Calculus I Answers

1. (a) False. Should be $\ln(xy) = \ln x + \ln y$. Then works provided $x > 0$ and $y > 0$.
   (b) True.
   (c) False, should be $\cos(x + y) = \cos x \cos y - \sin x \sin y$.
   (d) True.
   (e) True, provided $x \neq 1$.
   (f) False. Should be $\frac{n(n+1)}{2}$.

2. (a) False.
   (b) False.
   (c) True.
   (d) False.
   (e) True.
   (f) True.
   (g) True.
   (h) True.

3. (a) 0 exactly. (To get these marks, you need both the '0' and the 'exactly'.)
(b) -0.66 ± 0.01. (To get these marks, you need both the approximate calculation and an estimate of the error. More accurate calculations, e.g., '-0.656986599 ± 10^{-9}' equally acceptable.)
(c) 0.66 ± 0.01.
(d) -0.34 ± 0.01 (or -0.346573590 ± 10^{-9}, etc.).
(e) No such animal.
(f) 2980 ± 1 (or 2980.957987 ± 10^{-6}, etc.).
(g) 403 ± 1 (or 403.4287935 ± 10^{-7}, etc.).
(h) -0.5 exactly.
(i) 1.73 ± 0.01 (or 1.732050808 ± 10^{-9}, etc.).

4. (a) True.
(b) False. Try $a = c = -1$, $b = d = 0$.
(c) True.
(d) True.
(e) True.
(f) False. Try $a = 0$, $b = -1$.

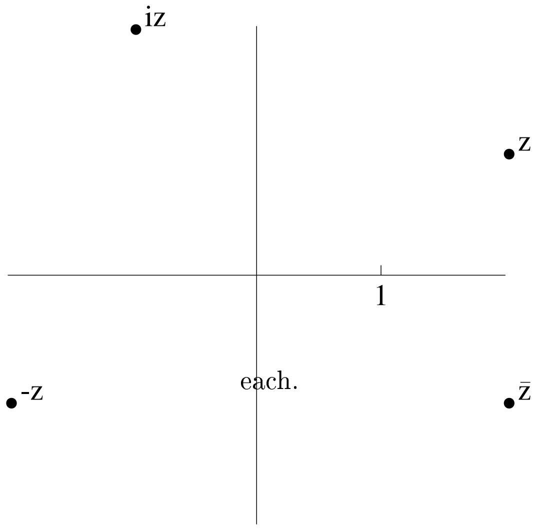
5. (a) $w + z$ is D .
wz is A. ($z \simeq -i$, so multiplication by z is approximately a clockwise rotation through a right angle).
$w - z$ is C.
$\frac{w}{z}$ is E. (Division by z is multiplication by $\frac{1}{z} \simeq i$, so is approximately an anticlockwise rotation through a right angle.)
$\frac{1}{wz}$ is F. (To get from A to F, we take the reciprocal of the modulus and reflect in the real axis.)

(b) D has smallest modulus,
   C has greatest.

(c) $\arg(wz) = \arg w + \arg z$,
   $\arg(\frac{w}{z}) \neq \arg w - \arg z$,
   $\arg(\frac{z}{w}) \neq \arg z - \arg w$ (because $\arg z \simeq -\frac{\pi}{2}$, $\arg w \simeq \frac{3\pi}{4}$ so $\arg w - \arg z > \pi$, $\arg z - \arg w < -\pi$ and must be wrong).

6. $-3-2i$;
$7-4i$;
$-7+17i$(allow if either real or imaginary part is correct);
$-\frac{1}{2}+\frac{1}{2}i$(allow if either real or imaginary part is correct).

7.



each.

8. (a) i (or $0+1i$).
(b) $-\frac{7}{41}-\frac{22}{41}i$.
(c) $-\frac{7}{41}+\frac{22}{41}i$.
(d) 1 (or $1+0i$).

9. (a) 5, 2.21.
(b) 5, -2.21
(c) 5, 0.
(d) 0.2, -2.21.

10. (a) True.
(b) False, $w=z=i$. (Of course there are innumerable possibilities here – check with your class teacher.)
(c) False , $w=z=i$.
(d) True.
(e) True.
(f) False, $z=-1$.

11. (a) If $2i=(a+bi)^2=a^2-b^2+2abi$, we must have $a^2-b^2=0$ and $2ab=2$. Now $a=\pm b$ and $\pm a^2=1$, so $a=\pm 1$ and $b=\pm 1$. Thus the two solutions are $1+i$, $-1-i$.
(b) The standard formula for quadratics still works:

$$\begin{aligned} z &= \frac{-2\pm\sqrt{2^2-4.(1-\frac{i}{2}).1}}{2(1-\frac{i}{2})} \\ &= \frac{-2\pm\sqrt{4-4+2i}}{2-i)} \\ &= \frac{-2\pm(1+i)}{2-i)}. \end{aligned}$$

So the two solutions are

$$z=\frac{-2+1+i}{2-i}=-\frac{3}{5}+\frac{1}{5}i, \quad z=\frac{-2-1-i}{2-i}=-1-i.$$

12. (a) The imaginary axis.

(b) The ‘unit circle’, that is, the circle with centre 0 and radius 1.

(c) The straight line running NW-SE through 0, that is, the perpendicular bisector of the line segment from 1 to $-i$.

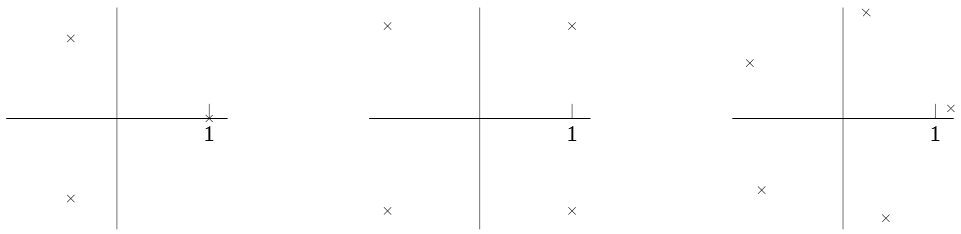
(d) The circle centre 1 radius 1.

Remember! a *modulus* is a *distance*.

13.

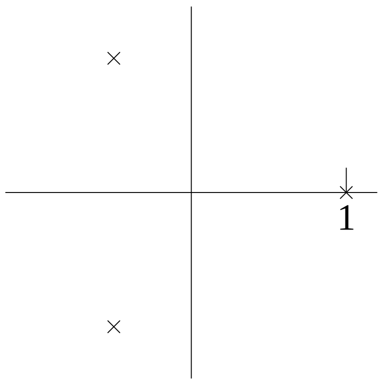
	real part	imaginary part	modulus	argument
(a) $z^3 = 1$	1	0	1	0
	-0.5	0.87	1	2.09
	-0.5	-0.87	1	-2.09
(b) $z^4 = -4$	1	1	1.41	0.79
	-1	1	1.41	2.36
	-1	-1	1.41	-2.36
	1	-1	1.41	-0.79
(c) $z^5 = 2 + i$	1.17	0.11	1.17	0.09
	0.26	1.15	1.17	1.35
	-1.01	0.60	1.17	2.61
	-0.88	-0.78	1.17	-2.42
	0.46	-1.08	1.17	-1.16.

(In part (c), $|2+i| = \sqrt{5} \simeq 2.36$, $\arg(2+i) = \arctan\frac{1}{2} \simeq 0.46$.)



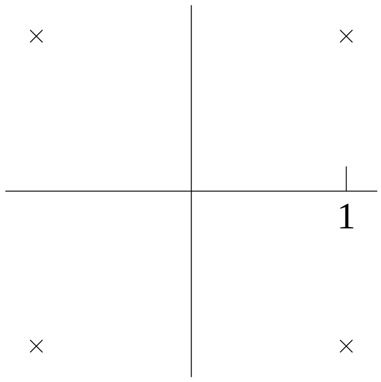
14.

real part	imaginary part	modulus	argument
1	0	1	0
-0.5	0.87	1	2.09
-0.5	-0.87	1	-2.09



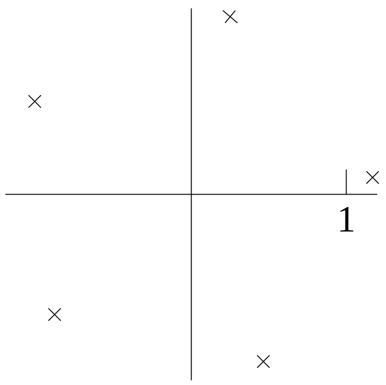
15.

real part	imaginary part	modulus	argument
1	1	1.41	0.79
-1	1	1.41	2.36
-1	-1	1.41	-2.36
1	-1	1.41	-0.79



16.

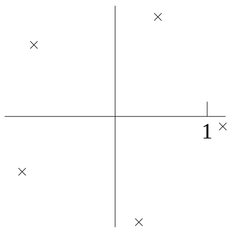
real part	imaginary part	modulus	argument
1.17	0.11	1.17	0.09
0.26	1.15	1.17	1.35
-1.01	0.60	1.17	2.61
-0.88	-0.78	1.17	-2.42
0.46	-1.08	1.17	-1.16.

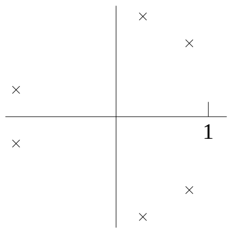


($|2+i| = \sqrt{5} \simeq 2.36$, $\arg(2+i) = \arctan \frac{1}{2} \simeq 0.46$.)

17.

	real part	imaginary part	modulus	argument
(a) $z^5 = 2 - i$	1.17	-0.11	1.17	-0.09
	0.26	-1.15	1.17	-1.35
	-1.01	-0.60	1.17	-2.61
	-0.88	0.78	1.17	2.42
	0.46	1.08	1.17	1.16
(b) $z^6 + 2z^3 + 2 = 0$				
$(z^3 = -1 + i)$	0.79	0.79	1.12	0.79
	-1.08	0.29	1.12	2.88
	0.29	-1.08	1.12	-1.31
$(z^3 = -1 - i)$	0.79	-0.79	1.12	-0.79
	-1.08	-0.29	1.12	-2.88
	0.29	1.08	1.12	1.31





18.

(a) $\frac{1}{2}x - \frac{1}{4}\sin 2x$	$\frac{1}{2}(1 - \cos 2x)$ *or* $\sin^2 x$
(b) $\cos x^2$	$-2x \sin x^2$
(c) $\tan x - x$	$\frac{1}{\cos^2 x} - 1$ *or* $\tan^2 x$
(d) $\exp 2x \sin 3x$	$\exp 2x(2\sin 3x + 3\cos 3x)$
(e) $\exp(\frac{1}{x})$	$-\frac{1}{x^2}\exp(\frac{1}{x})$
(f) $x \ln x - x$	$\ln x$
(g) $\cosh x = \frac{1}{2}(\exp(x) + \exp(-x))$	$\sinh x$
(h) $\sinh x = \frac{1}{2}(\exp(x) - \exp(-x))$	$\cosh x$
(i) $\tanh x = \frac{\sinh x}{\cosh x}$	$\frac{1}{\cosh^2 x}$ *or* $1 - \tanh^2 x$
(j) $\ln(x + \sqrt{x^2 - 1})$	$\frac{1}{\sqrt{x^2-1}}$
(k) $\frac{1}{2}\ln(\frac{1+x}{1-x})$	$\frac{1}{1-x^2}$
(l) $x^\alpha = \exp(\alpha \ln x)$	$\alpha x^{\alpha-1}$
(m) $\alpha^x = \exp(x \ln \alpha)$	$\alpha^x \ln \alpha$
(n) $\ln(\lvert x\rvert)$	$\frac{1}{x}$
(o) $-\ln(\lvert\cos x\rvert)$	$\tan x$
(p) $\ln(\frac{1+\sin x}{\lvert\cos x\rvert})$	$\frac{1}{\cos x}$.

Acceptable alternatives: e^{2x}, $e^{1/x}$ in place of $\exp 2x$, $\exp\frac{1}{x}$ etc.
$\frac{1}{2}(e^x - e^{-x})$ in place of $\sinh x$, $\frac{1}{2}(e^x + e^{-x})$ in place of $\cosh x$.
$\frac{4}{e^{2x}+2+e^{-2x}}$ in place of $\frac{1}{\cosh^2 x}$.

19.

(a) $\ln(x - \sqrt{x^2-1})$ $\quad -\frac{1}{\sqrt{x^2-1}}$

(b) $\ln(|\sin x|)$ $\quad \cot x$

(c) $\ln(x + \sqrt{x^2+1})$ $\quad \frac{1}{\sqrt{x^2+1}}$

(d) $\sqrt{\frac{1}{x} - 1}$ $\quad -\frac{1}{2x\sqrt{x-x^2}}$.

20.
(a) $|x^2 - 1|$

domain:	$\mathbb{R}$
zeroes:	± 1
stationary points:	0
local maxima:	0
global minima:	± 1
set of values:	$[0, \infty[$.

Alternative expressions often permissible; check with class teacher, especially in reference to the 'set of values'.

