Normed Spaces Problems

1. For each of the following statements, say whether it is true or false:
(a) $x \leq y \Rightarrow |x| \leq |y|$
(b) $x^2 \leq |x|^2$ for every $x \in \mathbb{R}$
(c) $\frac{|x|}{|y|} = \left|\frac{x}{y}\right|$ for every $x \in \mathbb{R}$, $y \neq 0$
(d) $x < y \Rightarrow x \leq y$
(e) $x \geq y$ & $y < z \Rightarrow x \leq z$
(f) $x \leq y$ & $y < z \Rightarrow x < z$
(g) $x < y$, $x \neq 0$ & $y \neq 0 \Rightarrow \frac{1}{x} > \frac{1}{y}$
(h) $x \leq x^2$ for every $x \geq 0$
(i) $x < y$ & $w \leq z \Rightarrow x + w < y + z$
(j) $x \leq y$ & $w \geq z \Rightarrow x - w \leq y - z$
(k) $0 < x^2 < y^2 \Rightarrow \frac{1}{-|x|} < \frac{1}{-|y|}$
(Throughout, x, y, z and w are supposed to be real numbers.) [A]

2. Let (X, ρ) be a metric space and $f : X \to \mathbb{R}$ a continuous function. Suppose that there is some $a \in \mathbb{R}$ such that $\{x : x \in X, f(x) \geq a\}$ is compact and non-empty. Show that there is an $x_0 \in X$ such that $f(x) \leq f(x_0)$ for every $x \in X$. [AA]

3. Let $(X_1, \rho_1), \dots, (X_n, \rho_n)$ be metric spaces with product X. Show that a subset K of X is compact iff it is closed and $\pi_j[K]$ is a compact subset of X_j for each $j \leq n$, setting $\pi_j(x) = \xi_j$ whenever $x = (\xi_1, \dots, \xi_n) \in X$. [AB]

4. Show that ρ_1 and ρ_∞ are metrics on $\mathbb{R}^n$ for any $n \in \mathbb{N}$, where $\rho_1(x, y) = \sum_{i=1}^n |\xi_i - \eta_i|$, $\rho_\infty(x, y) = \max_{1 \leq i \leq n} |\xi_i - \eta_i|$ for $x = (\xi_1, \dots, \xi_n)$, $y = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$. [AC]

5. Sketch the sets $N_1(0) = \{x : \rho(x, 0) \leq 1\}$ in $\mathbb{R}^2$ for $\rho = \rho_1$, ρ_2, ρ_∞, where $\rho_1(x, y) = \sum_{i=1}^2 |\xi_i - \eta_i|$, $\rho_2(x, y) = \sqrt{\sum_{i=1}^2 |\xi_i - \eta_i|^2}$, $\rho_\infty(x, y) = \max_{1 \leq i \leq 2} |\xi_i - \eta_i|$ for $x = (\xi_1, \xi_2)$, $y = (\eta_1, \eta_2) \in \mathbb{R}^2$. [AD]

6. Show that ρ_1, ρ_2, ρ_∞ are equivalent metrics on $\mathbb{R}^n$, where $\rho_1(x, y) = \sum_{i=1}^n |\xi_i - \eta_i|$, $\rho_2(x, y) = \sqrt{\sum_{i=1}^n |\xi_i - \eta_i|^2}$, $\rho_\infty(x, y) = \max_{1 \leq i \leq n} |\xi_i - \eta_i|$ for $x = (\xi_1, \dots, \xi_n)$, $y = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$. (Hint: show that $\rho_\infty(x, y) \leq \rho_1(x, y) \leq n\rho_\infty(x, y)$, $\rho_\infty(x, y) \leq \rho_2(x, y) \leq \sqrt{n}\rho_\infty(x, y)$ for all x, $y \in \mathbb{R}^n$.) [AE]

