\cos \frac{2\pi}{3}$
(c) $\sin 3 \cos 4 + \sin 4 \cos 3$	(f) $\exp(2^3)$	(i) $\tan \frac{\pi}{3}$.

[AC]

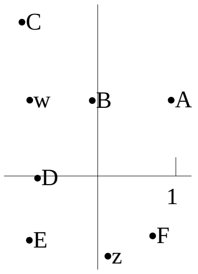
4. Which of the following are true for all real numbers a, b, …? For those which are sometimes false, give examples of numbers for which they don't work.

(a) If $a \leq b$ and $c < d$ then $a + c < b + d$.
(b) If $a \leq b$ and $c \leq d$ then $ac \leq bd$.
(c) If $a^2 \leq b^2$ and $c^2 \leq d^2$ then $a^2c^2 \leq b^2d^2$.
(d) If $a < b < 0$ then $\frac{1}{b} \leq \frac{1}{a}$.
(e) If $a < b < 0$ and $0 < c < d$ then $ad < bc$.
(f) If $a^2 < b^2$ then $a \leq b$. [AD]

5. On the following diagram,
(a) identify which of the points A, $B, \ldots, F$ correspond to $w+z$, wz, $w-z$, $\frac{w}{z}$, $\frac{1}{wz}$;
(b) identify which of the points $A, \ldots, F$ has smallest modulus and which has greatest;
(c) say which of the formulae

$$\arg(wz) = \arg w + \arg z, \quad \arg(\tfrac{w}{z}) = \arg w - \arg z, \quad \arg(\tfrac{z}{w}) = \arg z - \arg w$$

(if any) are true.



[BA]

6. Find $w+z$, $w-z$, wz and $\frac{w}{z}$ in the form $a+bi$ where $w = 2-3i$, $z = -5+i$. [BB]

7. On an Argand diagram plot z, $-z$, $\bar{z}$, iz where $z = 2+i$. [BC]

8. Calculate the following, giving answers in the form $a+bi$:
(a)$\left(\frac{1+i}{\sqrt{2}}\right)^2$ (b)$\frac{2-3i}{4+5i}$ (c)$\frac{2+3i}{4-5i}$ (d)$\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3$. [BD]

9. Find the moduli and arguments, correct to two decimal places, of the following:
(a) $-3+4i$ (b) $\overline{-3+4i}$ (c) $|-3+4i|$ (d) $\frac{1}{-3+4i}$
(remember: radians!) [BE]

10. Which of the following are true for all w, $z \in \mathbb{C}$? When the answer is 'no', give examples of particular w, z for which the formula makes sense, but is false.
(a) $\mathcal{R}\mathrm{e}(w+z) = \mathcal{R}\mathrm{e}(w) + \mathcal{R}\mathrm{e}(z)$ (d) $|wz| = |w||z|$
(b) $\mathcal{I}\mathrm{m}(wz) = \mathcal{I}\mathrm{m}(w)\,\mathcal{I}\mathrm{m}(z)$ (e) $|\bar{z}| = |z|$
(c) $\arg(w+z) = \arg(w) + \arg(z)$ (f) $\arg(\bar{z}) = -\arg z$.

[BF]

11. (a) Find the two complex square roots of $2i$, expressing them in the form $a+bi$.
(b) Find all the complex numbers z such that $(1-\frac{i}{2})z^2 + 2z + 1 = 0$, expressing them in the form $a+bi$.
[BG]

12. Describe the following sets in geometric terms:
(a) $\{z : z \in \mathbb{C},\ \bar{z} = -z\}$ (c) $\{z : z \in \mathbb{C},\ |z+i| = |z-1|\}$
(b) $\{z : z \in \mathbb{C},\ \bar{z} = \frac{1}{z}\}$ (d) $\{z : z \in \mathbb{C},\ |z-1| = 1\}$.

[BH]

13. Solve the following equations for the complex number z. In each case give the real part, the imaginary part, the modulus and the argument of each solution in decimal form correct to two decimal places, and sketch the solutions on an Argand diagram:

(a) $z^3 = 1$ (b) $z^4 = -4$ (c) $z^5 = 2+i$

[CA]

14. Solve the equation $z^3 = 1$ for the complex number z. Give the real part, the imaginary part, the modulus and the argument of each solution in decimal form correct to two decimal places, and sketch the solutions on an Argand diagram.

[CAA]

15. Solve the equation $z^4 = -4$ for the complex number z. Give the real part, the imaginary part, the modulus and the argument of each solution in decimal form correct to two decimal places, and sketch the solutions on an Argand diagram.

[CAB]

16. Solve the equation $z^5 = 2 + i$ for the complex number z. Give the real part, the imaginary part, the modulus and the argument of each solution in decimal form correct to two decimal places, and sketch the solutions on an Argand diagram.

[CAC]

17. As question 14 for

(a) $z^5 = 2 - i$ (b) $z^6 + 2z^3 + 2 = 0$

[CB]

18. Find formulae for the derivatives with respect to x of the following:

(a) $\frac{1}{2}x - \frac{1}{4}\sin 2x$
(b) $\cos x^2$
(c) $\tan x - x$
(d) $\exp 2x \sin 3x$
(e) $\exp(\frac{1}{x})$
(f) $x \ln x - x$
(g) $\cosh x = \frac{1}{2}(\exp(x) + \exp(-x))$
(h) $\sinh x = \frac{1}{2}(\exp(x) - \exp(-x))$
(i) $\tanh x = \dfrac{\sinh x}{\cosh x}$
(j) $\ln(x + \sqrt{x^2 - 1})$
(k) $\frac{1}{2}\ln(\frac{1+x}{1-x})$
(l) $x^\alpha = \exp(\alpha \ln x)$
(m) $\alpha^x = \exp(x \ln \alpha)$
(n) $\ln(|x|)$
(o) $-\ln(|\cos x|)$
(p) $\ln(\frac{1+\sin x}{|\cos x|})$

[CC]

19. Find formulae for the derivatives with respect to x of the following:

(a) $\ln(x - \sqrt{x^2 - 1})$
(b) $\ln(|\sin x|)$
(c) $\ln(x + \sqrt{x^2 + 1})$
(d) $\sqrt{\frac{1}{x} - 1}$.