(b) $\frac{x}{|x|}$

domain:	$\mathbb{R} \setminus \{0\}$
zeroes:	none
stationary points:	all $x \neq 0$
local maxima:	all $x \neq 0$
global minima:	all $x < 0$
set of values:	$\{-1, 1\}$.

Allow a blank to count as 'none'.

(c) $x \ln x$

domain:	$]0, \infty[$
zeroes:	1
stationary points:	$\frac{1}{e}$
local maxima:	none
global minima:	$\frac{1}{e}$
set of values:	$\left[-\frac{1}{e}, \infty\right[$.

(d) $\exp(-\frac{1}{2}x^2)$

domain:	$\mathbb{R}$
zeroes:	none
stationary points:	0
local maxima:	0
global minima:	none
set of values:	$]0,1]$.

(e) $\frac{1}{x^2-1}$

domain:	$\mathbb{R}\setminus\{-1,1\}$
zeroes:	none
stationary points:	0
local maxima:	0
global minima:	none
set of values:	$]-\infty,-1]\cup]0,\infty[$.

(f) $\tanh x$

domain:	$\mathbb{R}$
zeroes:	0
stationary points:	none
local maxima:	none
global minima:	none
set of values:	$]-1,1[$.

(g) $\sin x + \cos x = \sqrt{2}\cos(x-\frac{\pi}{4})$

domain:	$\mathbb{R}$
zeroes:	$k\pi-\frac{\pi}{4}$, any $k\in\mathbb{Z}$
stationary points:	$k\pi+\frac{\pi}{4}$, any $k\in\mathbb{Z}$
local maxima:	$2k\pi+\frac{\pi}{4}$, any $k\in\mathbb{Z}$
global minima:	$2k\pi-\frac{3\pi}{4}$, any $k\in\mathbb{Z}$
set of values:	$[-\sqrt{2},\sqrt{2}]$.

(Note for class teachers: In first five items of each part, insist on exactly correct answers, but be tolerant of eccentric formulations. In last item, be generous with the first mark, stingy with the second.)

21. (a) $|x^2 - 1|$

domain:	$\mathbb{R}$
zeroes:	± 1
stationary points:	0
local maxima:	0
global minima:	± 1
set of values:	$[0, \infty[$.

Alternative expressions often permissible; check with class teacher, especially in reference to the 'set of values'.

(b) $\frac{x}{|x|}$

domain:	$\mathbb{R} \setminus \{0\}$
zeroes:	none
stationary points:	all $x \neq 0$
local maxima:	all $x \neq 0$
global minima:	all $x < 0$
set of values:	$\{-1, 1\}$.

Allow a blank to count as 'none'.

22. (a) $\frac{1}{x^2-1}$

domain:	$\mathbb{R} \setminus \{-1, 1\}$
zeroes:	none
stationary points:	0
local maxima:	0
global minima:	none
set of values:	$]-\infty, -1] \cup]0, \infty[$.

(b) $\tanh x$

domain:	$\mathbb{R}$
zeroes:	0
stationary points:	none
local maxima:	none
global minima:	none
set of values:	$]-1, 1[$.

(c) $\sin x + \cos x = \sqrt{2}\cos(x - \frac{\pi}{4})$

domain:	$\mathbb{R}$
zeroes:	$k\pi - \frac{\pi}{4}$, any $k \in \mathbb{Z}$
stationary points:	$k\pi + \frac{\pi}{4}$, any $k \in \mathbb{Z}$
local maxima:	$2k\pi + \frac{\pi}{4}$, any $k \in \mathbb{Z}$
global minima:	$2k\pi - \frac{3\pi}{4}$, any $k \in \mathbb{Z}$
set of values:	$[-\sqrt{2}, \sqrt{2}]$.

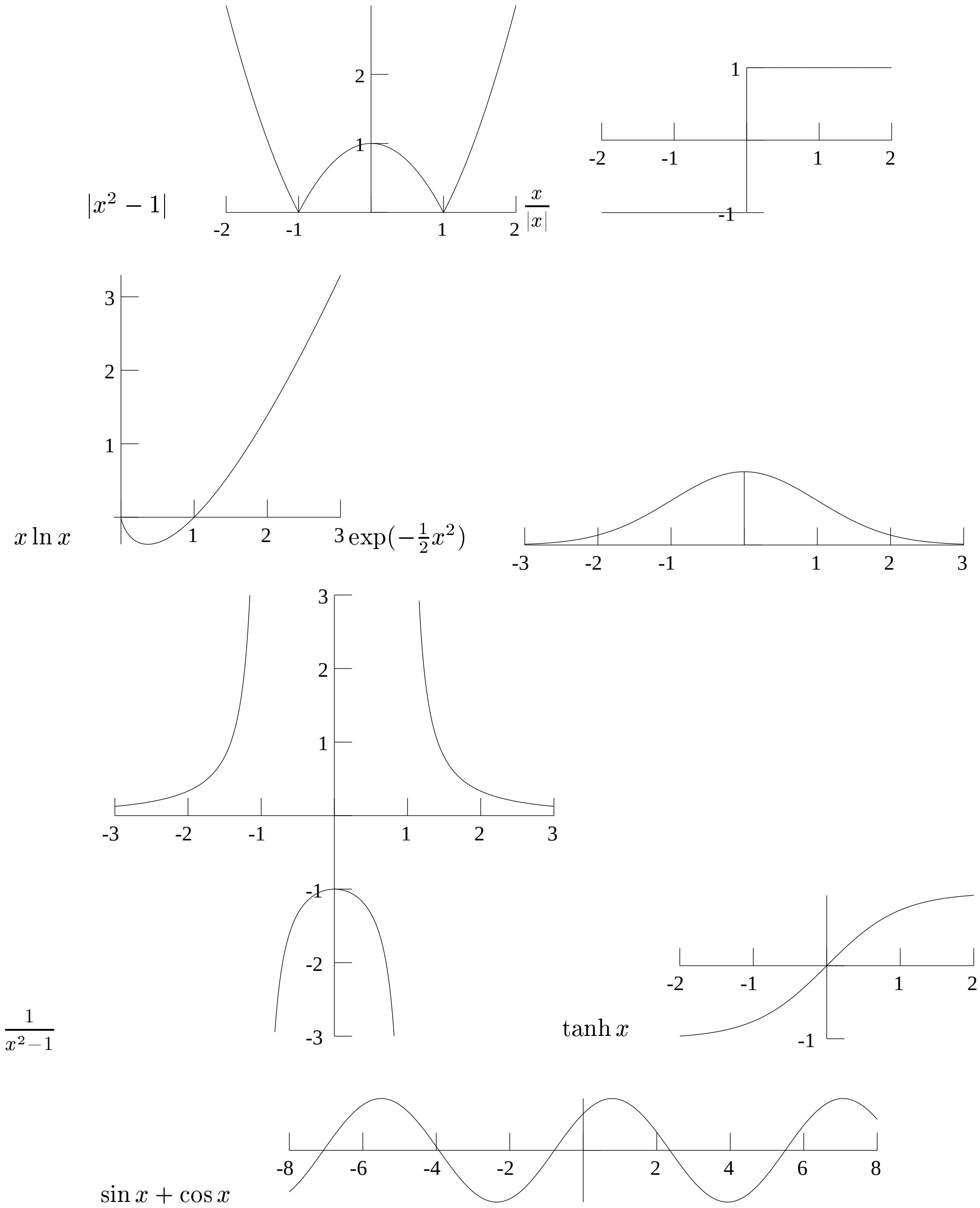
23. (a) $x \ln x$

domain:	$]0,\infty[$
zeroes:	1
stationary points:	$\frac{1}{e}$
local maxima:	none
global minima:	$\frac{1}{e}$
set of values:	$\left[-\frac{1}{e},\infty\right[$.

(b) $\exp(-\frac{1}{2}x^2)$

domain:	$\mathbb{R}$
zeroes:	none
stationary points:	0
local maxima:	0
global minima:	none
set of values:	$]0,1]$.

24.



25.

(a) $x^4 - 2x^2 + 1 = (x^2 - 1)^2$

domain:	$\mathbb{R}$
zeroes:	± 1
stationary points:	$0, \pm\sqrt{\frac{2}{3}}$
local maxima:	0
global minima:	$\pm\sqrt{\frac{2}{3}}$
set of values:	$[0, \infty[$

(b) $x - \ln x$

domain:	$]0, \infty[$
zeroes:	none
stationary points:	1
local maxima:	none
global minima:	1
set of values:	$[1, \infty[$

(c) $\ln \sin x$

domain:	$\{x : \sin x > 0\} = \{x : \exists\, k \in \mathbb{Z}, 2k\pi < x < (2k+1)\pi\}$
zeroes:	$2k\pi + \frac{\pi}{2}$, any $k \in \mathbb{Z}$
stationary points:	$2k\pi + \frac{\pi}{2}$, any $k \in \mathbb{Z}$
local maxima:	$2k\pi + \frac{\pi}{2}$, any $k \in \mathbb{Z}$
global minima:	none
set of values:	$]-\infty, 0]$

(d) $\ln |\cos x|$

domain:	$\{x : \cos x \neq 0\} = \{x : x$ is not an odd multiple of $\frac{\pi}{2}\}$
zeroes:	multiples of π
stationary points:	multiples of π
local maxima:	multiples of π
global minima:	none
set of values:	$]-\infty, 0]$

(e) xe^x

domain:	$\mathbb{R}$
zeroes:	0
stationary points:	-1
local maxima:	none
global minima:	-1
set of values:	$\left[-\frac{1}{e}, \infty\right[$

(f) $\sinh x$

domain:	$\mathbb{R}$
zeroes:	0
stationary points:	none
local maxima:	none
global minima:	none
set of values:	$\mathbb{R}$

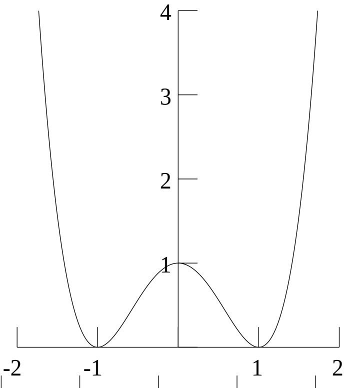
(g) $\cosh x$

domain:	$\mathbb{R}$
zeroes:	none
stationary points:	0
local maxima:	none
global minima:	0
set of values:	$[1, \infty[$

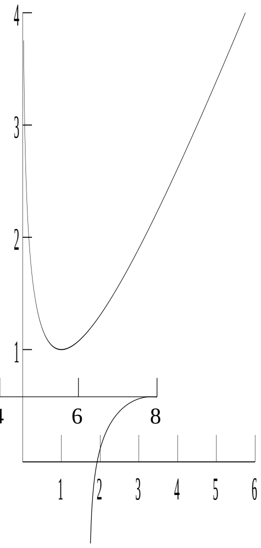
(h) $\frac{\ln x}{x}$

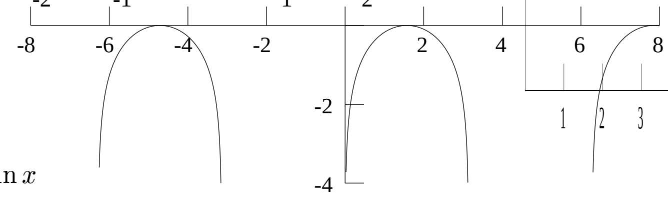
domain:	$]0, \infty[$
zeroes:	1
stationary points:	e
local maxima:	e
global minima:	none
set of values:	$\left]-\infty, \frac{1}{e}\right]$