7. Let $X = C([0, 1])$, the space of continuous real-valued functions on $[0, 1]$. For f, $g \in X$ write $\rho_\infty(f, g) = \sup_{0 \leq \alpha \leq 1} |f(\alpha) - g(\alpha)|$. Show that ρ_∞ is a metric on X. [AF]

8. Let (X, ρ) be any metric space. Show that σ_1 and σ_2 are metrics on X, both equivalent to ρ, where

$$\sigma_1(x, y) = \min(1, \rho(x, y)) \text{ for } x,\ y \in X,$$

$$\sigma_2(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)} \text{ for } x,\ y \in X.$$

[AG]

9. Find the closures of the following sets:
(a) in $\mathbb{R}$, with the usual metric:
$[0, 1]$ $\quad]0, 1[$ $\quad]0, 1]$ $\quad \mathbb{Q}$ $\quad \mathbb{R} \setminus \mathbb{Q}$ $\quad \mathbb{N}$ $\quad \mathbb{Z}$ $\quad \{\frac{1}{n} : n \in \mathbb{N}\}$.
(b) in $\mathbb{R}^2$, with the Euclidean metric:
$\{(x, 0) : x \in \mathbb{R}\}$ $\quad \{(x, \sin \frac{1}{x}) : x > 0\}$
$\{(x, y) : 0 < x < 1, 0 < y < 1\}$ $\quad \{(x, y) : x + y \in \mathbb{Q}\}$
$\{(x, y) : \exists\, n \in \mathbb{N}, n(x^2 + y^2) = 1\}$ $\quad \{(x, y) : x^2 + y^2 \in \mathbb{N}\}$
Are any of these sets open? [AH]

10. In a metric space (X,ρ), we say that a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ **converges** to a point $x\in X$ if

$$\text{for every } \epsilon>0 \text{ there is an } n_0\in\mathbb{N} \text{ such that } \rho(x_n,x)\le\epsilon \text{ for every } n\ge n_0.$$

Show that there can be at most one such x (so that we can safely call x 'the' limit of the sequence). [AI]

11. Sketch the following sets in the line:
(a) $\{x : x\in\mathbb{R},\ x^2-2<0\}$
(b) $\{x : x\in\mathbb{R},\ |x-2|\le 2\}$
(c) $\{x : x\in\mathbb{R},\ x^3-x\ge 0\}$
(d) $\{x : x\in\mathbb{R},\ \frac{1}{x}\ge 1\}$
and in the plane:
(e) $\{(x,y) : x-3y\ge 0\}$
(f) $\{(x,y) : x^2+y^2<2\}$
(g) $\{(x,y) : x^2+3y^2<0\}$. [B]

12. For each of the sets A-I below, say (i) is it non-empty? (ii) is it bounded above? (iii) is it bounded below? (iv) what is its supremum, if any? (v) what is its infimum, if any? (vi) does it contain its supremum, or infimum, or both?
(Here $\mathbb{N}=\{1,2,3,\dots\}$, $\mathbb{Z}=\{0,\pm1,\pm2,\dots\}$, $\mathbb{Q}$ is the set of all rational numbers.)
$A=\{n : n\in\mathbb{Z}, n^2<100\}$
$B=\{x : x\in\mathbb{R},\ x^2+2x+1\ge 0\}$
$C=\{x : x\in\mathbb{R},\ 0\le x\le\pi\}$
$D=\{x : x\in\mathbb{Q},\ 0\le x\le\pi\}$
$E=\mathbb{N}$
$F=\{n : n\in\mathbb{Z},\ -n>n\}$
$G=\{x : x\in\mathbb{R},\ x\ne 0,\ \frac{1}{x}\in\mathbb{N}\}$
$H=]0,\pi[$
$I=[0,1]\setminus\mathbb{Q}$. [C]