[CD]

20. Each of the following formulae may be used to define a real function f. Draw up a table, on one sheet of paper, with seven rows and six columns, showing in each case

(i) the natural domain $\operatorname{dom} f$ of the function;
(ii) the points where it takes the value 0;
(iii) its stationary points;
(iv) its local maxima;
(v) its global minima;
(vi) its set of values $\{f(x) : x \in \operatorname{dom} f\}$.

(a) $|x^2 - 1|$ (b) $\frac{x}{|x|}$ (c) $x \ln x$ (d) $\exp(-\frac{1}{2}x^2)$

(e) $\frac{1}{x^2-1}$ (f) $\tanh x$ (g) $\sin x + \cos x$.

(Hint for (g): express $\sin x + \cos x$ as $r\cos(x + \theta)$.) [DA]

21. Each of the following formulae may be used to define a real function f. For each formula state
(i) the natural domain $\operatorname{dom} f$ of the function;
(ii) the points where it takes the value 0;
(iii) its stationary points;
(iv) its local maxima;
(v) its global minima;
(vi) its set of values $\{f(x) : x \in \operatorname{dom} f\}$.
(a) $|x^2 - 1|$ (b) $\frac{x}{|x|}$ [DAA]

22. As question 21, with the functions
(a) $\frac{1}{x^2-1}$ (b) $\tanh x$ (c) $\sin x + \cos x$.
(Hint for (c): express $\sin x + \cos x$ as $r\cos(x+\theta)$.) [DAB]

23. As question 21, with the functions
(a) $x \ln x$ (b) $\exp(-\frac{1}{2}x^2)$. [DAC]

24. Give rough sketches of the functions in questions 21 to 23. [DB]

25. As question 20 with
(a) $x^4 - 2x^2 + 1$ (b) $x - \ln x$ (c) $\ln \sin x$ (d) $\ln|\cos x|$
(e) xe^x (f) $\sinh x$ (g) $\cosh x$ (h) $\frac{\ln x}{x}$
[DC]

26. As question 21 with
(a) $x^4 - 2x^2 + 1$ (b) $x - \ln x$. [DCA]

27. As question 21 with
(a) $\ln \sin x$ (b) $\ln|\cos x|$. [DCB]

28. As question 21 with
(a) xe^x (b) $\sinh x$. [DCC]

29. As question 21 with
(a) $\cosh x$ (b) $\frac{\ln x}{x}$. [DCD]

30. Sketch each of the following functions (defined on $\mathbb{R}$):
(a) $\sin x \cos x$ (b) $\sin 10x + \sin 11x$ (c) $\sin x + \sin(\sqrt{2}x)$
[DD]

31. For each of the following pairs of functions f and g, describe
(i) the domain and set of values of the composition $f \circ g$;
(ii) the domain and set of values of the composition $g \circ f$;
(iii) the set of values of the quotient f/g.
(a) $\operatorname{dom} f = \mathbb{R}$, $f(x) = \sin x$; $\operatorname{dom} g = \mathbb{R} \setminus \{0\}$, $g(x) = 1/x$.
(b) $\operatorname{dom} f = \mathbb{R}$, $f(x) = x^2$; $\operatorname{dom} g = [0, \infty[$, $g(x) = \sqrt{x}$.
(c) $\operatorname{dom} f = \mathbb{R} \setminus \{0\}$, $f(x) = \frac{1}{x} - 1$; $\operatorname{dom} g = [0, \infty[$, $g(x) = \sqrt{x}$. [EA]

32. For the following functions f and g, describe
(i) the domain and set of values of the composition $f \circ g$;
(ii) the domain and set of values of the composition $g \circ f$;
(iii) the set of values of the quotient f/g.

$$\operatorname{dom} f = \mathbb{R},\ f(x) = \sin x;\ \operatorname{dom} g = \mathbb{R} \setminus \{0\},\ g(x) = 1/x.$$

[EAA]

33. As question 32, with
$$\operatorname{dom} f = \mathbb{R},\ f(x) = x^2;\ \operatorname{dom} g = [0,\infty[,\ g(x) = \sqrt{x}.$$
[EAB]

34. As question 32, with
$$\operatorname{dom} f = \mathbb{R}\setminus\{0\},\ f(x) = \tfrac{1}{x} - 1;\ \operatorname{dom} g = [0,\infty[,\ g(x) = \sqrt{x}.$$
[EAC]

35. As question 31, with
(a) $\operatorname{dom} f =]0,\infty[,\ f(x) = \ln x;\ \operatorname{dom} g = \mathbb{R},\ g(x) = \exp(\alpha x)$, where $\alpha \in \mathbb{R}$.
(Do the answers depend on the value of α?)
(b) $\operatorname{dom} f = \{x : \cos x \neq 0\},\ f(x) = \tan x;\ \operatorname{dom} g = \mathbb{R}\setminus\{0\},\ g(x) = \ln|x|$.
(c) $\operatorname{dom} f = \mathbb{R},\ f(x) = \cos x;\ \operatorname{dom} g = [-1,1],\ g(x) = \arccos x$.
(I write $\arccos x$ for what your calculators probably call $\cos^{-1} x$.)
[EB]

36. As question 32, with
$$\operatorname{dom} f =]0,\infty[,\ f(x) = \ln x;\ \operatorname{dom} g = \mathbb{R},\ g(x) = \exp(\alpha x),$$
where $\alpha \in \mathbb{R}$. (Do the answers depend on the value of α?) [EBA]