$x^4 - 2x^2 + 1$

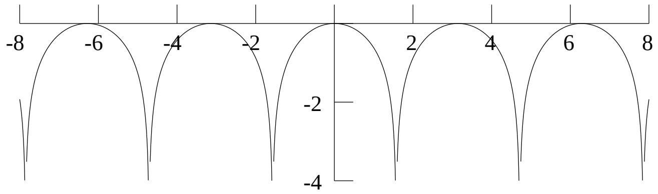


$x - \ln x$

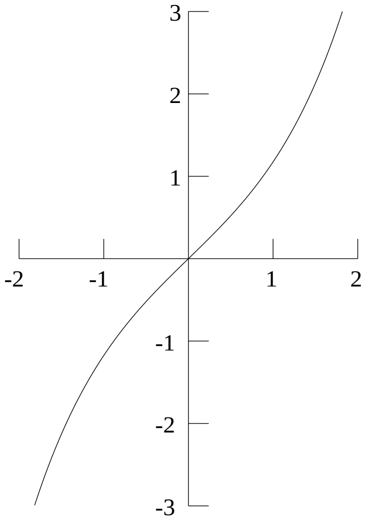


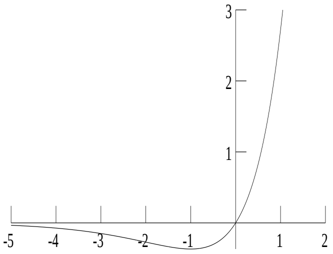


$\ln \sin x$



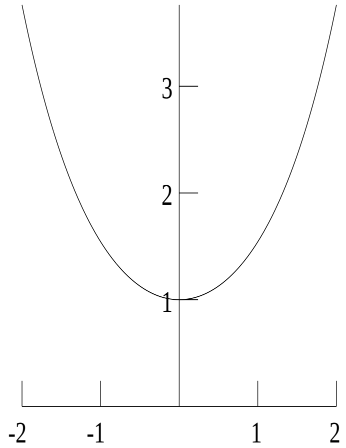
$\ln |\cos x|$





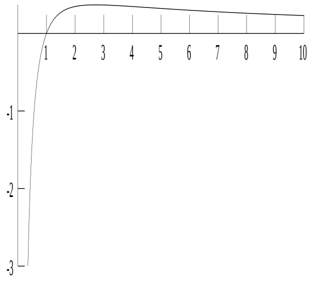
xe^x

$\sinh x$



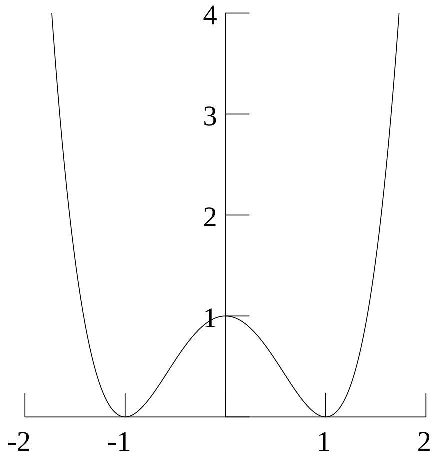
$\cosh x$

$\frac{\ln x}{x}$



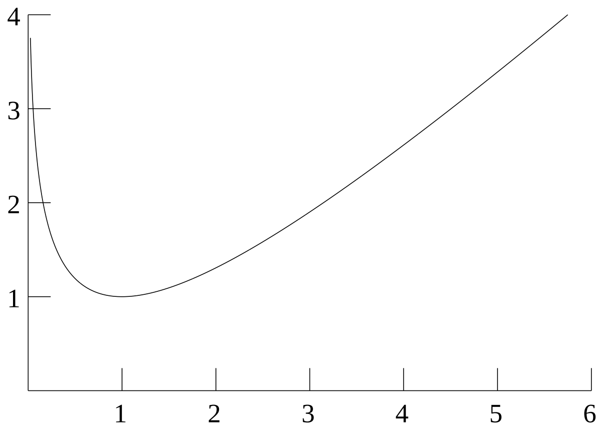
26. (a) $x^4 - 2x^2 + 1 = (x^2-1)^2$

domain:	$\mathbb{R}$
zeroes:	± 1
stationary points:	$0, \pm\sqrt{\frac{2}{3}}$
local maxima:	0
global minima:	$\pm\sqrt{\frac{2}{3}}$
set of values:	$[0, \infty[$



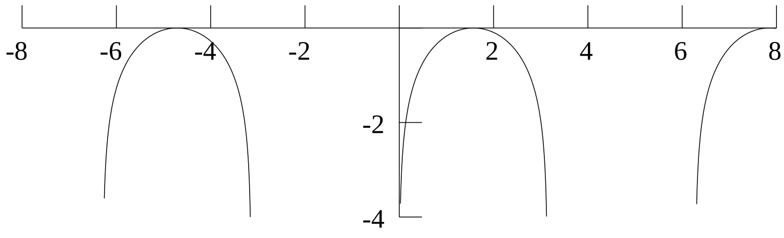
(b) $x - \ln x$

domain:	$]0, \infty[$
zeroes:	none
stationary points:	1
local maxima:	none
global minima:	1
set of values:	$[1, \infty[$



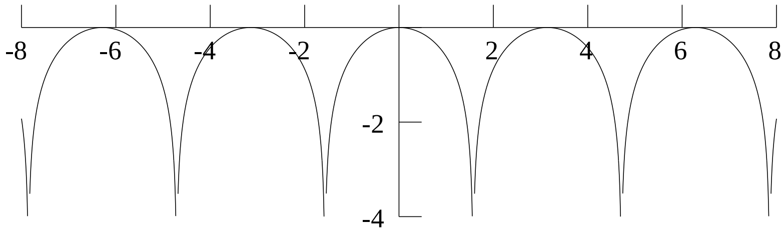
27. (a) $\ln \sin x$

domain:	$\{x : \sin x > 0\} = \{x : \exists\, k \in \mathbb{Z},\, 2k\pi < x < (2k+1)\pi\}$
zeroes:	$2k\pi + \frac{\pi}{2}$, any $k \in \mathbb{Z}$
stationary points:	$2k\pi + \frac{\pi}{2}$, any $k \in \mathbb{Z}$
local maxima:	$2k\pi + \frac{\pi}{2}$, any $k \in \mathbb{Z}$
global minima:	none
set of values:	$]-\infty, 0]$



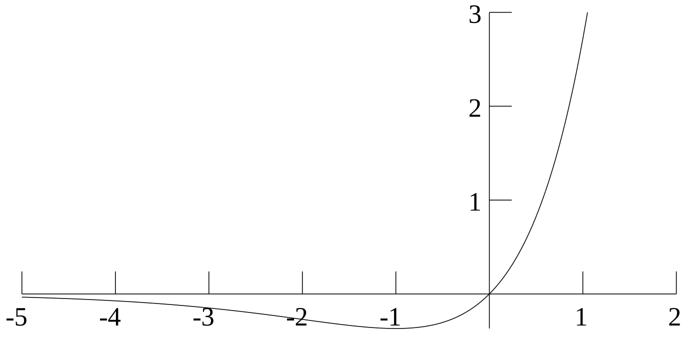
(b) $\ln|\cos x|$

domain:	$\{x : \cos x \neq 0\} = \{x : x \text{ is not an odd multiple of } \frac{\pi}{2}\}$
zeroes:	multiples of π
stationary points:	multiples of π
local maxima:	multiples of π
global minima:	none
set of values:	$]-\infty, 0]$



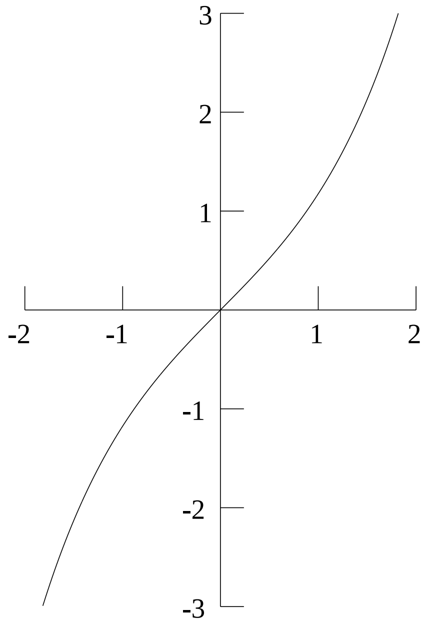
28. (a) xe^x

domain:	$\mathbb{R}$
zeroes:	0
stationary points:	-1
local maxima:	none
global minima:	-1
set of values:	$\left[-\frac{1}{e}, \infty\right[$



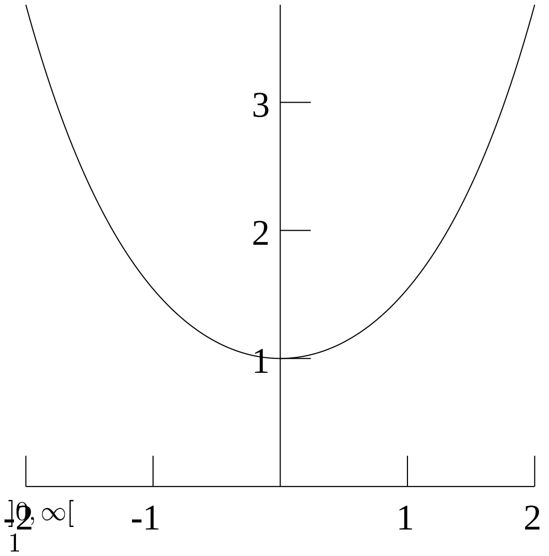
(b) $\sinh x$

domain:	$\mathbb{R}$
zeroes:	0
stationary points:	none
local maxima:	none
global minima:	none
set of values:	$\mathbb{R}$



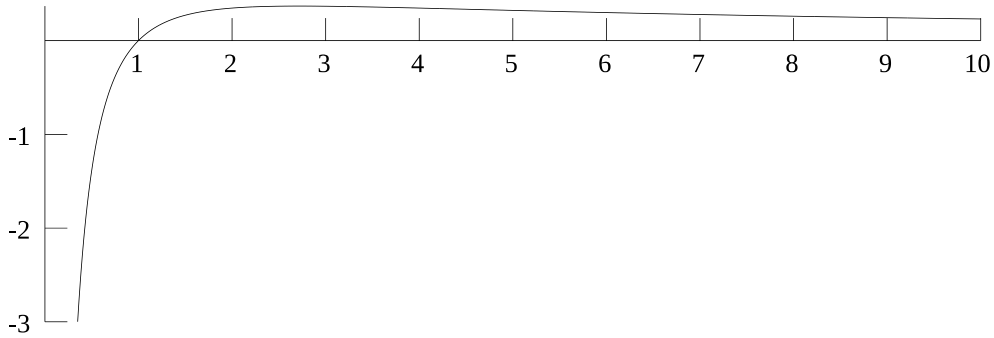
29. (a) $\cosh x$

domain:	$\mathbb{R}$
zeroes:	none
stationary points:	0
local maxima:	none
global minima:	0
set of values:	$[1, \infty[$

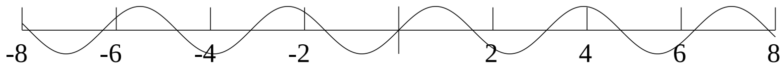


(b) $\frac{\ln x}{x}$

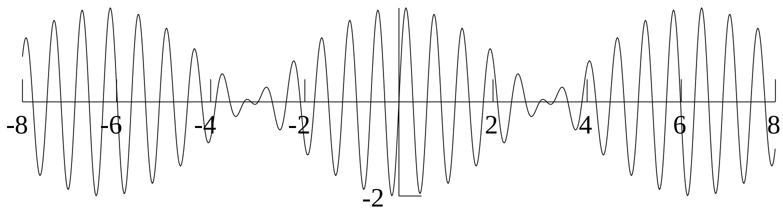
domain:	$]0, \infty[$
zeroes:	1
stationary points:	e
local maxima:	e
global minima:	none
set of values:	$]-\infty, \frac{1}{e}]$



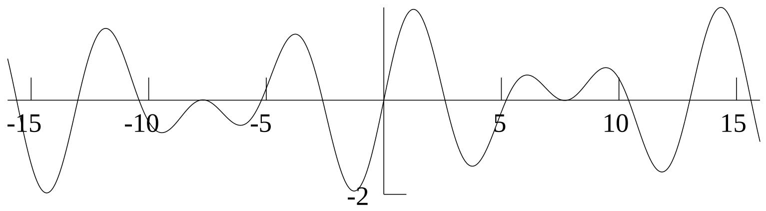
30. (a)



$\sin x \cos x = \frac{1}{2} \sin 2x$



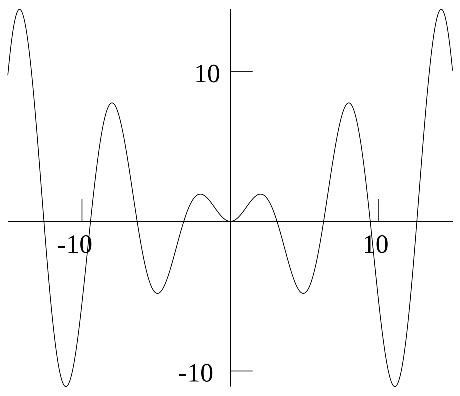
$\sin 10x + \sin 11x = 2 \sin \frac{21x}{2} \cos \frac{x}{2}$



$\sin x + \sin x\sqrt{2}$

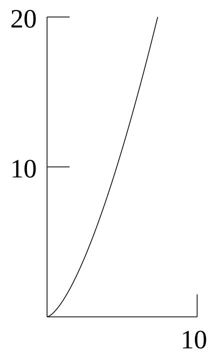
31. (a) $\operatorname{dom}(f\circ g) = \mathbb{R} \setminus \{0\}$, set of values $[-1,1]$.

$\operatorname{dom}(g\circ f) = \{x : \sin x \neq 0\}$ *or* $\{x : x \text{ is not a multiple of } \pi\}$ *or* $\{x : \frac{x}{\pi} \notin \mathbb{Z}\}$,
set of values $]-\infty,1] \cup [1,\infty[$ *or* $\mathbb{R} \setminus]-1,1[$ *or* $\{x : |x| \geq 1\}$.
$\mathbb{R}$.



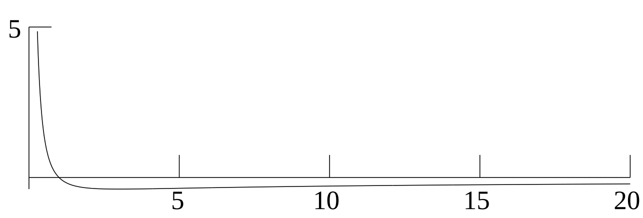
$x \sin x$

(b) $\operatorname{dom}(f\circ g) = [0,\infty[$, set of values $[0,\infty[$.
$\operatorname{dom}(g\circ f) = \mathbb{R}$, set of values $[0,\infty[$.
$]0,\infty[$.