13. For sets $X,\ Y\subseteq\mathbb{R}$ write $X+Y=\{x+y : x\in X, y\in Y\}$, $X-Y=\{x-y : x\in X, y\in Y\}$. Describe the sets $A+B$, $C+C$, $C-C$, $C+H$, $E+H$ where A, B, C, E, H are taken from the previous question. Show that $\sup(X+Y)=\sup X+\sup Y$ whenever $\sup X$ and $\sup Y$ are defined. State (without proof) corresponding results concerning $\inf(X+Y)$, $\sup(X-Y)$, $\inf(X-Y)$. [D]

14. What is meant by saying that $\lim_{n\to\infty}x_n=x$, where x and x_n are real numbers? Show from your definition that
(i) if $\lim_{n\to\infty}x_n=x$ and $a\le x_n\le b$ for every $n\in\mathbb{N}$, then $a\le x\le b$;
(ii) if $\lim_{n\to\infty}x_n=x$ and $\lim_{n\to\infty}y_n=y$ and $x_n\le y_n$ for every $n\in\mathbb{N}$, then $x\le y$.
Is it true that if $\lim_{n\to\infty}x_n=x$ and $a<x_n<b$ for every $n\in\mathbb{N}$, then $a<x<b$? Justify your answer.
[If you use any properties of convergent sequences here, you should prove them.] [E]

15. Draw up a table (with six rows and four columns) to show for each of the sequences below whether (i) it is monotonic (in which case state whether it is non-decreasing or non-increasing) (ii) it is bounded above or below (in which case give its supremum and infimum if you can) (iii) it is convergent (in which case give the limit if you can) and (iv) if the sequence is convergent, give the least integer n_0 you can find such that $|x_n-x|\le\frac{1}{100}$ for every $n\ge n_0$.
$a_n=\frac{n}{n+1}$ for $n\ge 0$;
$b_n=n$ for even $n\ge 0$, $\frac{1}{n}$ for odd $n\ge 1$;
$c_n=10^{-5}$ if $0\le n\le 10$, n^{-n} if $n>10$;
$d_n=n-\sqrt{n^2+n}$ for $n\ge 0$;
$e_n=\sin n$ for $n\ge 0$;
$f_0=1$, $f_{n+1}=1+1/f_n$ for $n\ge 0$.
[F]

16. Define *contraction* on a metric space. If $a<b$ in $\mathbb{R}$ and $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on $]a,b[$, show that f is a contraction iff $\sup_{t\in]a,b[}|f(t)|<1$. Give an example of a contraction $f:\mathbb{R}\to\mathbb{R}$ which is not differentiable. [FA]

17. State and prove the Contraction Mapping Theorem.

Consider the equation $x^5 + 2x = \cos x$, where $x \in \mathbb{R}$. Express this in the form $x = f(x)$, where $f : \mathbb{R} \to \mathbb{R}$ is a contraction (prove it!). Use the associated iterative scheme to find a solution correct to three decimal places. [FB]

18. Show that for a differentiable function $f : \mathbb{R} \to \mathbb{R}$ and $\alpha \geq 0$ the following are equivalent: (i) $|f(x) - f(y)| \leq \alpha|x - y|$ for all x, $y \in \mathbb{R}$ (ii) $|f'(x)| \leq \alpha$ for all $x \in \mathbb{R}$. [FC]

19. (a) Let (X, ρ) be a non-empty complete metric space and $f : X \to X$ a continuous function with the property that, for some $\alpha \in [0, 1[$,

$$\rho(f(x), f^2(x)) \leq \alpha\rho(x, f(x)) \text{ for every } x \in X.$$

Show that f has a fixed point. Is this necessarily unique? Give a proof or a counter-example.