37. As question 32, with
$$\operatorname{dom} f = \{x : \cos x \neq 0\},\ f(x) = \tan x;\ \operatorname{dom} g = \mathbb{R}\setminus\{0\},\ g(x) = \ln|x|.$$
[EBB]

38. As question 32, with
$$\operatorname{dom} f = \mathbb{R},\ f(x) = \cos x;\ \operatorname{dom} g = [-1,1],\ g(x) = \arccos x.$$
(I write $\arccos x$ for what your calculators probably call $\cos^{-1} x$.) [EBC]

39. Give the following limits:
(a) $\lim_{x\to 1} |x^2-1|$ (b) $\lim_{x\downarrow 0} x\ln x$ (c) $\lim_{x\uparrow -1} \frac{1}{x^2-1}$
(d) $\lim_{x\to\infty} x^4 - 2x^2 + 1$ (e) $\lim_{x\uparrow\pi} \ln\sin x$ (f) $\lim_{x\to\infty} \sinh x$
(g) $\lim_{x\to\infty} \frac{\ln x}{x}$
[EC]

40. Find the following limits:
(a) $\lim_{x\to 1} |x^2-1|$ (b) $\lim_{x\downarrow 0} x\ln x$ (c) $\lim_{x\uparrow -1} \frac{1}{x^2-1}$.
[ECA]

41. Find the following limits:
(a) $\lim_{x\to\infty} x^4 - 2x^2 + 1$ (b) $\lim_{x\uparrow\pi} \ln\sin x$.
[ECB]

42. Find the following limits:
(a) $\lim_{x\to\infty} \sinh x$ (b) $\lim_{x\to\infty} \frac{\ln x}{x}$.
[ECC]

43. Find the following limits:

(a) $\lim_{x\to-\infty} |x^2-1|$ (b) $\lim_{x\to\infty} \frac{x}{|x|}$ (c) $\lim_{x\to-\infty} \frac{x}{|x|}$

(d) $\lim_{x\to\infty} \exp(-\frac{1}{2}x^2)$ (e) $\lim_{x\downarrow-1} \frac{1}{x^2-1}$ (f) $\lim_{x\to\infty} \frac{1}{x^2-1}$

(g) $\lim_{x\to\infty} \tanh x$ (h) $\lim_{x\downarrow 0} x-\ln x$ (i) $\lim_{x\to\infty} x-\ln x$

(j) $\lim_{x\to\frac{\pi}{2}} \ln|\cos x|$ (k) $\lim_{x\to-\infty} xe^x$ (l) $\lim_{x\to-\infty} \cosh x$

(m) $\lim_{x\downarrow 0} \frac{\ln x}{x}$

[ED]

44. Find the following limits (allowing $\pm\infty$); when no limit exists, say so plainly.

(a) $\lim_{x\to-4}(x+3)^{143}$ (b) $\lim_{x\to 0} \frac{1}{x}$ (c) $\lim_{x\to 0} |\frac{1}{x}|$

(d) $\lim_{x\to 0} \frac{1}{x^2}$ (e) $\lim_{x\to 0} \sqrt{x}$ (f) $\lim_{x\to 1} \sin x$

(g) $\lim_{x\to\infty} \cos^2 x$ (h) $\lim_{x\to\infty} \sin^2 x$ (i) $\lim_{x\to\infty} \cos^2 x+\sin^2 x$

(j) $\lim_{x\to 0} \frac{|x|^{3/2}}{x}$ (k) $\lim_{x\to 2} \frac{x^2-4}{x-2}$ (l) $\lim_{x\to-2} \frac{x^3+8}{x^2-4}$

(m) $\lim_{x\downarrow\frac{\pi}{2}} \tan x$ (n) $\lim_{x\to-\infty} \arctan x$.

[FA]

45. Find the following limits (allowing $\pm\infty$); when no limit exists, say so plainly.

(a) $\lim_{x\to-4}(x+3)^{143}$ (b) $\lim_{x\to 0} |\frac{1}{x}|$ (c) $\lim_{x\to 1} \sin x$

(d) $\lim_{x\to\infty} \cos^2 x+\sin^2 x$ (e) $\lim_{x\to 2} \frac{x^2-4}{x-2}$ (f) $\lim_{x\downarrow\frac{\pi}{2}} \tan x$.

[FAA]

46. Find the following limits (allowing $\pm\infty$); when no limit exists, say so plainly.

(a) $\lim_{x\to 0} \frac{1}{x}$ (b) $\lim_{x\to 0} \frac{1}{x^2}$ (c) $\lim_{x\to 0} \sqrt{x}$

(d) $\lim_{x\to\infty} \cos^2 x$ (e) $\lim_{x\to\infty} \sin^2 x$ (f) $\lim_{x\to 0} \frac{|x|^{3/2}}{x}$

(g) $\lim_{x\to-2} \frac{x^3+8}{x^2-4}$ (h) $\lim_{x\to-\infty} \arctan x$.

[FAB]

47. As question 44, with

(a) $\lim_{x\to-\infty} \frac{1}{x}\sin x$ (b) $\lim_{x\to\infty} x^{\pi}$ (c) $\lim_{x\downarrow 1} \frac{1+x}{1-x}$

(d) $\lim_{x\downarrow 0} x^{-\sqrt{2}}$ (e) $\lim_{x\to\infty} 8x^3\exp(-x)\sin x$ (f) $\lim_{x\to-\infty} x^{10}\exp(-\frac{x}{2})$

(g) $\lim_{x\to\infty} \cosh^2 x$ (h) $\lim_{x\to\infty} \sinh^2 x$ (i) $\lim_{x\to\infty} \cosh x+\sinh x$

(j) $\lim_{x\to\infty} \cosh x-\sinh x$ (k) $\lim_{x\to\infty} \cosh^2 x-\sinh^2 x$ (l) $\lim_{x\to\infty}(\frac{1}{\pi})^x$.