$\frac{x^2}{\sqrt{x}} = x^{3/2}$

(c) $\operatorname{dom}(f\circ g) =]0,\infty[$, set of values $]-1,\infty[$.
$\operatorname{dom}(g\circ f) =]0,1]$, set of values $[0,\infty[$.
$\left[-\frac{2}{3\sqrt{3}},\infty\right[$.

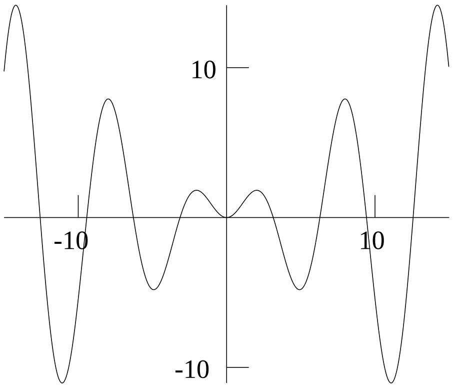


$\frac{\frac{1}{x}-1}{\sqrt{x}} = \frac{1}{x\sqrt{x}} - \frac{1}{x}$

32. $\mathrm{dom}(f \circ g) = \mathbb{R} \setminus \{0\}$, set of values $[-1, 1]$.

$\mathrm{dom}(g \circ f) = \{x : \sin x \neq 0\}$ *or* $\{x : x$ is not a multiple of $\pi\}$ *or* $\{x : \frac{x}{\pi} \notin \mathbb{Z}\}$,
set of values $]-\infty, -1] \cup [1, \infty[$ *or* $\mathbb{R} \setminus]-1, 1[$ *or* $\{x : |x| \geq 1\}$.

$\mathbb{R}$.

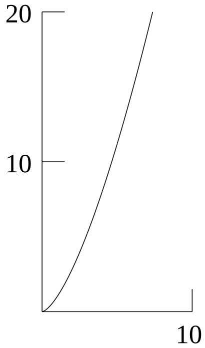


$x \sin x$

33. $\mathrm{dom}(f \circ g) = [0, \infty[$, set of values $[0, \infty[$.

$\mathrm{dom}(g \circ f) = \mathbb{R}$, set of values $[0, \infty[$.

$]0, \infty[$.

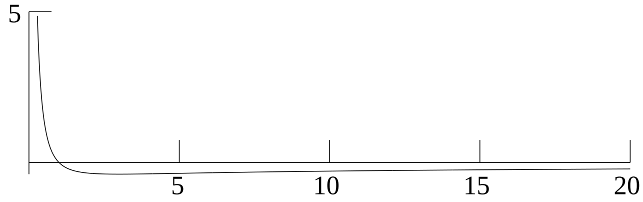


$\frac{x^2}{\sqrt{x}} = x^{3/2}$

34. $\operatorname{dom}(f\circ g) = \,]0,\infty[$, set of values $]-1,\infty[$.

$\operatorname{dom}(g\circ f) = \,]0,1]$, set of values $[0,\infty[$.

$\left[-\frac{2}{3\sqrt{3}},\infty\right[$.

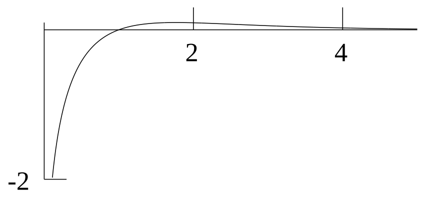


$$\frac{\frac{1}{x}-1}{\sqrt{x}} = \frac{1}{x\sqrt{x}} - \frac{1}{\sqrt{x}}$$

35. (a) $\mathrm{dom}(f{\circ}g) = \mathbb{R}$, set of values $\mathbb{R}$ if $\alpha \neq 0$, $\{0\}$ if $\alpha = 0$.

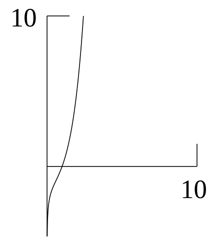
$\mathrm{dom}(g{\circ}f) =]0, \infty[$, set of values $]0, \infty[$ if $\alpha \neq 0$, $\{1\}$ if $\alpha = 0$.

(if $\alpha > 0$) $\left]-\infty, \frac{\ln\gamma}{\exp(\alpha\gamma)}\right]$ where $\gamma > 1$ is the unique solution of the equation $\gamma \ln \gamma = \frac{1}{\alpha}$. (The function $x \mapsto \dfrac{\ln x}{\exp(\alpha x)}$ has a global maximum at γ.)



$$\frac{\ln x}{e^x}$$

(if $\alpha \leq 0$) $\mathbb{R}$.

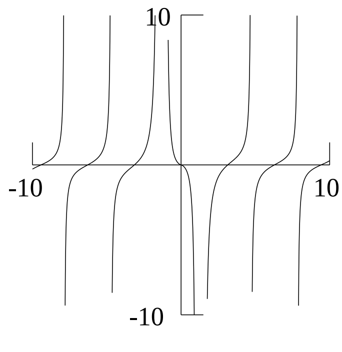


$$\frac{\ln x}{e^{-x}} = e^x \ln x$$

(b) $\mathrm{dom}(f{\circ}g) = \{x : x \neq 0,\ x \neq \pm\exp(k\pi + \frac{\pi}{2}) \text{ for every } k \in \mathbb{Z}\}$, set of values $\mathbb{R}$.

$\mathrm{dom}(g{\circ}f) = \{x : x \text{ is not a multiple of } \frac{\pi}{2}\}$, set of values $\mathbb{R}$.

$\mathbb{R}$.



$$\frac{\tan x}{\ln|x|}$$

(c) $\mathrm{dom}(f{\circ}g) = [-1, 1]$, set of values $[-1, 1]$.

$\mathrm{dom}(g{\circ}f) = \mathbb{R}$, set of values $[0, \pi]$.

$[\frac{1}{\pi}, \infty[$.

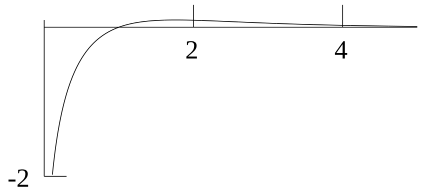
(No diagram available.)

$$\frac{\cos x}{\arccos x}$$

36. $\mathrm{dom}(f \circ g) = \mathbb{R}$, set of values $\mathbb{R}$ if $\alpha \neq 0$, $\{0\}$ if $\alpha = 0$.

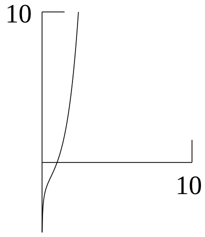
$\mathrm{dom}(g \circ f) =]0, \infty[$, set of values $]0, \infty[$ if $\alpha \neq 0$, $\{1\}$ if $\alpha = 0$.

(if $\alpha > 0$) $\left]-\infty, \frac{\ln \gamma}{\exp(\alpha\gamma)}\right]$ where $\gamma > 1$ is the unique solution of the equation $\gamma \ln \gamma = \frac{1}{\alpha}$. (The function $x \mapsto \frac{\ln x}{\exp(\alpha x)}$ has a global maximum at γ.)



$\frac{\ln x}{e^x}$

(if $\alpha \leq 0$) $\mathbb{R}$.

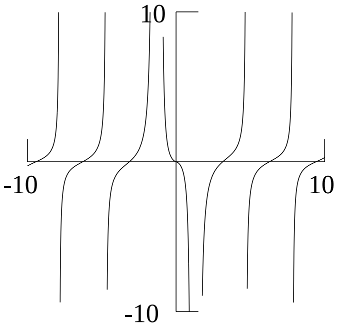


$\frac{\ln x}{e^{-x}} = e^x \ln x$

37. $\mathrm{dom}(f \circ g) = \{x : x \neq 0,\ x \neq \pm\exp(k\pi + \frac{\pi}{2}) \text{ for every } k \in \mathbb{Z}\}$, set of values $\mathbb{R}$.

$\mathrm{dom}(g \circ f) = \{x : x \text{ is not a multiple of } \frac{\pi}{2}\}$, set of values $\mathbb{R}$.

$\mathbb{R}$.



$\frac{\tan x}{\ln |x|}$

38. $\mathrm{dom}(f \circ g) = [-1, 1]$, set of values $[-1, 1]$.

$\mathrm{dom}(g \circ f) = \mathbb{R}$, set of values $[0, \pi]$.

$\left[\frac{1}{\pi}, \infty\right[$.

(No diagram available.)

$$\frac{\cos x}{\arccos x}$$

39. (a) $\lim_{x\to 1} |x^2 - 1| = 0$.

(No diagram available.)

(b) $\lim_{x\downarrow 0} x \ln x = 0$.

(No diagram available.)

(c) $\lim_{x\uparrow -1} \frac{1}{x^2-1} = \infty$.

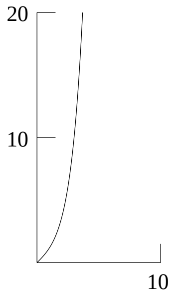
(No diagram available.)

(d) $\lim_{x\to\infty} x^4 - 2x^2 + 1 = \infty$.

(No diagram available.)

(e) $\lim_{x\uparrow\pi} \ln \sin x = -\infty$.

(No diagram available.)



(f) $\lim_{x\to\infty} \sinh x = \infty$.

(g) $\lim_{x\to\infty} \frac{\ln x}{x} = 0$.

(No diagram available.)

40. (a) $\lim_{x\to 1} |x^2 - 1| = 0$.

(No diagram available.)

(b) $\lim_{x\downarrow 0} x \ln x = 0$.

(No diagram available.)

(c) $\lim_{x\uparrow -1} \frac{1}{x^2-1} = \infty$.

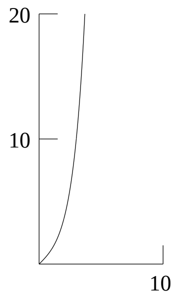
(No diagram available.)

41. (a) $\lim_{x\to\infty} x^4 - 2x^2 + 1 = \infty$.

(No diagram available.)

(b) $\lim_{x\uparrow\pi} \ln \sin x = -\infty$.

(No diagram available.)



42. (a) $\lim_{x\to\infty} \sinh x = \infty$.

(b) $\lim_{x\to\infty} \dfrac{\ln x}{x} = 0$.

(No diagram available.)

43. (a) $\lim_{x\to-\infty} |x^2 - 1| = 0$.
(b) $\lim_{x\to\infty} \dfrac{x}{|x|} = 1$.
(c) $\lim_{x\to-\infty} \dfrac{x}{|x|} = -1$.
(d) $\lim_{x\to\infty} \exp(-\frac{1}{2}x^2) = 0$.
(e) $\lim_{x\downarrow-1} \dfrac{1}{x^2-1} = \infty$.
(f) $\lim_{x\to\infty} \dfrac{1}{x^2-1} = 0$.
(g) $\lim_{x\to\infty} \tanh x = 1$.
(h) $\lim_{x\downarrow 0} x - \ln x = 0$.
(i) $\lim_{x\to\infty} x - \ln x = \infty$.
(j) $\lim_{x\to\frac{\pi}{2}} \ln|\cos x| = 0$.
(k) $\lim_{x\to-\infty} xe^x = 0$.
(l) $\lim_{x\to-\infty} \cosh x = \infty$.
(m) $\lim_{x\downarrow 0} \dfrac{\ln x}{x} = -\infty$.

44. (a) $\lim_{x\to -4}(x+3)^{143} = (-1)^{143} = -1$.
(b) $\lim_{x\to 0} \frac{1}{x}$ does not exist
($\lim_{x\uparrow 0} \frac{1}{x} = -\infty$, $\lim_{x\downarrow 0} \frac{1}{x} = \infty$)
(c) $\lim_{x\to 0} |\frac{1}{x}| = \infty$
(d) $\lim_{x\to 0} \frac{1}{x^2} = \infty$
(e) $\lim_{x\to 0} \sqrt{x}$ does not exist
($\sqrt{x}$ undefined for $x \doteq 0$, $x < 0$)
(f) $\lim_{x\to 1} \sin x = \sin 1$
(g) $\lim_{x\to\infty} \cos^2 x$ does not exist
(oscillates between 0 and 1)
(h) $\lim_{x\to\infty} \sin^2 x$ does not exist
(i) $\lim_{x\to\infty} \cos^2 x + \sin^2 x = 1$
(j) $\lim_{x\to 0} \frac{|x|^{3/2}}{x} = 0$
(k) $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} x+2 = 4$
(l) $\lim_{x\to -2} \frac{x^3+8}{x^2-4} = \lim_{x\to -2} \frac{x^2-2x+4}{x-2} = -3$
(m) $\lim_{x\downarrow \frac{\pi}{2}} \tan x = -\infty$
(n) $\lim_{x\to -\infty} \arctan x = -\frac{\pi}{2}$

45. (a) $\lim_{x\to -4}(x+3)^{143} = (-1)^{143} = -1$.
(b) $\lim_{x\to 0} |\frac{1}{x}| = \infty$
(c) $\lim_{x\to 1} \sin x = \sin 1$
(d) $\lim_{x\to\infty} \cos^2 x + \sin^2 x = 1$
(e) $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} x+2 = 4$
(f) $\lim_{x\downarrow \frac{\pi}{2}} \tan x = -\infty$

46. (a) $\lim_{x\to 0} \frac{1}{x}$ does not exist ($\lim_{x\uparrow 0} \frac{1}{x} = -\infty$, $\lim_{x\downarrow 0} \frac{1}{x} = \infty$).
(b) $\lim_{x\to 0} \frac{1}{x^2} = \infty$.
(c) $\lim_{x\to 0} \sqrt{x}$ does not exist ($\sqrt{x}$ is undefined for $x \doteq 0$, $x < 0$).
(d) $\lim_{x\to\infty} \cos^2 x$ does not exist (oscillates between 0 and 1).
(e) $\lim_{x\to\infty} \sin^2 x$ does not exist.
(f) $\lim_{x\to 0} \frac{|x|^{3/2}}{x} = 0$.
(g) $\lim_{x\to -2} \frac{x^3+8}{x^2-4} = \lim_{x\to -2} \frac{x^2-2x+4}{x-2} = -3$.
(h) $\lim_{x\to -\infty} \arctan x = -\frac{\pi}{2}$.