(b) Suppose that X is a set and $f : X \to X$ is a function such that $f^n : X \to X$ has a unique fixed point, where $n \geq 2$ is an integer. What, if anything, can we say about the fixed points of f? [FD]

20. State Baire's theorem for complete metric spaces. Show that if (X, ρ) is a complete metric space and $\langle G_n \rangle_{n \in \mathbb{N}}$ is a sequence of open sets with $\bigcap_{n \in \mathbb{N}} G_n = \emptyset$, then $\bigcap_{n \in \mathbb{N}} \overline{G_n}$ does not include any non-empty open set. [FE]

21. Let (X, ρ) be a complete metric space and $f : X \to \mathbb{R}$ a function such that $\{x : f(x) < a\}$ is open for every $a \in \mathbb{R}$. Show that $\{x : f \text{ is continuous at } x\}$ is dense in X. (*Hint:* set $G_q = \{x : f(x) < q\}$, $H_q = G_q \cup (X \setminus \overline{G_q})$ for $q \in \mathbb{Q}$. Show that $E = \bigcap_{q \in \mathbb{Q}} H_q$ is dense in X. Show that f is continuous at every point of E.) [FF]

22. In any metric space X, a $\mathbf{G}_\delta$ **set** is a set expressible as $\bigcap_{n \in \mathbb{N}} G_n$ for some sequence $\langle G_n \rangle_{n \in \mathbb{N}}$ of open sets.

(a) Show that if (X, ρ) and (Y, σ) are metric spaces and $f : X \to Y$ is any function, then

$$\{x : f \text{ is continuous at } x\}$$

is a G_δ subset of X. (*Hint:* show that it is

$$\bigcap_{k \in \mathbb{N}} \bigcup \{H : H \subseteq X \text{ is open, } \sigma(f(x), f(y)) \leq \tfrac{1}{k} \text{ for all } x,\, y \in H\}.)$$

(b) Show that in a complete metric space, the intersection of two dense G_δ sets is dense.

(c) Let (X, ρ) be a complete metric space, and f, g two real-valued functions on X such that $\{x : f \text{ is continuous at } x\}$ and $\{x : g \text{ is continuous at } x\}$ are both dense. Show that $\{x : f + g \text{ is continuous at } x\}$ is dense. [FG]

23. The following began as a correct proof that

$$\text{if } \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y \neq 0, \text{ then } \lim_{n\to\infty} \tfrac{x_n}{y_n} \text{ exists } = \tfrac{x}{y}.$$

A malicious person replaced many of the signs $\le$, $<$, $\ge$, $>$ with $\bigcirc$. Write it out again with the inequalities as they should be, in a different colour from the rest.

Let $\epsilon > 0$. Set

$$\eta = \min(\tfrac{1}{2}|y|, \tfrac{\epsilon|y|^2}{2(|x|+|y|)}) \bigcirc 0.$$

Because $\lim_{n\to\infty} x_n = x$, there is an $n_1 \in \mathbb{N}$ such that $|x_n - x| \le \eta$ for every $n \ge n_1$. Because $\lim_{n\to\infty} y_n = y$, there is an $n_2 \in \mathbb{N}$ such that $|y_n - y| \bigcirc \eta$ for every $n \bigcirc n_2$. Set $n_0 = \max(n_1, n_2) \in \mathbb{N}$. Then if $n \ge n_0$ we have $n \bigcirc n_1$ and $n \bigcirc n_2$, so that $|x_n - x| \bigcirc \eta$ and $|y_n - y| \bigcirc \eta$ and, in the first place,

$$|y_n| \ge |y| - |y_n - y| \bigcirc |y| - \eta \bigcirc |y| - \tfrac{1}{2}|y| = \tfrac{1}{2}|y| \bigcirc 0;$$

in particular, $y_n \neq 0$ and x_n/y_n is defined. Next, still taking $n \bigcirc n_0$,

$$\begin{aligned}
|\frac{x_n}{y_n} - \frac{x}{y}| = |\frac{x_n y - x y_n}{y_n y}| &= \frac{|x_n y - x y_n|}{|y_n||y|} \\
&\bigcirc \frac{|x_n y - x y_n|}{\frac{1}{2}|y|^2} = \frac{2}{|y|^2}|x_n y - x y_n| \\
&= \frac{2}{|y|^2}|(x_n - x)y + x(y - y_n)| \bigcirc \frac{2}{|y|^2}(|(x_n - x)y| + |x(y - y_n)|) \\
&= \frac{2}{|y|^2}(|x_n - x||y| + |x||y - y_n|) \bigcirc \frac{2}{|y|^2}(\eta|y| + |x|\eta) \\
&= \frac{2(|x| + |y|)}{|y|^2}\eta \bigcirc \epsilon.
\end{aligned}$$

This is true for every $n \bigcirc n_0$. Because ϵ is arbitrary, $\lim_{n\to\infty} x_n/y_n$ exists $= x/y$.