[FB]

48. Find the following limits (allowing $\pm\infty$); when no limit exists, say so plainly.

(a) $\lim_{x\downarrow 1} \frac{1+x}{1-x}$ (b) $\lim_{x\downarrow 0} x^{-\sqrt{2}}$ (c) $\lim_{x\to\infty} 8x^3\exp(-x)\sin x$

(d) $\lim_{x\to\infty}(\frac{1}{\pi})^x$.

[FBA]

49. Find the following limits (allowing $\pm\infty$); when no limit exists, say so plainly.

(a) $\lim_{x\to-\infty} \frac{1}{x}\sin x$ (b) $\lim_{x\to\infty} x^{\pi}$

(c) $\lim_{x\to-\infty} x^{10}\exp(-\frac{x}{2})$ (d) $\lim_{x\to\infty} \cosh^2 x$

(e) $\lim_{x\to\infty} \sinh^2 x$ (f) $\lim_{x\to\infty} \cosh x+\sinh x$

(g) $\lim_{x\to\infty} \cosh x-\sinh x$ (h) $\lim_{x\to\infty} \cosh^2 x-\sinh^2 x$.

[FBB]

50. As question 44, with

(a) $\lim_{x\to\infty} \frac{x^4-x^3+2x^2-7}{18x^3-12x-99}$ (b) $\lim_{x\to 0} \frac{x^4-x^3+2x^2-7}{18x^3-12x-99}$ (c) $\lim_{x\to-\infty} \frac{x^4-x^3+2x^2-7}{18x^3-12x-99}$

(d) $\lim_{x\to\infty} \ln x \tanh x$ (e) $\lim_{x\downarrow 0} \ln x + \sqrt{x}$ (f) $\lim_{x\downarrow 0} \frac{x\ln x+1}{e^x}$

(g) $\lim_{x\to\infty} \frac{x^2+1}{x^2-1} + \tanh x$ (h) $\lim_{x\to\infty} x + e^{-x}$ (i) $\lim_{x\to-\infty} xe^{-x}$

(j) $\lim_{x\to 0} \frac{\sin x}{x^3}$.

[FC]

51. Find the following limits (allowing $\pm\infty$); when no limit exists, say so plainly.

(a) $\lim_{x\to-\infty} \frac{x^4-x^3+2x^2-7}{18x^3-12x-99}$ (b) $\lim_{x\downarrow 0} \frac{x\ln x+1}{e^x}$ (c) $\lim_{x\to\infty} \frac{x^2+1}{x^2-1} + \tanh x$

(d) $\lim_{x\to 0} \frac{\sin x}{x^3}$.

[FCA]

52. Find the following limits (allowing $\pm\infty$); when no limit exists, say so plainly.

(a) $\lim_{x\to\infty} \frac{x^4-x^3+2x^2-7}{18x^3-12x-99}$ (b) $\lim_{x\to 0} \frac{x^4-x^3+2x^2-7}{18x^3-12x-99}$ (d) $\lim_{x\to\infty} \ln x \tanh x$

(e) $\lim_{x\downarrow 0} \ln x + \sqrt{x}$ (h) $\lim_{x\to\infty} x + e^{-x}$ (i) $\lim_{x\to-\infty} xe^{-x}$.

[FCB]

53. As question 44, with

(a) $\lim_{x\to\infty} \exp x - x^{10}$ (b) $\lim_{x\to-\infty} x + e^{-x}$ (c) $\lim_{x\to\infty} \ln x + \sin x$

(d) $\lim_{x\to\infty} e^{-x} \frac{\sin x}{|\sin x|}$ (e) $\lim_{x\to-\infty} \frac{x}{\tanh x} + x^2$ (f) $\lim_{x\to\infty} x - \ln x$

(g) $\lim_{x\downarrow 0} \sqrt{x} \ln x$ (h) $\lim_{x\downarrow 0} e^{-x} \ln x$ (i) $\lim_{x\to\infty} e^{-x} \ln x$

(j) $\lim_{x\to 1} \frac{e^x}{x^3-2x^2+x}$.

[FD]

54. Find the following limits (allowing $\pm\infty$); when no limit exists, say so plainly.

(a) $\lim_{x\to\infty} \exp x - x^{10}$ (b) $\lim_{x\to\infty} \ln x + \sin x$ (c) $\lim_{x\to\infty} e^{-x} \frac{\sin x}{|\sin x|}$

(d) $\lim_{x\to-\infty} \frac{x}{\tanh x} + x^2$ (e) $\lim_{x\to 1} \frac{e^x}{x^3-2x^2+x}$.

[FDA]

55. Find the following limits (allowing $\pm\infty$); when no limit exists, say so plainly.

(a) $\lim_{x\to-\infty} x + e^{-x}$ (b) $\lim_{x\to\infty} x - \ln x$ (c) $\lim_{x\downarrow 0} \sqrt{x} \ln x$

(d) $\lim_{x\downarrow 0} e^{-x} \ln x$ (e) $\lim_{x\to\infty} e^{-x} \ln x$.