47. (a) $\lim_{x\to-\infty} \frac{1}{x}\sin x = 0$
(b) $\lim_{x\to\infty} x^{\pi} = \infty$
(c) $\lim_{x\downarrow 1} \frac{1+x}{1-x} = \infty$
(d) $\lim_{x\downarrow 0} x^{-\sqrt{2}} = \infty$
(e) $\lim_{x\to\infty} 8x^3 \exp(-x)\sin x = 0$
(exponentials beat powers)
(f) $\lim_{x\to-\infty} x^{10}\exp(-\frac{x}{2}) = 0$
(g) $\lim_{x\to\infty} \cosh^2 x = \infty$
(h) $\lim_{x\to\infty} \sinh^2 x = \infty$
(i) $\lim_{x\to\infty} \cosh x + \sinh x = \infty$
(j) $\lim_{x\to\infty} \cosh x - \sinh x = 0$
(k) $\lim_{x\to\infty} \cosh^2 x - \sinh^2 x = 1$
(l) $\lim_{x\to\infty} (\frac{1}{\pi})^x = 0$

48. (a) $\lim_{x\downarrow 1} \frac{1+x}{1-x} = -\infty$.

(b) $\lim_{x\downarrow 0} x^{-\sqrt{2}} = \infty$.

(c) $\lim_{x\to\infty} 8x^3 \exp(-x)\sin x = 0$
(exponentials beat powers).

(d) $\lim_{x\to\infty} (\frac{1}{\pi})^x = 0$.

49. (a) $\lim_{x\to-\infty} \frac{1}{x}\sin x = 0$.
(b) $\lim_{x\to\infty} x^{\pi} = \infty$ (exponentials beat powers).
(c) $\lim_{x\to-\infty} x^{10}\exp(-\frac{x}{2}) = 0$.
(d) $\lim_{x\to\infty} \cosh^2 x = \infty$.
(e) $\lim_{x\to\infty} \sinh^2 x = \infty$.
(f) $\lim_{x\to\infty} \cosh x + \sinh x = \infty$.
(g) $\lim_{x\to\infty} \cosh x - \sinh x = 0$.
(h) $\lim_{x\to\infty} \cosh^2 x - \sinh^2 x = 1$.

50. (a) $\lim_{x\to\infty} \frac{x^4-x^3+2x^2-7}{18x^3-12x-99} = \infty$
(b) $\lim_{x\to 0} \frac{x^4-x^3+2x^2-7}{18x^3-12x-99} = \frac{7}{99}$
(c) $\lim_{x\to-\infty} \frac{x^4-x^3+2x^2-7}{18x^3-12x-99} = -\infty$
(think: $\lim_{x\to-\infty} \frac{x}{18}$.)
(d) $\lim_{x\to\infty} \ln x \tanh x = \infty.1 = \infty$
(e) $\lim_{x\downarrow 0} \ln x + \sqrt{x} = -\infty + 0 = -\infty$
(f) $\lim_{x\downarrow 0} \frac{x\ln x+1}{e^x} = \frac{0+1}{1} = 1$
(g) $\lim_{x\to\infty} \frac{x^2+1}{x^2-1} + \tanh x = 1+1 = 2$
(h) $\lim_{x\to\infty} x + e^{-x} = \infty + 0 = \infty$
(i) $\lim_{x\to-\infty} xe^{-x} = -\infty.\infty = \infty$
(j) $\lim_{x\to 0} \frac{\sin x}{x^3} = \lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{1}{x^2} = 1.\infty = \infty$

51. (a) $\lim_{x\to-\infty}\frac{x^4-x^3+2x^2-7}{18x^3-12x-99}=-\infty$
(think: $\lim_{x\to-\infty}\frac{x}{18}$).

(b) $\lim_{x\downarrow 0}\frac{x\ln x+1}{e^x}=\frac{0+1}{1}=1$.

(c) $\lim_{x\to\infty}\frac{x^2+1}{x^2-1}+\tanh x=1+1=2$.

(d) $\lim_{x\to 0}\frac{\sin x}{x^3}=\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{1}{x^2}=1.\infty=\infty$.

52. (a) $\lim_{x\to\infty}\frac{x^4-x^3+2x^2-7}{18x^3-12x-99}=\infty$.
(b) $\lim_{x\to 0}\frac{x^4-x^3+2x^2-7}{18x^3-12x-99}=\frac{7}{99}$.
(d) $\lim_{x\to\infty}\ln x\tanh x=\infty.1=\infty$.
(e) $\lim_{x\downarrow 0}\ln x+\sqrt{x}=-\infty+0=-\infty$.
(h) $\lim_{x\to\infty}x+e^{-x}=\infty+0=\infty$.
(i) $\lim_{x\to-\infty}xe^{-x}=-\infty.\infty=\infty$.

53. (a) $\lim_{x\to\infty}\exp x-x^{10}=\infty$
(exponentials beat powers)
(b) $\lim_{x\to-\infty}x+e^{-x}=\infty$
(c) $\lim_{x\to\infty}\ln x+\sin x=\infty$
(d) $\lim_{x\to\infty}e^{-x}\frac{\sin x}{|\sin x|}$ does not exist
(undefined for many large values of x)
(e) $\lim_{x\to-\infty}\frac{x}{\tanh x}+x^2=\infty$
(x^2 dominates)
(f) $\lim_{x\to\infty}x-\ln x=\infty$
(g) $\lim_{x\downarrow 0}\sqrt{x}\ln x=0$
(powers beat logs)
(h) $\lim_{x\downarrow 0}e^{-x}\ln x=1.-\infty=-\infty$
(i) $\lim_{x\to\infty}e^{-x}\ln x=0$
(j) $\lim_{x\to 1}\frac{e^x}{x^3-2x^2+x}=\lim_{x\to 1}\frac{e^x}{x}\lim_{x\to 1}\frac{1}{(x-1)^2}=e.\infty=\infty$

54. (a) $\lim_{x\to\infty}\exp x-x^{10}=\infty$
(exponentials beat powers).

(b) $\lim_{x\to\infty}\ln x+\sin x=\infty$.

(c) $\lim_{x\to\infty}e^{-x}\frac{\sin x}{|\sin x|}$ does not exist
(undefined for many large values of x).

(d) $\lim_{x\to-\infty}\frac{x}{\tanh x}+x^2=\infty$
(x^2 dominates).

(e) $\lim_{x\to 1}\frac{e^x}{x^3-2x^2+x}=\lim_{x\to 1}\frac{e^x}{x}\lim_{x\to 1}\frac{1}{(x-1)^2}=e.\infty=\infty$

55. (a) $\lim_{x\to-\infty}x+e^{-x}=\infty$.
(b) $\lim_{x\to\infty}x-\ln x=\infty$.
(c) $\lim_{x\downarrow 0}\sqrt{x}\ln x=0$ (powers beat logs).
(d) $\lim_{x\downarrow 0}e^{-x}\ln x=1.-\infty=-\infty$.
(e) $\lim_{x\to\infty}e^{-x}\ln x=0$.

56. (a)$\lim_{x\downarrow\pi}\frac{\sin x}{|\sin x|}=\lim_{y\uparrow 0}\frac{y}{|y|}=-1$

(b) $\lim_{x\to 0}\exp(\frac{1}{x^2})=\lim_{y\to\infty}\exp(y)=\infty$

(c) $\lim_{x\to\infty}\ln(x+\sin x)=\lim_{y\to\infty}\ln y=\infty$

(d) $\lim_{x\to-\infty}\exp(\frac{1}{x})=\lim_{y\uparrow 0}\exp(y)=1$

(e) $\lim_{x\to\infty}\sqrt{\frac{\sin x}{x}}$ does not exist

($\sqrt{\frac{\sin x}{x}}$ undefined for too many x)

(f) $\lim_{x\uparrow 0}\sinh\frac{1}{x}=\lim_{y\to-\infty}\sinh y=-\infty$

(g) $\lim_{x\uparrow\pi}\sin x\ln(\sin x)=\lim_{y\downarrow 0}y\ln y=0$

(h) $\lim_{x\uparrow 0}\exp(\frac{1}{x})=\lim_{y\to-\infty}\exp(y)=0$

(i) $\lim_{x\downarrow 0}\frac{1}{x\ln x}=\lim_{y\uparrow 0}\frac{1}{y}=-\infty$

(j) $\lim_{x\to 0}\ln(\frac{2\sin x}{x})=\lim_{y\to 2}\ln y=\ln 2$

(k) $\lim_{x\downarrow 0}x^x=\lim_{x\downarrow 0}\exp(x\ln x)=\lim_{y\uparrow 0}\exp y=1$

(l) $\lim_{x\to\infty}x^x e^{-x^2/2}=\lim_{x\to\infty}\exp(x\ln x-\frac{1}{2}x^2)=\lim_{y\to-\infty}\exp(y)=0$

(m) $\lim_{x\to 2}\ln(\frac{x^2-4}{x-2})=\lim_{x\to 2}\ln(x+2)=\ln 4$

(n) $\lim_{x\to 0}|\ln(e^{1/x}-x)|=\infty$

$$\begin{aligned}(\lim_{x\uparrow 0}|\ln(e^{1/x}-x)| &= \lim_{y\downarrow 0}|\ln(e^{-1/y}+y)|\\ &= \lim_{z\downarrow 0}|\ln z|=\infty,\\ \lim_{x\downarrow 0}|\ln(e^{1/x}-x)| &= \lim_{y\to\infty}|\ln y|=\infty.)\end{aligned}$$

57. (a) $\lim_{x\to 0}\exp(\frac{1}{x^2})=\lim_{y\to\infty}\exp(y)=\infty.$

(b) $\lim_{x\uparrow 0}\exp(\frac{1}{x})=\lim_{y\to-\infty}\exp(y)=0.$

(c) $\lim_{x\to 0}\ln(\frac{2\sin x}{x})=\lim_{y\to 2}\ln y=\ln 2.$

(d) $\lim_{x\downarrow 0}x^x=\lim_{x\downarrow 0}\exp(x\ln x)=\lim_{y\uparrow 0}\exp y=1.$

(e) $\lim_{x\to\infty}x^x e^{-x^2/2}=\lim_{x\to\infty}\exp(x\ln x-\frac{1}{2}x^2)=\lim_{y\to-\infty}\exp(y)=0.$

(f) $\lim_{x\to 0}|\ln(e^{1/x}-x)|=\infty$

$$\begin{aligned}(\lim_{x\uparrow 0}|\ln(e^{1/x}-x)| &= \lim_{y\downarrow 0}|\ln(e^{-1/y}+y)|\\ &= \lim_{z\downarrow 0}|\ln z|=\infty,\\ \lim_{x\downarrow 0}|\ln(e^{1/x}-x)| &= \lim_{y\to\infty}|\ln y|=\infty).\end{aligned}$$

58. (a)$\lim_{x\downarrow\pi} \frac{\sin x}{|\sin x|} = \lim_{y\uparrow 0} \frac{y}{|y|} = -1$.
(b) $\lim_{x\to\infty} \ln(x+\sin x) = \lim_{y\to\infty} \ln y = \infty$.
(c) $\lim_{x\to-\infty} \exp(\frac{1}{x}) = \lim_{y\uparrow 0} \exp(y) = 1$.
(d) $\lim_{x\to\infty} \sqrt{\frac{\sin x}{x}}$ does not exist ($\sqrt{\frac{\sin x}{x}}$ is undefined for too many x).
(e) $\lim_{x\uparrow 0} \sinh \frac{1}{x} = \lim_{y\to-\infty} \sinh y = -\infty$.
(f) $\lim_{x\uparrow\pi} \sin x \ln(\sin x) = \lim_{y\downarrow 0} y \ln y = 0$.
(g) $\lim_{x\downarrow 0} \frac{1}{x\ln x} = \lim_{y\uparrow 0} \frac{1}{y} = -\infty$.
(h) $\lim_{x\to 2} \ln(\frac{x^2-4}{x-2}) = \lim_{x\to 2} \ln(x+2) = \ln 4$.

59. (No solution available.)