[G]

24. Define *linear space* over $\mathbb{R}$. What is meant by the (real) space ℓ^2? Show that it is a linear space. [GA]

25. What is a *norm* on a real linear space? Define the norms $\| \|_1$ on ℓ^1 and $\| \|_\infty$ on $C[a,b]$, and show that they are norms. [GB]

26. Define the terms *norm* (on a real linear space), *metric*, *convergent sequence*, *Cauchy sequence* (in a metric space). Explain (with proof) how a norm gives rise to a metric. Give an example of a normed space with a Cauchy sequence which is not convergent. [GC]

27. For a real $m \times n$ matrix $\mathbf{A} = \langle \alpha_{ij} \rangle_{1\le i\le m, 1\le j\le n}$ write $\|\mathbf{A}\|_{\mathrm{HS}} = \sqrt{\sum_{i\le m, j\le n} |\alpha_{ij}|^2}$. Show that if $x \in \mathbb{R}^n$ then $\|\mathbf{A}x\|_2 \le \|\mathbf{A}\|_{\mathrm{HS}} \|x\|_2$. (*Hint.* Show that (i) $\sum_j |\xi_j| \sqrt{\sum_i |\alpha_{ij}|^2} \le \|\mathbf{A}\|_{\mathrm{HS}} \|x\|_2$ (ii) $|\sum_{ijk} \alpha_{ij}\alpha_{ik}\xi_j\xi_k| \le \sum_{jk} \sqrt{\sum_i |\alpha_{ij}|^2} \sqrt{\sum_i |\alpha_{ik}|^2} |\xi_j||\xi_k|$.) [GD]

28. For each of the following sets J and families $\mathcal{A}$ of sets, say (i) whether $J \subseteq \bigcup \mathcal{A}$, and if not find a point of $J \setminus \bigcup \mathcal{A}$ (ii) whether there is a finite collection $I_0, \ldots, I_n$ of members of $\mathcal{A}$ such that $J \subseteq I_0 \cup \ldots \cup I_n$, and if so describe such a collection.

(a) $J = [0,1]$, $\mathcal{A} = \{]\frac{1}{n}, 2[: n \in \mathbb{N}\}$.

(b) $J =]0,1]$, $\mathcal{A} = \{]\frac{1}{n}, 2[: n \in \mathbb{N}\}$.

(c) $J = [10^{-25}, 1]$, $\mathcal{A} = \{]\frac{1}{n}, 2[: n \in \mathbb{N}\}$.

(d) $J = [0,1]$, $\mathcal{A} = \{[a,b] : \text{either } a < b < \frac{1}{2} \text{ or } \frac{1}{2} \le a < b\}$. [H]

29. What is meant by saying that two norms are *equivalent*? Show that on $\mathbb{R}^n$ all the three norms $\| \|_1$, $\| \|_2$ and $\| \|_\infty$ are equivalent, but that on $C[0,1]$ no pair of these norms are equivalent. [HA]

30. Give two definitions of *closed set* in a metric space, one in terms of open sets and one in terms of convergent sequences.

Show that the space $\mathbf{c}$ of convergent real sequences is a closed subset of the space ℓ^∞ of bounded real sequences.

Give an example of a normed space with a linear subspace which is not a closed set. [HB]

31. (a) Let X be a normed space over $\mathbb{R}$. Describe (with proof) the metric associated with the norm.

(b) Show that the function $(x,y) \mapsto x+y : X \times X \to X$ is continuous, if $X \times X$ is given its product metric. Is it uniformly continuous?