[FDB]

56. As question 44, with

(a)$\lim_{x\downarrow \pi} \frac{\sin x}{|\sin x|}$ (b) $\lim_{x\to 0} \exp(\frac{1}{x^2})$ (c) $\lim_{x\to\infty} \ln(x + \sin x)$

(d) $\lim_{x\to-\infty} \exp(\frac{1}{x})$ (e) $\lim_{x\to\infty} \sqrt{\frac{\sin x}{x}}$ (f) $\lim_{x\uparrow 0} \sinh \frac{1}{x}$

(g) $\lim_{x\uparrow \pi} \sin x \ln(\sin x)$ (h) $\lim_{x\uparrow 0} \exp(\frac{1}{x})$ (i) $\lim_{x\downarrow 0} \frac{1}{x\ln x}$

(j) $\lim_{x\to 0} \ln(\frac{2\sin x}{x})$ (k) $\lim_{x\downarrow 0} x^x$ (l) $\lim_{x\to\infty} x^x e^{-x^2/2}$

(m) $\lim_{x\to 2} \ln(\frac{x^2-4}{x-2})$ (n) $\lim_{x\to 0} |\ln(e^{1/x} - x)|$.

[FE]

57. Find the following limits (allowing $\pm\infty$); when no limit exists, say so plainly.

(a) $\lim_{x\to 0}\exp(\frac{1}{x^2})$ (b) $\lim_{x\uparrow 0}\exp(\frac{1}{x})$ (c) $\lim_{x\to 0}\ln(\frac{2\sin x}{x})$

(d) $\lim_{x\downarrow 0} x^x$ (e) $\lim_{x\to\infty} x^x e^{-x^2/2}$ (f) $\lim_{x\to 0}|\ln(e^{1/x}-x)|$.

[FEA]

58. Find the following limits (allowing $\pm\infty$); when no limit exists, say so plainly.

(a)$\lim_{x\downarrow\pi}\frac{\sin x}{|\sin x|}$ (b) $\lim_{x\to\infty}\ln(x+\sin x)$ (c) $\lim_{x\to-\infty}\exp(\frac{1}{x})$

(d) $\lim_{x\to\infty}\sqrt{\frac{\sin x}{x}}$ (e) $\lim_{x\uparrow 0}\sinh\frac{1}{x}$ (f) $\lim_{x\uparrow\pi}\sin x\ln(\sin x)$

(g) $\lim_{x\downarrow 0}\frac{1}{x\ln x}$ (h) $\lim_{x\to 2}\ln(\frac{x^2-4}{x-2})$.

[FEB]

59. Find the following limits (allowing $\pm\infty$); when no limit exists, say so plainly.

(a) $\lim_{x\to 0}\frac{1}{x}\ln(1+x)$ (b) $\lim_{x\to\infty} x\ln(1+\frac{1}{x})$ (c) $\lim_{x\to\infty}(1+\frac{1}{x})^x$

(d) $\lim_{x\to\infty}(1+\frac{\alpha}{x})^{\beta x}$ (e) $\lim_{x\to 0}\frac{\sin x}{x}$ (f) $\lim_{x\to 0}\frac{\sin 5x}{3x}$

(g) $\lim_{x\to 0}(\frac{1}{\sin x}-\frac{1}{x})$ (h) $\lim_{x\to 0}(\frac{1}{\sin^2 x}-\frac{1}{x^2})$ (i) $\lim_{x\to 0}\frac{e^x-1}{x}$

(j) $\lim_{x\to 0}(\frac{1}{e^x-1}-\frac{1}{x})$

(k) $\lim_{x\to 0}\frac{x-\ln(1+x)}{x^2}$ (l) $\lim_{x\to 0}\frac{x^2}{\sin^2 x}$

(m) $\lim_{x\to 0}\frac{x-\ln(1+x)}{\sin^2 x}$ (n) $\lim_{x\to 3}\frac{3x-9}{x^2-3x+7}$ (o) $\lim_{x\downarrow -3}\frac{\sqrt{9-x^2}}{x+3}$

(p) $\lim_{x\to 0}\frac{\sqrt{a+x}-\sqrt{a}}{x}$ where $a>0$ (q) $\lim_{x\to\pi}\frac{1+\cos x}{\tan^2 x}$ (r) $\lim_{x\to 1}\frac{x^3+x-2}{x^2-2x+1}$.

[GA]

60. Find the following limits (allowing $\pm\infty$); when no limit exists, say so plainly.

(a) $\lim_{x\to 0}\frac{1}{x}\ln(1+x)$ (b) $\lim_{x\to\infty} x\ln(1+\frac{1}{x})$ (c) $\lim_{x\to\infty}(1+\frac{1}{x})^x$

(d) $\lim_{x\to\infty}(1+\frac{\alpha}{x})^{\beta x}$ (e) $\lim_{x\to 0}\frac{\sin 5x}{3x}$.

[GAA]

61. Find the following limits (allowing $\pm\infty$); when no limit exists, say so plainly.

(a) $\lim_{x\to 0}\frac{e^x-1}{x}$ (b) $\lim_{x\to 0}(\frac{1}{e^x-1}-\frac{1}{x})$ (c) $\lim_{x\to 0}\frac{x-\ln(1+x)}{\sin^2 x}$

(d) $\lim_{x\to 3}\frac{3x-9}{x^2-3x+7}$ (e) $\lim_{x\to 1}\frac{x^3+x-2}{x^2-2x+1}$.

[GAB]

62. Find the following limits (allowing $\pm\infty$); when no limit exists, say so plainly.

(a) $\lim_{x\to 0}\frac{\sin x}{x}$ (b) $\lim_{x\to 0}(\frac{1}{\sin x}-\frac{1}{x})$ (c) $\lim_{x\to 0}(\frac{1}{\sin^2 x}-\frac{1}{x^2})$

(d) $\lim_{x\to 0}\frac{x-\ln(1+x)}{x^2}$.

[GAC]

63. Find the following limits (allowing $\pm\infty$); when no limit exists, say so plainly.

(a) $\lim_{x\to 0}\frac{x^2}{\sin^2 x}$ (b) $\lim_{x\downarrow -3}\frac{\sqrt{9-x^2}}{x+3}$ (c) $\lim_{x\to 0}\frac{\sqrt{a+x}-\sqrt{a}}{x}$ where $a>0$

(d) $\lim_{x\to\pi}\frac{1+\cos x}{\tan^2 x}$.