60. (a) $\lim_{x\to 0} \frac{1}{x} \ln(1+x) = 1$.

$$(\frac{\lim_{x\to 0} \ln(1+x)}{\lim_{x\to 0} x} = \frac{0}{0},$$

$$\lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1.)$$

(b) $\lim_{x\to\infty} x \ln(1+\frac{1}{x}) = 1$.

$$(\lim_{x\to\infty} x \ln(1+\frac{1}{x}) = \lim_{y\downarrow 0} \frac{1}{y} \ln(1+y) = 1.)$$

(c) $\lim_{x\to\infty} (1+\frac{1}{x})^x = e$.

$$(\lim_{x\to\infty} (1+\frac{1}{x})^x = \lim_{x\to\infty} \exp(x\ln(1+\frac{1}{x})) = \exp 1 = e.)$$

(d) $\lim_{x\to\infty} (1+\frac{\alpha}{x})^{\beta x} = \exp(\alpha\beta)$.

$$(\lim_{x\to 0} \frac{\beta\ln(1+\alpha x)}{x} = \lim_{x\to 0} \frac{\frac{\alpha\beta}{1+\beta x}}{1} = \alpha\beta,$$

$$\lim_{x\to\infty} \beta x \ln(1+\frac{\alpha}{x}) = \alpha\beta,$$

$$\lim_{x\to\infty} \exp\big(\beta x\ln(1+\frac{\alpha}{x})\big) = \exp(\alpha\beta).)$$

(e) $\lim_{x\to 0} \frac{\sin 5x}{3x} = \frac{5}{3}$.

61. (a) $\lim_{x\to 0}\frac{e^x-1}{x}=1.$

(b) $\lim_{x\to 0}(\frac{1}{e^x-1}-\frac{1}{x})=-\frac{1}{2}.$

$$\Bigg(\lim_{x\to 0}(\frac{1}{e^x-1}-\frac{1}{x})=\lim_{x\to 0}\frac{x-e^x+1}{x(e^x-1)}$$

$$=\lim_{x\to 0}\frac{x-e^x+1}{x^2}\lim_{x\to 0}\frac{x}{e^x-1}$$

$$=\lim_{x\to 0}\frac{1-e^x}{2x}=\lim_{x\to 0}\frac{-e^x}{2}=-\frac{1}{2}$$

or

$$\lim_{x\to 0}(\frac{1}{e^x-1}-\frac{1}{x})=\lim_{x\to 0}\frac{x-e^x+1}{xe^x-x}$$

$$=\lim_{x\to 0}\frac{1-e^x}{xe^x+e^x-1}=\lim_{x\to 0}\frac{-e^x}{xe^x+2e^x}=-\frac{1}{2}.\Bigg)$$

(c) $\lim_{x\to 0}\frac{x-\ln(1+x)}{\sin^2 x}=\frac{1}{2}.$

(d) $\lim_{x\to 3}\frac{3x-9}{x^2-3x+7}=0.$

(e) $\lim_{x\to 1}\frac{x^3+x-2}{x^2-2x+1}$ does not exist.

$$(\frac{x^3+x-2}{x^2-2x+1}=\frac{x^2+x+2}{x-1}\sim\frac{4}{x-1}.)$$

62. (a) $\lim_{x\to 0}\frac{\sin x}{x}=1$.

(b)

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)&=\lim_{x\to 0}\frac{x-\sin x}{x\sin x}=\lim_{x\to 0}\frac{x-\sin x}{x^2}\lim_{x\to 0}\frac{x}{\sin x}\\&=\lim_{x\to 0}\frac{1-\cos x}{2x}=\lim_{x\to 0}\frac{\sin x}{2}=0.\end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)&=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x}=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^4}\lim_{x\to 0}x^2\sin^2 x\\&=\lim_{x\to 0}\frac{2x-2\sin x\cos x}{4x^3}=\lim_{x\to 0}\frac{2x-\sin 2x}{4x^3}\\&=\lim_{x\to 0}\frac{2-2\cos 2x}{12x^2}=\lim_{x\to 0}\frac{1-\cos 2x}{6x^2}\\&=\lim_{x\to 0}\frac{2\sin 2x}{12x}=\frac{1}{6}\lim_{x\to 0}\frac{\sin 2x}{2x}\\&=\frac{1}{6}\lim_{y\to 0}\frac{\sin y}{y}=\frac{1}{6}.\end{aligned}$$

(d)

$$\begin{aligned}\lim_{x\to 0}\frac{x-\ln(1+x)}{x^2}&=\lim_{x\to 0}\frac{1-\frac{1}{1+x}}{2x}=\lim_{x\to 0}\frac{x}{2x(1+x)}\\&=\lim_{x\to 0}\frac{1}{2(1+x)}=\frac{1}{2}.\end{aligned}$$

63. (a) $\lim_{x\to 0} \frac{x^2}{\sin^2 x} = 1.$

(b)

$$\lim_{x\downarrow -3} \frac{\sqrt{9-x^2}}{x+3} = \lim_{y\downarrow 0} \frac{\sqrt{9-(y-3)^2}}{y}$$

(substituting $y = x+3$, $x = y-3$)

$$= \lim_{y\downarrow 0} \frac{\sqrt{6y-y^2}}{y} = \lim_{y\downarrow 0} \sqrt{\frac{6}{y} - 1} = \lim_{z\to\infty} \sqrt{z}$$

(substituting $z = \frac{6}{y} - 1$)

$$= \infty$$

or

$$\lim_{x\downarrow -3} \frac{\sqrt{9-x^2}}{x+3} = \lim_{x\downarrow -3} \sqrt{\frac{3-x}{3+x}} = \infty$$

because $\lim_{x\downarrow -3} \frac{3-x}{3+x} = 6 \lim_{x\downarrow -3} \frac{1}{3+x} = \infty$.

(c) If $a > 0$,

$$\lim_{x\to 0} \frac{\sqrt{a+x} - \sqrt{a}}{x} = \lim_{x\to 0} \frac{\frac{1}{2\sqrt{a+x}}}{1} = \frac{1}{2\sqrt{a}}.$$

(d) $\lim_{x\to\pi} \frac{1+\cos x}{\tan^2 x} = \lim_{x\to\pi} \frac{-\sin x}{2\tan x \operatorname{cosec}^2 x} = -\frac{1}{2} \lim_{x\to\pi} \cos x = 1.$

64. (No solution available.)

65. (a) Not injective.

(b) Injective; $\operatorname{dom} f^{-1} = \mathbb{R}$;

$$f^{-1}(y) = \sqrt[3]{\frac{1}{2}(y + \sqrt{y^2 + \frac{4}{27}})} + \sqrt[3]{\frac{1}{2}(y - \sqrt{y^2 + \frac{4}{27}})}.$$

(Congratulations, but no extra marks, to students who knew how to solve the cubic.)

(c) Injective; $\operatorname{dom} f^{-1} =]0, \infty[$; $f^{-1}(y) = \exp(-\sqrt{y})$.
(If $(\ln x)^2 = y$ then $\ln x = \pm\sqrt{y}$, $x = \exp(\pm\sqrt{y})$; but in our case $x < 1$ so $\ln x < 0$ and we must take the negative square root. for $\exp\sqrt{y}$.)

(d) Injective; $\operatorname{dom} f^{-1} = \mathbb{R}$; $f^{-1}(y) = \ln(y + \sqrt{y^2+1})$.
(If $\frac{e^x - e^{-x}}{2} = y$, then $(e^x)^2 - 2ye^x - 1 = 0$ so $e^x = y \pm \sqrt{y^2+1}$; since $e^x > 0$ we must take the positive square root and $x = \ln(y + \sqrt{y^2+1})$. for $\ln(y \pm \sqrt{y^2+1})$.)

(e) Injective; $\operatorname{dom} f^{-1} = [0, \infty[$; $f^{-1}(y) = \ln(\sqrt{y+1} - \sqrt{y})$.
(If $\sinh^2 x = y$, then $\sinh x = \pm\sqrt{y}$, $x = \ln(\pm\sqrt{y} + \sqrt{y+1})$; but in this case $x \le 0$ so $\sinh x \le 0$ and we must take the negative square root.)

66. (a) Injective; dom $f^{-1} = [0, \infty[$, $f^{-1}(y) = y^2$.

(b) Injective; dom $f^{-1} = \mathbb{R}$, $f^{-1}(y) = \sqrt[3]{y}$.

(c) Not injective.

(d) Not injective.

(e) Not injective.

(f) Injective; dom $f^{-1} =]1, \infty[$; $f^{-1}(y) = \ln\ln(y + \sqrt{y^2 - 1})$.

(g) Injective; dom $f^{-1} =]-1, 1[$; $f^{-1}(y) = \frac{1}{2}\ln(\frac{1+y}{1-y})$.

(h) Injective; dom $f^{-1} = \mathbb{R}$.

(i) Not injective.

(j) Injective; dom $f^{-1} =]-\sin 1, \sin 1[$; $f^{-1}(y) = \frac{1}{2}\ln(\frac{1+\arcsin y}{1-\arcsin y})$.

67. (No solution available.)

68. dom $f = [1, \infty[$; it is injective; dom $f^{-1} = [0, \infty[$; $f^{-1}(y) = \cosh y$.

69. dom $f = [1, \infty[$; it is injective ; dom $f^{-1} =]-\infty, 0]$; $f^{-1}(y) = \cosh y$.
(If $y = \ln(x - \sqrt{x^2 - 1})$ then $e^y = x - \sqrt{x^2 - 1}$, $\quad (e^y - x)^2 = x^2 - 1$, $\quad e^{2y} - 2e^y x + 1 = 0$, $\quad x = \frac{e^{2y}+1}{2e^y} = \frac{e^y + e^{-y}}{2}$.)

70. 1·76 . $\quad 1{\cdot}755\ln(1{\cdot}755) \simeq 0{\cdot}987$, $\quad 1{\cdot}765\ln(1{\cdot}765) \simeq 1{\cdot}002$.

71. (No solution available.)

72. (No solution available.)

73. 0·94. $\quad f(0{\cdot}935) \simeq -0{\cdot}003 < 0$, $\quad f(0{\cdot}945) \simeq 0{\cdot}02 > 0$.
(There are infinitely many correct alternatives.)

74. The set of values of f' must include the interval $[-1, 2]$, because there is an $x_1 \in]-1, 0[$ such that $f'(x_1) = \frac{f(0)-f(-1)}{0-(-1)} = -1$, there is an $x_2 \in]0, 1[$ such that $f'(x_2) = \frac{f(1)-f(0)}{1-0} = 2$, and (because f' is continuous) its set of values includes the whole interval $[-1, 2]$.

(Congratulations, but no extra mark, to students who noticed that the set of values must properly include the interval $[-1, 2]$.)

75. It must include $[\frac{1}{5}, \frac{1}{3}]$..
Set $g(x) = x^2$ for $x \geq 1$. By Cauchy's Mean Value Theorem there are an $x_1 \in]1, 2[$ such that

$$\frac{f'(x_1)}{2x_1} = \frac{f'(x_1)}{g'(x_1)} = \frac{f(2)-f(1)}{g(2)-g(1)} = \frac{1}{3}$$

and an $x_2 \in]2, 3[$ such that

$$\frac{f'(x_2)}{2x_2} = \frac{f'(x_2)}{g'(x_2)} = \frac{f(3)-f(2)}{g(3)-g(2)} = \frac{1}{5} \quad .$$

Because $x \mapsto \frac{f'(x)}{2x}$ is continuous, it must also take every value between $\frac{1}{5}$ and $\frac{1}{3}$.

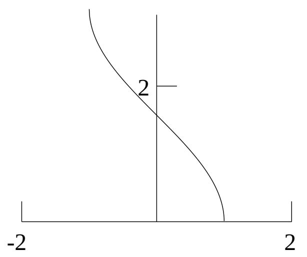
(Congratulations, but no extra mark, to anyone who observed that the set of values of $f'(x)/2x$ must properly include $[\frac{1}{5}, \frac{1}{3}]$.)

76. There must be some $x_1 \in \,]0,1[$ such that $f'(x_1) = \frac{f(1)-f(0)}{1-0} = -1$, and some $x_2 \in \,]1,2[$ such that $f'(x_2) = \frac{f(2)-f(1)}{2-1} = 2$. Now there must be some $x \in \,]x_1, x_2[$ such that

$$f''(x) = \frac{f'(x_2)-f'(x_1)}{x_2-x_1} = \frac{3}{x_2-x_1} > \frac{3}{2},$$

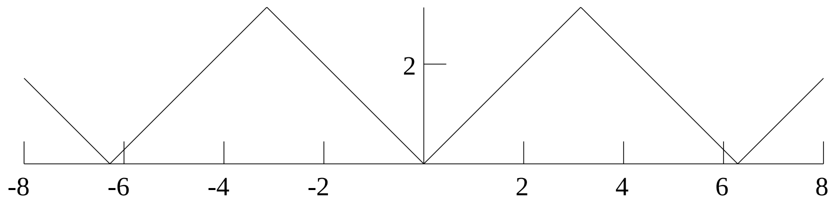
because $x_2 - x_1 < 2$.

77. (i) $[-1,1]$.
(ii) $[0,\pi]$.
(iii) $]-1,1[$.
(iv) $-\frac{1}{\sqrt{1-x^2}}$.
(v)



.