(c) Show that the function $(\alpha, x) \mapsto \alpha x : \mathbb{R} \times X \to X$ is continuous, if $\mathbb{R} \times X$ is given its product metric. Is it uniformly continuous? [HC]

32. (a) What is meant by saying that a subset of a normed space is *bounded*? What is meant by saying that two norms on a given linear space are *equivalent*?

(b) Let X be a linear space over $\mathbb{R}$ and $\| \|$, $\| \|'$ two norms on X. Show that if one of the following is true, so are the others:

(i) the two norms are equivalent;
(ii) the two norms give rise to the same bounded sets;
(iii) the two norms give rise to equivalent metrics;
(iv) the two norms give rise to the same convergent sequences;
(v) the two norms give rise to the same Cauchy sequences;
*(vi) the two norms give rise to the same totally bounded sets;
*(vii) the two norms give rise to the same complete sets;
*(viii) the two norms give rise to the same compact sets. [HD]

33. Let (X, ρ) be a metric space, $x_0 \in X$. Is the function $x \mapsto \rho(x, x_0) : X \to \mathbb{R}$ continuous? ($\mathbb{R}$ is of course given its usual metric.) Justify your answer. [I]

34. Define *complete* metric space, *Banach* space.

Let S be the linear space of those real sequences $x = \langle \xi_i \rangle_{i \in \mathbb{N}}$ with only finitely many non-zero terms. Let $\| \|$ be a norm on S and set

$$X = \{x : x \in \mathbb{R}^{\mathbb{N}},\ \sup_{n \in \mathbb{N}} \|P_n x\| < \infty\},$$

writing $\mathbb{R}^{\mathbb{N}}$ for the linear space of all real sequences and defining $P_n : \mathbb{R}^{\mathbb{N}} \to S$ by saying that $P_n x = (\xi_1, \xi_2, \dots, \xi_n, 0, 0, \dots)$ if $x = (\xi_1, \dots) \in \mathbb{R}^{\mathbb{N}}$. Show that X is a linear space and is a Banach space under the norm

$$\|x\| = \sup_{n \in \mathbb{N}} \|P_n x\|.$$

(You may assume that for each $n \in \mathbb{N}$ there are α_n, β_n such that $0 < \alpha_n \le \beta_n < \infty$ and $\alpha_n \|x\|_\infty \le \|x\| \le \beta_n \|x\|_\infty$ whenever $x \in S$ is of the form $x = (\xi_1, \xi_2, \dots, \xi_n, 0, 0, \dots)$.) [IA]

35. Without appealing to [IA], show that ℓ^1 and $C[a,b]$, with appropriate norms (to be described), are complete. [IB]

36. For any set I, write ℓ^∞_I for the set of bounded functions from I to $\mathbb{R}$. For $x \in \ell^\infty_I$ write $\|x\|_\infty = \sup_{i \in I} |x(i)|$. Show that $\| \|_\infty$ is a norm and that ℓ^∞_I is a Banach space. [IC]

37. Let X be a Banach space and write $\ell^1(X)$ for the set of sequences $x = (\xi_1, \xi_2, \dots)$ where ξ_1, $\xi_2, \dots$ belong to X and $\|x\|_1 = \sum_{i=1}^\infty \|\xi_i\| < \infty$. Show that $\ell^1(X)$ is a linear space (describing carefully the linear operations), that $\| \|_1$ is a norm, and that $\ell^1(X)$ is complete. [ID]

38. (a) Define *linear space* over $\mathbb{R}$.

(b) Show that if X is any linear space over $\mathbb{R}$, and I any set, then X^I, with appropriate operations (to be defined), is a linear space over $\mathbb{R}$.