[GAD]

64. Which of the following functions are injective (= one-to-one)? For those which are, give the domains of their inverses, and formulae for the inverse functions when you can.
(a) dom $f = [0, \infty[$, $f(x) = \sqrt{x}$.
(b) dom $f = \mathbb{R}$, $f(x) = x^3$.
(c) dom $f = \mathbb{R}$, $f(x) = x^6$.
(d) dom $f = \mathbb{R}$, $f(x) = x^3 - x$.
(e) dom $f = \mathbb{R}$, $f(x) = x^3 + x$.
(f) dom $f = \mathbb{R}$, $f(x) = \cosh x$.
(g) dom $f =]0, \infty[$, $f(x) = (\ln x)^2$.
(h) dom $f =]0, 1[$, $f(x) = (\ln x)^2$.
(i) dom $f = \mathbb{R}$, $f(x) = \sinh x$.
(j) dom $f =]-\infty, 0]$, $f(x) = \sinh x^2$.
(k) dom $f = \mathbb{R}$, $f(x) = \cosh(\exp x)$.
(l) dom $f = \mathbb{R}$, $f(x) = \tanh x$.
(m) dom $f = \mathbb{R}$, $f(x) = x + \cos x$.
(n) dom $f = \mathbb{R}$, $f(x) = \tanh(\sin x)$.
(o) dom $f = \mathbb{R}$, $f(x) = \sin(\tanh x)$. [GB]

65. Which of the following functions are injective (= one-to-one)? For those which are, give the domains of their inverses, and formulae for the inverse functions when you can.
(a) dom $f = \mathbb{R}$, $f(x) = x^3 - x$.
(b) dom $f = \mathbb{R}$, $f(x) = x^3 + x$.
(c) dom $f =]0, 1[$, $f(x) = (\ln x)^2$.
(d) dom $f = \mathbb{R}$, $f(x) = \sinh x$.
(e) dom $f =]-\infty, 0]$, $f(x) = (\sinh x)^2$.
[GBA]

66. Which of the following functions are injective (= one-to-one)? For those which are, give the domains of their inverses, and formulae for the inverse functions when you can.
(a) dom $f = [0, \infty[$, $f(x) = \sqrt{x}$.
(b) dom $f = \mathbb{R}$, $f(x) = x^3$.
(c) dom $f = \mathbb{R}$, $f(x) = x^6$.
(d) dom $f = \mathbb{R}$, $f(x) = \cosh x$.
(e) dom $f =]0, \infty[$, $f(x) = (\ln x)^2$.
(f) dom $f = \mathbb{R}$, $f(x) = \cosh(\exp x)$.
(g) dom $f = \mathbb{R}$, $f(x) = \tanh x$.
(h) dom $f = \mathbb{R}$, $f(x) = x + \cos x$.
(i) dom $f = \mathbb{R}$, $f(x) = \tanh(\sin x)$.
(j) dom $f = \mathbb{R}$, $f(x) = \sin(\tanh x)$. [GBD]

67. For each of the following formulae, what is the natural domain on which it defines a real function? Is it injective? If so, what is the domain of its inverse, and can you give a formula for the inverse function?
(a) $\ln(x + \sqrt{x^2 - 1})$.
(b) $\ln(x - \sqrt{x^2 - 1})$. [GC]

68. For the formula $\ln(x + \sqrt{x^2 - 1})$, what is the natural domain on which it defines a real function? Is it injective? If so, what is the domain of its inverse, and can you give a formula for the inverse function? [GCA]

69. For the formula $\ln(x - \sqrt{x^2 - 1})$, what is the natural domain on which it defines a real function? Is it injective? If so, what is the domain of its inverse, and can you give a formula for the inverse function? [GCB]

70. Find a solution of the equation $x \ln x = 1$, correct to two decimal places. Justify your answer by calculating $x \ln x$ for two values of x, one just above and one just below the answer you present. [HA]

71. Find the following limits (allowing $\pm\infty$); when no limit exists, say so plainly.
(a)$\lim_{x\to\infty}\frac{1}{x}(\ln x)^2 \quad \lim_{x\downarrow 0}\frac{x^2-\ln x}{x} \quad \lim_{x\to\infty}\frac{x^2-\ln x}{x}$

72. Find the following limits (allowing $\pm\infty$); when no limit exists, say so plainly.
(a)$\lim_{x\to\infty}\frac{1}{x}(\ln x)^2 \quad \lim_{x\downarrow 0}\frac{x^2-\ln x}{x} \quad \lim_{x\to\infty}\frac{x^2-\ln x}{x}$

73. Find a solution of the equation $x^2 = \sin\frac{1}{x}$, correct to two decimal places. Justify your answer by calculating $f(x) = x^2 - \sin\frac{1}{x}$ for two values of x, one just above and one just below the answer you present. [HB]

74. Suppose we know that f is a differentiable real function, defined (at least) on $[-1,1]$, and that its derivative f' is continuous. Suppose we know that $f(-1) = 1$, $f(0) = 0$ and $f(1) = 2$. What can we say about the set of values of f'? Give reasons for your answer. [HC]

75. Suppose we know that f is a differentiable real function, defined (at least) on $[1,3]$, and that its derivative f' is continuous. Suppose we know that $f(1) = 1$, $f(2) = 2$ and $f(3) = 3$. What can we say about the set of values of $f'(x)/2x$? Justify your answer. [HD]

76. Suppose we know that f is a twice-differentiable real function, defined (at least) on $[0,2]$, and that $f(0) = 1$, $f(1) = 0$ and $f(2) = 2$. Show that there is some $x \in]0,2[$ such that $f''(x) > \frac{3}{2}$. [HE]