(vi) $\mathbb{R}$.
(vii)



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78. The set of values of f must include $[1,2]$ and be included in $[\frac{2}{3},\frac{7}{3}]$.

(We must have $-1 \le \frac{f(a)-1}{a-0} \le 2$, that is, $1-a \le f(a) \le 1+2a$, and also $-1 \le \frac{2-f(a)}{1-a} \le 2$, that is, $a-1 \le 2-f(a) \le 2-2a$, that is, $2a-2 \le f(a)-2 \le 1-a$, that is, $2a \le f(a) \le 3-a$ for every $a \in [0,1]$. So the graph of f is trapped inside the parallelogram bounded by the lines $y = 1-x$, $y = 1+2x$, $y = 2x$ and $y = 3-x$, which has vertices $(0,1)$, $(\frac{2}{3},\frac{7}{3})$, $(1,2)$, $(\frac{1}{3},\frac{2}{3})$.)

(Congratulations, but no extra mark, to anyone who noticed that f cannot take either the value $\frac{2}{3}$ or the value $\frac{7}{3}$, so that its set of values must be included in $\left]\frac{2}{3},\frac{7}{3}\right[$.)

79. (a) No limit.

(b) 0.

(c) 0 , for any x .

(d) 0 .

80. (a) 1.

(b) ∞.

(c) Undefined if $x < 0$; 0 if $x = 0$; 1 if $x > 0$.

81. (a)

$$\begin{aligned}\sum_{r=0}^{n} 2r - 1 &= 2\sum_{r=0}^{n} r - (n+1) \\ &= 2\frac{n(n+1)}{2} - (n+1) = n^2 - 1;\end{aligned}$$

$$\lim_{n\to\infty} \sum_{r=0}^{n} 2r - 1 = \lim_{n\to\infty} n^2 - 1 = \infty.$$

(b)

$$\sum_{r=1}^{n} \frac{5}{2.4^r} = \frac{\frac{5}{8} - \frac{5}{2.4^{n+1}}}{1 - \frac{1}{4}} = \frac{4}{3}\left(\frac{5}{8} - \frac{5}{2.4^{n+1}}\right);$$

(also acceptable: $\frac{5}{6}(1 - 4^{-n})$, etc.)

$$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{5}{2.4^r} = \lim_{n\to\infty} \frac{4}{3}\left(\frac{5}{8} - \frac{5}{2.4^{n+1}}\right) = \frac{5}{6}.$$

82. Provided $x \neq \frac{3}{2}$,

$$\sum_{r=0}^{n} \left(\frac{2x}{3}\right)^r = \frac{\left(\frac{2x}{3}\right)^0 - \left(\frac{2x}{3}\right)^{n+1}}{1 - \frac{2x}{3}} = \frac{3}{3-2x}\left(1 - \left(\frac{2x}{3}\right)^{n+1}\right);$$

$$\begin{aligned}\lim_{n\to\infty} \sum_{r=0}^{n} \left(\frac{2x}{3}\right)^r &= \lim_{n\to\infty} \frac{3}{3-2x}\left(1 - \left(\frac{2x}{3}\right)^{n+1}\right) \\ &= \frac{3}{3-2x} \quad \text{if } |x| < \frac{3}{2},, \\ &= \infty \quad \text{if } x > \frac{3}{2}, \\ &\text{undefined if } x \leq -\frac{3}{2}.\end{aligned}$$

If $x = \frac{3}{2}$, then

$$\begin{aligned}\sum_{r=0}^{n} \left(\frac{2x}{3}\right)^r &= n + 1 \\ &\to \infty \text{ as } n \to \infty.\end{aligned}$$

83. (a) $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \sum_{r=1}^{n} \frac{1}{r} - \frac{1}{r+1} = 1 - \frac{1}{n+1} \to 1$ as $n \to \infty$.

(b) $\sum_{r=1}^{n} r(r-1) = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = \frac{n(n^2-1)}{3} \to \infty$ as $n \to \infty$.

(c) $\sum_{r=0}^{n} (-1)^r = \frac{(-1)^0 - (-1)^{n+1}}{1-(-1)} = \frac{1}{2}(1 - (-1)^{n+1})$ has no limit as $n \to \infty$.

(d) $\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = \sum_{r=1}^{n} \left(\frac{1}{r^2} - \frac{1}{(r+1)^2}\right) = 1 - \frac{1}{(n+1)^2} \to 1$ as $n \to \infty$.

84. (a) $\lim_{r\to\infty} |\frac{x_{r+1}}{x_r}| = \lim_{r\to\infty} |\frac{r+1}{2^{r+1}}\frac{2^r}{r}| = \lim_{r\to\infty} \frac{r+1}{2r} = \frac{1}{2}$,
so the series is absolutely summable .

(b) $\lim_{r\to\infty} |\frac{x_{r+1}}{x_r}| = \lim_{r\to\infty} \frac{(\sqrt{5}-1)(r^2+1)}{(r+1)^2+1} = \sqrt{5}-1$,
so the series is not absolutely summable (because $\sqrt{5}-1 > 1$).

(c) $\lim_{r\to\infty} |\frac{x_{r+1}}{x_r}| = \lim_{r\to\infty} \frac{(r^2-2r+1)\sqrt{r+1}}{((r+1)^2-2(r+1)+1)\sqrt{r}} = 1,$
$\lim_{r\to\infty} |r^{3/2}x_r| = 1,$
so the series is absolutely summable.

(d) $\lim_{r\to\infty} |\frac{x_{r+1}}{x_r}| = \lim_{r\to\infty} \frac{r^2}{(r+1)^2}\frac{|\sin(r+1)\theta|}{|\sin r\theta|}$ is undefined ,
but the series is absolutely summable (because $|x_r| \le \frac{1}{r^2}$ for every r).

(e) $\lim_{r\to\infty} |\frac{x_{r+1}}{x_r}| = \lim_{r\to\infty} \frac{\sqrt{r(r+1)}}{\sqrt{(r+1)(r+2)}} = 1$,
$\lim_{r\to\infty} |rx_r| = 1$,
so the series is not absolutely summable.

85. (a) $\lim_{r\to\infty} |\frac{x_{r+1}}{x_r}| = \lim_{r\to\infty} \frac{r(r+1)}{(r+1)(r+2)} = 1,$
$\lim_{r\to\infty} |r^2 x_r| = 1,$
so the series is absolutely summable.

(b) $\lim_{r\to\infty} |\frac{x_{r+1}}{x_r}| = \lim_{r\to\infty} \frac{r\sqrt{r}}{(r+1}\sqrt{r+1}) = 1,$
$\lim_{r\to\infty} r^{3/2}x_r = 1,$
so the series is absolutely summable.

(c) $\lim_{r\to\infty} |\frac{x_{r+1}}{x_r}| = 1,$
$\lim_{r\to\infty} |rx_r| = 1,$
so the series is not absolutely summable.

(d) $\lim_{r\to\infty} |\frac{x_{r+1}}{x_r}| = \lim_{r\to\infty} \frac{((r+1)!)^2(2r)!}{(r!)^2(2(r+1))!} = \lim_{r\to\infty} \frac{(r+1)^2}{(2r+1)(2r+2)} = \frac{1}{4},$
so the series is absolutely summable.

(e) $\lim_{r\to\infty} |\frac{x_{r+1}}{x_r}| = \lim_{r\to\infty} \frac{100}{\sqrt{r+1}} = 0,$
so the series is absolutely summable.

(f) $\lim_{r\to\infty} |\frac{x_{r+1}}{x_r}| = \lim_{r\to\infty} \frac{r+1}{2^{2r+1}} = 0,$
so the series is absolutely summable.

(g) $\lim_{r\to\infty} |\frac{x_{r+1}}{x_r}| = \lim_{r\to\infty} \frac{2^{r+1}}{(r+1)!}\frac{r!}{2^r} = \lim_{r\to\infty} \frac{2}{r+1} = 0,$
so the series is absolutely summable.

86. (a) $\lim_{r\to\infty}\frac{\ln r}{\ln(r+1)}=1$,

there is no α such that $\lim_{r\to\infty}\frac{r^\alpha}{\ln r}$ is defined in $]0,\infty[$,

the sequence is not absolutely summable (because $\frac{1}{\ln r}\ge\frac{1}{r}$ for $r\ge 2$).

(b) $\lim_{r\to\infty} rx_r=\lim_{r\to\infty}\exp(-\frac{1}{r}\ln r)=1$,
so

$$\begin{aligned}\lim_{r\to\infty}\frac{x_{r+1}}{x_r}&=\lim_{r\to\infty}\frac{r+1}{r}\lim_{r\to\infty}\frac{x_{r+1}}{x_r}\\&=\lim_{r\to\infty}\frac{(r+1)x_{r+1}}{rx_r}=\frac{1}{1}=1\end{aligned}$$

and the series is not absolutely summable.

(c)(i)

$$|\frac{x_{r+1}}{x_r}|=\exp(\frac{1}{r}\sum_{k=1}^r\ln k-\frac{1}{r+1}\sum_{k=1}^{r+1}\ln k)=\exp(\frac{1}{r(r+1)}\sum_{k=1}^r\ln k-\frac{1}{r+1}\ln(r+1)).$$

Now

$$\frac{1}{r(r+1)}\sum_{k=1}^r\ln k\le\frac{1}{r(r+1)}\int_1^r\ln t\,dt=\frac{r\ln r-r+1}{r(r+1)}\to 0$$

as $r\to\infty$, while also $\frac{\ln(r+1)}{r+1}\to 0$ as $r\to\infty$, so $\frac{x_{r+1}}{x_r}\to 1$ as $r\to\infty$.

(ii) $\ln(\sqrt[r]{r!})=\frac{1}{r}\sum_{k=1}^r\ln k$ lies between $\frac{1}{r}\int_1^r\ln t\,dt=\ln r-1+\frac{1}{r}$ and $\frac{1}{r}\int_1^{r+1}\ln t\,dt=\frac{r+1}{r}\ln(r+1)-1$. So $\ln(rx_r)$ lies between $\ln r-\ln r+1-\frac{1}{r}=1-\frac{1}{r}$ and $\ln r-\frac{r+1}{r}\ln(r+1)+1=1+\ln\frac{r}{r+1}-\frac{1}{r}\ln(r+1)$; since both of these converge to 1, $rx_r\to e$ as $r\to\infty$.

(iii) Consequently $\langle x_r\rangle_{r\in\mathbb{N}}$ is not summable.

(d)

$$\begin{aligned}\lim_{r\to\infty}r^2(1-\cos\frac{1}{r})&=\lim_{x\to\infty}x^2(1-\cos\frac{1}{x})=\lim_{y\downarrow 0}\frac{1-\cos y}{y^2}\\&=\lim_{y\downarrow 0}\frac{\sin y}{2y}=\frac{1}{2}.\end{aligned}$$

Consequently the series is absolutely summable, and also

$$\lim_{r\to\infty}|\frac{x_{r+1}}{x_r}|=\lim_{r\to\infty}\frac{r^2}{(r+1)^2}=1.$$

87. (a) $\lim_{r\to\infty}|\frac{(-1)^r}{(-1)^{r+1}}| = 1$.

(b) $\lim_{r\to\infty}\frac{(r^2+r+1)(r+3)}{((r+1)^2+(r+1)+1)(r+2)} = 1$.

(c) $\lim_{r\to\infty}\frac{2^r\sqrt{(r+1)^2+1}}{\sqrt{r^2+1}2^{r+1}} = \frac{1}{2}$.

(d) $\lim_{r\to\infty}\frac{r!}{(r+1)!} = 0$.

(e) $\lim_{r\to\infty}\frac{3^{-r}}{3^{-r-1}} = 3$.

(f)

$$\lim_{r\to\infty}\frac{r^r(r+1)!}{r!(r+1)^{r+1}} = \lim_{r\to\infty}\frac{r^r}{(r+1)^r} = \frac{1}{\lim_{r\to\infty}(1+\frac{1}{r})^r} = \frac{1}{e}.$$

(g) $\lim_{r\to\infty}\frac{r!(r+1)^{r+1}}{r^r(r+1)!} = e$.

(h) $\sum_{r=0}^\infty 2^r w^r$ has radius of convergence $\lim_{r\to\infty}\frac{2^r}{2^{r+1}} = \frac{1}{2}$, so is summable if $|w| < \frac{1}{2}$, not if $|w| > \frac{1}{2}$. $\sum_{r=0}^\infty 2^r x^{2r} = \sum_{r=0}^\infty 2^r(x^2)^r$ is therefore summable if $|x^2| < \frac{1}{2}$ (that is, if $|x| < \frac{1}{\sqrt{2}}$) and not if $|x^2| > \frac{1}{2}$ (that is, $|x| > \frac{1}{\sqrt{2}}$), and its radius of convergence is $\frac{1}{\sqrt{2}}$.

(i) $\sum_{r=0}^\infty 3^{-r}w^r$ has radius of convergence $\lim_{r\to\infty}|\frac{3^{-r}}{3^{-r-1}}| = 3$, so $\sum_{r=0}^\infty 3^{-r}x^{3r}$ has radius of convergence $\sqrt[3]{3}$.