39. Let X be a normed space. Let $\ell^1(X)$ be the space of all sequences $x = (\xi_1, \xi_2, \dots)$ in X such that $\|x\|_1 = \sum_{i=1}^{\infty} \|\xi_i\| < \infty$.
(a) Show that $\ell^1(X)$ is a linear subspace of $X^{\mathbb{N}}$ and that $\| \|_1$ is a norm on $\ell^1(X)$.
(b) Show that if X is a Banach space so is $\ell^1(X)$.

40. Let (X, ρ) be a metric space, and write $C_b(X)$ for the space of bounded continuous real-valued functions on X. Show that $C_b(X)$ is a closed linear subspace of ℓ_X^∞ (see q. IC). [IE]

41. Let X be a metric space. Write $C_{bu}(X)$ for the set of bounded uniformly continuous real-valued functions on X. Show that $C_{bu}(X)$ is a closed linear subspace of the normed space $C_b(X)$ of all bounded continuous functions on X. [IF]

42. Let X be the set of functions $f : [0,1] \to \mathbb{R}$ such that $f(0) = 0$ and

$$\|f\| = \sup\{\textstyle\sum_{i=1}^{k} |f(t_i) - f(t_{i-1})| : 0 = t_0 \le t_1 \le \dots \le t_k = 1\}$$

is finite. Show that X is a linear subspace of $\mathbb{R}^{[0,1]}$ and that $\| \|$ is a norm on X under which X is a Banach space. [IG]

43. Give $\mathbb{R}$ its usual metric and $\mathbb{R}^2$ its Euclidean metric. Show that the following functions are continuous:
(a) $(\xi_1, \xi_2) \mapsto \xi_1 + \xi_2 : \mathbb{R}^2 \to \mathbb{R}$
(b) $(\xi_1, \xi_2) \mapsto \max(\xi_1, \xi_2) : \mathbb{R}^2 \to \mathbb{R}$
(c) $x \mapsto (x, x^2) : \mathbb{R} \to \mathbb{R}^2$. [J]

44. State and prove Picard's theorem on the solution of integral equations of the form $x(t) = x_0 + \int_{t_0}^{t} f(u, x(u))du$. What iterative scheme is associated with such an equation?
Consider the equation $\frac{dx}{dt} = x^2$, $x(0) = 1$. Describe the associated integral equation and the first few approximations to the solution given by your iterative scheme. Compare these with the exact solution. [JA]

45. Let X be the space of continuous real-valued functions on $[0,1]$, with the norm $\| \|_\infty$. For $x \in X$, $t \in [0,1]$ write

$$(Tx)(t) = e^{-3\sqrt{t}} \int_0^t \cos(\tfrac{x(u)}{\sqrt{u}} e^{3\sqrt{u}})du.$$

(a) Explain why, taking the integral to be a Riemann integral, $(Tx)(t)$ is always defined and $Tx \in X$.
(b) Show that

$$|(Tx)(t) - (Ty)(t)| \le e^{-3\sqrt{t}} \|x - y\|_\infty \int_0^t \frac{e^{3\sqrt{u}}}{\sqrt{u}} du.$$

(c) Show that $\|Tx - Ty\|_\infty \le \frac{2}{3}\|x - y\|_\infty$ for all $x, y \in X$. (*Hint:* You will need to evaluate the integral of $e^{3\sqrt{u}}/\sqrt{u}$; use the substitution $v = \sqrt{u}$.)
(d) Show that there is a continuous function $h : [0,1] \to \mathbb{R}$ such that $h(0) = 0$ and $h'(t) = \cos(\frac{h(t)}{\sqrt{t}})$ for $t \in]0,1[$. [JB]