77. Consider the formula $\arccos x$. (i) What is the natural domain to take when using this formula to define a real function f? (ii) What is the set of values of f? (iii) What is the domain of the derivative f' of f? (iv) Find a formula for $f'(x)$. (v) Sketch the function f. (vi) What is the domain of the function $f\circ\cos$? (vii) Sketch $f\circ\cos$. [HF]

78. Suppose that we know that f is a continuous function with domain $[0,1]$ (exactly) which is differentiable at every point of $]0,1[$, and that (i) $f(0) = 1$, $f(1) = 2$ (ii) $-1 \le f'(x) \le 2$ for every $x \in]0,1[$. What can we deduce about the set of values of f? [HG]

79. Find the following limits; when no limit exists, say so plainly.
(a) $\lim_{n\to\infty}\frac{1}{2}(1-(-1)^n)$ (b) $\lim_{n\to\infty}\frac{2n-1}{3n^2+2}(1-(-1)^n)$
(c) $\lim_{n\to\infty}\frac{x^n}{n!}$ where $x \in \mathbb{R}$ (d) $\lim_{n\to\infty}\sin(n\pi + \frac{1}{n})$.
(In part (c), does the answer depend on the value of x?) [IAA]

80. Find the following limits; when no limit exists, say so plainly.
(a) $\lim_{n\to\infty}\frac{n}{n+1}$ (b) $\lim_{n\to\infty}\frac{n^n}{n!}$
(c) $\lim_{n\to\infty}\sqrt[n]{x}$ where $x \in \mathbb{R}$.
(In part (c), does the answer depend on the value of x?) [IAB]

81. Find formulae for the following sums, and their limits as $n\to\infty$; if no limit exists, say so plainly.
(a) $\sum_{r=0}^{n} 2r-1$ (b) $\sum_{r=1}^{n}\frac{5}{2}4^{-r}$.
[IBAA]

82. As question 81, with $\sum_{r=0}^{n}\left(\frac{2x}{3}\right)^r$. (Does the answer depend on the value of x?) [IBAB]

83. As question 81, with
(a) $\sum_{r=1}^{n}\frac{1}{r(r+1)}$ (hint: partial fractions) (b) $\sum_{r=1}^{n} r(r-1)$ (c) $\sum_{r=0}^{n}(-1)^r$
(d) $\sum_{r=1}^{n}\frac{2r+1}{r^2(r+1)^2}$.
[IBB]

84. For each of the following series $\sum_{r=\dots}^{\infty} x_r$,

(i) find $\lim_{r\to\infty}\left|\frac{x_{r+1}}{x_r}\right|$, if it exists;

(ii) if the limit exists and is equal to 1, find (if you can) an α such that $\lim_{r\to\infty}|r^\alpha x_r|$ exists and is neither 0 nor ∞;

(iii) say whether the series is absolutely summable.

(a) $\sum_{r=0}^{\infty}\frac{r}{2^r}$ (b) $\sum_{r=0}^{\infty}\frac{(\sqrt{5}-1)^r}{r^2+1}$ (c) $\sum_{r=2}^{\infty}\frac{\sqrt{r}}{r^2-2r+1}$ (d) $\sum_{r=1}^{\infty}\frac{\sin r\theta}{r^2}$

(e) $\sum_{r=1}^{\infty}\frac{1}{\sqrt{r(r+1)}}$.

(In part (d), does the answer depend on θ?) [ICA]

85. As question 84, with

(a) $\sum_{r=0}^{\infty}\frac{1}{r(r+2)}$ (b) $\sum_{r=1}^{\infty}\frac{1}{r\sqrt{r}}$ (c) $\sum_{r=1}^{\infty}\sin(-\frac{1}{r})$ (d) $\sum_{r=0}^{\infty}\frac{(r!)^2}{(2r)!}$

(e) $\sum_{r=0}^{\infty}\frac{(-100)^r}{\sqrt{r!}}$ (f) $\sum_{r=0}^{\infty}2^{-r^2}r!$ (g) $\sum_{r=0}^{\infty}\frac{2^r}{r!}$.

[ICB]

86. As question 84, with

(a) $\sum_{r=2}^{\infty}\frac{1}{\ln r}$ (b) $\sum_{r=1}^{\infty}r^{-(1+\frac{1}{r})}$ (c) $\sum_{r=1}^{\infty}\frac{1}{\sqrt[r]{r!}}$ (d) $\sum_{r=1}^{\infty}(1-\cos\frac{1}{r})$.

[ID]

87. Find the radii of convergence of the following power series.

(a) $\sum_{r=0}^{\infty}(-1)^r x^r$ (b) $\sum_{r=0}^{\infty}\frac{r^2+r+1}{r+2}x^r$ (c) $\sum_{r=0}^{\infty}\frac{2^r}{\sqrt{r^2+1}}x^r$

(d) $\sum_{r=0}^{\infty}r!x^r$ (e) $\sum_{r=0}^{\infty}3^{-r}x^r$ (f) $\sum_{r=0}^{\infty}\frac{r^r}{r!}x^r$

(g) $\sum_{r=0}^{\infty}\frac{r!}{r^r}x^r$ (h) $\sum_{r=0}^{\infty}2^r x^{2r}$ (i) $\sum_{r=0}^{\infty}3^{-r}x^{3r}$

(j) $\sum_{r=0}^{\infty}\frac{(1-\sqrt{5})^r}{r^2-2}x^{2r}$ (k) $\sum_{r=0}^{\infty}x^r\cos r\theta$ (l) $\sum_{r=0}^{\infty}\frac{(2r)!}{(r!)^2}x^r$

(m) $\sum_{r=0}^{\infty}\frac{r!(2r)!}{(3r)!}x^r$

(In part (k), does the answer depend on θ?) [JA]

88. Find the first three non-zero terms in the Maclaurin series of each of the following functions.

(a) $\ln(1-x)$ (b) $\frac{1}{1-x^2}$ (c) $\arctan\frac{1}{2}x$

(d) $\tan x$ (e) $\sin x^2$ (f) $\sin x\cos x$

(g) $\sinh 3x$ (h) $\exp(x+x^2)$ (i) $\cosh(x+1)$

(j) $\sqrt{1-x^2}$

(*hint for part(d)*: $\tan'(x) = 1+\tan^2 x$; find expressions for $\tan''(x)$, etc., in terms of $t = \tan x$ and $u = \tan'(x)$.)