(j) $\sum_{r=0}^\infty \frac{(1-\sqrt{5})^r}{r^2-2}w^r$ has radius of convergence

$$\lim_{r\to\infty}|\frac{(1-\sqrt{5})^r((r+1)^2-2)}{(r^2-2)(1-\sqrt{5})^{r+1}}| = \frac{1}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{4},$$

so $\sum_{r=0}^\infty \frac{(1-\sqrt{5})^r}{r^2-2}x^{2r}$ has radius of convergence $\sqrt{\frac{\sqrt{5}+1}{4}} = \frac{\sqrt{1+\sqrt{5}}}{2}$.

(k) $\sum_{r=0}^\infty x^r\cos r\theta$ has radius of convergence 1 (for any θ).

(l) $\lim_{r\to\infty}\frac{(2r)!(r+1)!(r+1)!}{r!r!(2r+2)!} = \lim_{r\to\infty}\frac{(r+1)^2}{(2r+1)(2r+2)} = \frac{1}{4}$.

(m) $\lim_{r\to\infty}\frac{r!(2r)!(3r+3)!}{(3r)!(r+1)!(2r+2)!} = \lim_{r\to\infty}\frac{(3r+1)(3r+2)(3r+3)}{(r+1)(2r+1)(2r+2)} = \frac{27}{4}$.

88. (a) $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$.

(b) $\frac{1}{1-x^2} = 1 + x^2 + x^4 + \dots$.

(c) $\arctan \frac{1}{2}x = \frac{x}{2} - \frac{x^3}{24} + \frac{x^5}{160} - \dots$.

(d) Write $t = \tan x$, $u = t' = 1 + t^2$. Then $t'' = u' = 2tt' = 2tu$. So

$$\begin{aligned}
t''' &= 2t'u + 2tu' = 2u^2 + 4t^2u,\\
t'''' &= 4uu' + 8tt'u + 4t^2u' = 8tu^2 + 8tu^2 + 8t^3u = 16tu^2 + 8t^3u,\\
t''''' &= 16t'u^2 + 32tuu' + 24t^2t'u + 8t^3u'\\
&= 16u^3 + 64t^2u^2 + 24t^2u^2 + 16t^4u\\
&= 16u^3 + 88t^2u^2 + 16t^4u.
\end{aligned}$$

At $x = 0$, $t = 0$ and $u = 1$ so $t' = 1$, $t'' = 0$, $t''' = 2$, $t'''' = 0$ and $t''''' = 16$; accordingly

$$\tan x = x + \frac{2}{3!}x^3 + \frac{16}{5!}x^5 + \dots = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots.$$

(e) $\sin x^2 = x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^{10} - \dots$.

(f) $\sin x \cos x = \frac{1}{2}\sin 2x = \frac{1}{2}(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \dots) = x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \dots$.

(g) $\sinh 3x = 3x + \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} + \dots = 3x + \frac{9}{2}x^3 + \frac{243}{40}x^5 + \dots$.

(h) $\exp(x + x^2) = 1 + (x + x^2) + \frac{1}{2}(x + x^2)^2 + \dots = 1 + x + \frac{3}{2}x^2 + \dots$.

(i) If $f(x) = \cosh(x+1)$ then $f'(x) = \sinh(x+1)$, $f''(x) = \cosh(x+1)$ so $f(0) = \cosh 1$, $f'(0) = \sinh 1$, $f''(0) = \cosh 1$ and

$$f(x) = \cosh 1 + x \sinh 1 + \frac{\cosh 1}{2}x^2 + \dots$$

for x near 0.

(j) $\sqrt{1-x^2} = (1-x^2)^{1/2} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 + \dots$.

89. (a) $\sin x = \sin 1 + \cos 1(x-1) - \frac{\sin 1}{2}(x-1)^2 - \dots$.

(b) $\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \dots$.

(c) $\frac{1}{x} = -1 - (x+1) - (x+1)^2 - \dots$.

(d) $x^3 - x = 2(x-1) + 3(x-1)^2 + (x-1)^3$.

90. (a)

$$\sum_{r=1}^{\infty}\frac{1}{r(r+2)}=\frac{1}{2}\sum_{r=1}^{\infty}(\frac{1}{r}-\frac{1}{r+2})$$
$$=\frac{1}{2}(1-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\frac{1}{4}-\frac{1}{6}+\dots)=\frac{3}{4}.$$

(b) $\sum_{r=0}^{\infty}\frac{2^r}{r!}=\exp 2.$

(c) $\sum_{r=0}^{\infty}(-1)^r x^r=\frac{1}{1+x}$ for $|x|<1$.

(d)

$$\sum_{r=0}^{\infty}x^r\cos r\theta=\mathcal{R}\mathrm{e}(\sum_{r=0}^{\infty}x^r e^{ir\theta})=\mathcal{R}\mathrm{e}(\frac{1}{1-xe^{i\theta}})$$
$$=\mathcal{R}\mathrm{e}(\frac{1-xe^{-i\theta}}{(1-xe^{i\theta})(1-xe^{-i\theta})})=\mathcal{R}\mathrm{e}(\frac{1-xe^{-i\theta}}{(1-xe^{i\theta}-xe^{-i\theta}+x^2)})$$
$$=\frac{1-x\cos\theta}{(1-2x\cos\theta+x^2)}$$

for $|x|<1$.

(e) $\sum_{r=1}^{\infty}rx^{r-1}=\frac{d}{dx}\sum_{r=0}^{\infty}x^r=\frac{d}{dx}\frac{1}{1-x}=\frac{1}{(1-x)^2}$ for $|x|<1$.

(f) $\sum_{r=0}^{\infty}(r+1)x^r=\sum_{r=1}^{\infty}rx^{r-1}=\frac{1}{(1-x)^2}$ for $|x|<1$.

(g) $\sum_{r=0}^{\infty}rx^r=x\sum_{r=1}^{\infty}rx^{r-1}=\frac{x}{(1-x)^2}$ for $|x|<1$.

(h) $\sum_{r=0}^{\infty}(r+1)(r+2)x^r=\frac{d^2}{dx^2}\sum_{r=0}^{\infty}x^r=\frac{2}{(1-x)^3}$ for $|x|<1$.

91. (a)

$$\sum_{r=0}^{\infty} r^2x^r = \sum_{r=0}^{\infty}(r+1)(r+2)x^r - 3\sum_{r=0}^{\infty} rx^r - 2\sum_{r=0}^{\infty} x^r$$
$$= \frac{2}{(1-x)^3} - 3\frac{x}{(1-x)^2} - 2\frac{1}{1-x} = \frac{2-3x(1-x)-2(1-x)^2}{(1-x)^3} = \frac{x+2x^2}{(1-x)^3}$$

for $|x| < 1$.

(b) $\sum_{r=0}^{\infty} \frac{r}{2^r} = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2.$

(c) $\sum_{r=0}^{\infty} \frac{(-100)^r}{(2r)!} = \cos 100.$

(d) $\sum_{r=0}^{\infty}(-1)^r x^{2r} = \frac{1}{1+x^2}$ for $|x| < 1$.

(e) $\sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1}x^{2r+1} = \arctan x$ for $|x| < 1$.

(f)

$$\frac{d}{dx}\sum_{r=0}^{\infty} \frac{1}{2r+1}x^{2r+1} = \sum_{r=0}^{\infty} x^{2r} = \frac{1}{1-x^2}$$

for $|x| < 1$;

$$\int \frac{1}{1-x^2}dx = \frac{1}{2}\int(\frac{1}{1-x} + \frac{1}{1+x})dx =_{\text{utc}} \frac{1}{2}(\ln(1+x) - \ln(1-x)) = \frac{1}{2}\ln\frac{1+x}{1-x};$$

since $\frac{1}{2}\ln\frac{1+x}{1-x} = 0$ when $x = 0$, $\sum_{r=0}^{\infty} \frac{1}{2r+1}x^{2r+1} = \frac{1}{2}\ln\frac{1+x}{1-x}$ when $|x| < 1$.

(g)

$$\sum_{r=0}^{\infty} \frac{1}{4r+1}x^{4r} = \frac{1}{2}\sum_{r=0}^{\infty} \frac{1+(-1)^r}{2r+1}x^{2r+1} = \frac{1}{2}(\arctan x + \frac{1}{2}\ln\frac{1+x}{1-x})$$

for $|x| < 1$.

(h) $\sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1}y^{2r} = \frac{1}{y}\arctan y$ if $0 < |y| < 1$. So if $0 < x < 1$, $\sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1}x^r = \frac{1}{\sqrt{x}}\arctan\sqrt{x}$.
$\sum_{r=0}^{\infty} \frac{1}{2r+1}y^{2r} = \frac{1}{2y}\ln\frac{1+y}{1-y}$ if $0 < |y| < 1$. If $-1 < x < 0$, then

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1}x^r = \sum_{r=0}^{\infty} \frac{1}{2r+1}(-x)^r = \frac{1}{2\sqrt{-x}}\ln\frac{1+\sqrt{-x}}{1-\sqrt{-x}}.$$

If $x = 0$ then $\sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1}x^r = 1$.

92. (No solution available.)

93. 1 in 2. (The pawns must be on squares of opposite colours, and if they are you can always manage it.)

94. (No solution available.)

95. About 49%, as in any other human population.

96. Start by choosing a door at random. Stick with your choice at the first offer to change, and switch at the second. This gives you three chances in four of winning. (You will win iff your first choice was *wrong*.)

97. (No solution available.)

98. If $OP = x$ and $OQ = y$, then $x^2 + y^2 = \alpha^2$; also, because the triangles APB, CBQ are similar, $(x-1)/1 = 1/(y-1)$, that is, $(x-1)(y-1) = 1$. Setting $u = (x+y)/\sqrt{2}$, $v = (x-y)/\sqrt{2}$ (that is, rotating the coordinate frame by 45°), these equations become $u^2 + v^2 = \alpha^2$, $(u + v - \sqrt{2})(u - v - \sqrt{2}) = 2$, that is, $(u - \sqrt{2})^2 - v^2 = 2$. Adding these equations, $2u^2 - 2\sqrt{2}u - \alpha^2 = 0$ and $u = \frac{1}{\sqrt{2}}(1 + \sqrt{1+\alpha^2})$. Now $v = \sqrt{\alpha^2 - u^2} = \frac{1}{2}(\alpha^2 - 2 - 2\sqrt{1+\alpha^2})$ and the values of x and y are

$$\frac{u \pm v}{\sqrt{2}} = \frac{1}{2}(1 + \sqrt{1+\alpha^2} \pm \sqrt{\alpha^2 - 2 - 2\sqrt{1+\alpha^2}}).$$

99. 1-d, 2-g, 3-c, 4-f, 5-b, 6-e, 7-a.

The first step is to count the occasions on which a word appears. Thus 'farmer' appears 4 times (in sentences 1,4,5,7), 'dentist' 4 times (2,3,4,7), 'dog' twice (1,2), 'cow' twice (2,3), 'bite' twice (2,5), 'hurt' twice (6,7), and 'step', 'take', 'sue' and 'tooth' once each. (Note that the 'take' in sentence 5 isn't the same as the 'take' in sentence 3; the phrase 'take a bite' matches the 'bite' in sentence 2.) To match these, we have chup-cha* (in sentences a,c,f,g), fila-cha* (a,b,d,f), tetro*or (a,e), dinge*or (b,g), fila* (c,g), muta* (d,g), and bap, sesnazor, chupi, ceclisor as singletons.

There is just one two-word sentence in Erewhonese, that is, sentence e, and this must correspond to sentence 6. In sentence e, we have one word (tetro*or) which appears again, so that must correspond to 'hurt', while 'chupi' means 'teeth'. The other sentence in which 'hurt' appears is 7, so that must match the other sentence in which 'tetro*or' appears, which is a. So we have 7-a and 6-e.

The words chup-cha* and fila-cha*, which appear four times, must mean 'dentist' and 'farmer', or the other way round. The sentences in which both occur are a, f, 4 and 7; since we have already matched 7 with a, we must have 4-f. So 'sesnazor' means 'sues'.

There is one four-word sentence in Erewhonese, sentence g, which must correspond to sentence 2. Since 'chup-cha*' appears in g and 'dentist' in 2, we know that 'chup-cha*' means 'dentist' and 'fila-cha*' means 'farmer'.

The remaining sentence with 'dentist' in it is 3, while the remaining sentence with 'chup-cha*' is c; so these go together. The double-occurrence word in 3 is 'cows', while the double-occurrence word in c is 'fila*'; so these go together, while 'sesnazor' must mean 'take'.

The remaining sentences are

(b) dingesor fila-cha bap.
(d) sqensesor fila-cha mutan.
(1) A farmer steps on a dog.
(5) A farmer takes a bite of bread.

Apart from 'fila-cha', which we know means 'farmer', we have 'dingesor' and 'mutan'; these appear also in sentence g, so must mean 'dog' and 'bite'. At this point, observe that in all the five sentences so far deciphered the verb comes first. So it's a fair assumption that 'dingesor' is also a verb, in which case we have 5-b and 1-d.

As a final confirmation, observe that in the verbs *sor is the singular form and *zor is the plural form. The use of the word 'bap' for 'bread' is a consequence of its introduction by a shipwrecked Scotsman in the eighteenth century.