[JB]

89. For each of the following functions f, find the first three non-zero terms in its Taylor series centered at the given point x_0:

(a) $f(x) = \sin x$, $x_0 = 1$ (b) $f(x) = \ln x$, $x_0 = 1$

(c) $f(x) = \frac{1}{x}$, $x_0 = -1$ (d) $f(x) = x^3 - x$, $x_0 = 1$.

[JC]

90. Find the sums of the following series:

(a) $\sum_{r=1}^{\infty} \frac{1}{r(r+2)}$ (b) $\sum_{r=0}^{\infty} \frac{2^r}{r!}$

(c) $\sum_{r=0}^{\infty}(-1)^r x^r$ for $|x|<1$ (d) $\sum_{r=0}^{\infty} x^r \cos r\theta$ for $|x|<1$

(e) $\sum_{r=1}^{\infty} r x^{r-1}$ for $|x|<1$ (f) $\sum_{r=0}^{\infty}(r+1)x^r$ for $|x|<1$

(g) $\sum_{r=0}^{\infty} r x^r$ for $|x|<1$ (h) $\sum_{r=0}^{\infty}(r+1)(r+2)x^r$ for $|x|<1$.

(*Hints*: for (a), use partial fractions; for (d), $x^r \cos r\theta = \mathcal{R}\mathrm{e}((xe^{i\theta})^r)$; for (e), differentiate $\sum_{r=0}^{\infty} x^r$.) [JDA]

91. Find the sums of the following series:

(a) $\sum_{r=0}^{\infty} r^2 x^r$ for $|x|<1$ (b) $\sum_{r=0}^{\infty} \frac{r}{2^r}$

(c) $\sum_{r=0}^{\infty} \frac{(-100)^r}{(2r)!}$ (d) $\sum_{r=0}^{\infty}(-1)^r x^{2r}$ for $|x|<1$

(e) $\sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} x^{2r+1}$ for $|x|<1$ (f) $\sum_{r=0}^{\infty} \frac{1}{2r+1} x^{2r+1}$ for $|x|<1$

(g) $\sum_{r=0}^{\infty} \frac{1}{4r+1} x^{4r}$ for $|x|<1$ (h) $\sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} x^r$ for $|x|<1$

(*Hint*: for (e), differentiate the series given.) [JDB]

92. How many people do you need to invite to a party in order that there should be a 95% chance that two of them have the same birthday? (You are not allowed to invite twins.) [KA]

93. Suppose two pawns are placed on a chessboard at random. What is the probability that you will be able to cover the remaining 62 squares by a set of 31 dominoes, each domino occupying two adjacent squares? [KB]

94. Find the coefficients of the Maclaurin series for $\tan x$ up to and including the coefficient of x^{10}. What do you observe? Can you explain it? [KC]

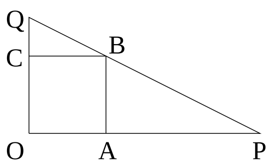
95. In the country of Xanadu, the authorities are much concerned to keep the population from growing, and if possible to reduce it. On the other hand, daughters are especially valued. A law is therefore passed that women may have as many children as they wish up to and including their first daughter, but must then stop. What proportion of the children in Xanadu will be girls? [KD]

96. In the four-door Monty Hall game, you are presented with four identical doors. Behind one of them is a prize. You have to choose one of the doors (without opening it). The master of ceremonies (who knows where the prize is) then opens one of the *other* doors, not the one with the prize behind it, and invites you to choose again; you are permitted to stay with your first choice if you wish. The MC then opens a second door (again, not the one you have currently chosen, and not the one with the prize) and this time invites you to open one of the two remaining doors; you win the prize if it is behind the door you open.

What should you do to get the best chance of winning, and what is your chance of the prize? [KE]

97. What is the largest number of regions into which the plane can be divided by four straight lines? [KF]

98. In the diagram



$OABC$ is a square of unit side and the line PQ has length α. What are the lengths OP, OQ? [KG]

99. In Erewhonese, nouns and verbs play much the same roles as they do in English or Greek or Chinese, but there are no words for ‘a’ or ‘the’ in their English meanings, and prepositions like ‘on’ and quantifiers like ‘some’ are indicated by inflexions, that is, variations in the form of the noun affected. Pronouns like ‘he’ are commonly omitted. Moreover, the rules for ordering words in a sentence, while perfectly straightforward, are different from most other languages.

Here are seven sentences in Erewhonese:

(a) tetrozor chup-chas fila-chan.
(b) dingesor fila-cha bap.
(c) sesnazor chup-chas filaf.
(d) sqensesor fila-cha mutan.
(e) tetrosor chupi.
(f) ceclisor fila-cha chup-chaf.
(g) dingezor muta fila’q chup-chan.

These sentences may be translated into English as the following, but in a different order.

(1) A farmer steps on a dog.
(2) A dog and a cow bite a dentist.
(3) Some dentists take some cows.
(4) A farmer sues some dentists.
(5) A farmer takes a bite of bread.
(6) He hurts some teeth.
(7) Some dentists hurt a farmer.

Which Erewhonese sentence matches each English sentence? [KI]