46. Let X be the space of bounded continuous real-valued functions on $\mathbb{R}$, with the norm $\|x\|_\infty = \sup_{t \in \mathbb{R}} |x(t)|$.
(a) Show that X is a Banach space.
(b) For $x \in X$, define $Tx : \mathbb{R} \to \mathbb{R}$ by setting

$$(Tx)(t) = e^{-t^2}\left(1 + \int_0^t \sin(ue^{u^2} x(u))du\right)$$

for $x \in X$. Show that

$$|(Tx)(t) - (Ty)(t)| \le e^{-t^2} \|x - y\|_\infty \int_0^{|t|} ue^{u^2} du$$

for all $x, y \in X$ and all $t \in \mathbb{R}$.
(c) Show that $Tx \in X$ for every $x \in X$, and that $\|Tx - Ty\|_\infty \le \frac{1}{2}\|x - y\|_\infty$ for all $x, y \in X$.
(d) Show that there is a differentiable function $h : \mathbb{R} \to \mathbb{R}$ such that $h(0) = 1$ and $h'(t) = \sin(th(t))$ for every $t \in \mathbb{R}$. [JC]

47. Let (X,ρ) be a metric space. For any set $A \subseteq X$, let ρ_A be the subspace metric on A.

(a) Show that if $F \subseteq X$ is closed, then a subset H of F is closed in the metric space (F,ρ_F) iff it is closed in (X,ρ).

(b) Show that if (Y,σ) is another metric space and $f : X \to Y$ is a function, then f is continuous iff $f^{-1}[H]$ is closed in X for every closed set $H \subseteq Y$.

(c) Suppose that E, F are closed subsets of X such that $E \cup F = X$. Let (Y,σ) be another metric space and $f : X \to Y$ a function such that $f{\restriction}E : E \to Y$ and $f{\restriction}F : F \to Y$ are continuous, where $f{\restriction}E$ is the restriction of f to E. Show that f is continuous. [K]

48. Give three definitions of *continuous function* between metric spaces (one in terms of ϵ and δ, one in terms of open sets, and one in terms of convergent sequences).

What is meant by saying that a subset of a normed space is *bounded*? Give (with proof) a necessary and sufficient condition, in terms of bounded sets, for a linear operator between normed spaces to be continuous.

Let $\mathbf{c}$ be the space of convergent real sequences and for $x = \langle \xi_i \rangle_{i\in\mathbb{N}} \in \mathbf{c}$ set $f(x) = \lim_{i\to\infty} \xi_i$. Show that $f : \mathbf{c} \to \mathbb{R}$ is linear and continuous. [KA]

49. (a) For a linear space Y and a set X, describe (with proof) the linear structure of the space Y^X of all functions from X to Y.

(b) For linear spaces X and Y, what is meant by saying that a function $T : X \to Y$ is *linear*? Show that the set $L(X;Y)$ of linear operators from X to Y is a linear subspace of Y^X.

(c) For normed spaces X and Y, what is meant by saying that a linear operator $T : X \to Y$ is *bounded*? Show that the set $B(X;Y)$ of bounded linear operators from X to Y is a linear subspace of $L(X;Y)$.

(d) For normed spaces X and Y, describe the *norm* of a bounded linear operator from X to Y. Show that this is indeed a norm on $B(X;Y)$, and that $B(X;Y)$ is complete if Y is. [KB]

50. Define the term *bounded* as applied (i) to a set in a normed space (ii) to a linear operator, and explain its relationship to continuity. Describe (without proof) the linear structure and norm on the space $B(X,Y)$ of bounded linear operators from one normed space to another. When is $B(X,Y)$ complete? Give an example of normed spaces X, Y and a linear operator from X to Y which is not bounded. [KC]

51. Let $\mathbf{A}$ be the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Regarding $\mathbf{A}$ as a linear operator from $\mathbb{R}^2$ to itself, find the norm of $\mathbf{A}$ in each of the nine possible cases in which $\mathbb{R}^2$ is given one of the norms $\| \|_1$, $\| \|_2$, $\| \|_\infty$ when regarded as the domain of $\mathbf{A}$, and another when regarded as the range. [KD]

52. If X is any set, a collection $\mathcal{A}$ of subsets of X has the **finite intersection property** if $A_0 \cap A_1 \cap \ldots \cap A_n \neq \emptyset$ for every finite list $A_0, \ldots, A_n$ of members of $\mathcal{A}$.

Show that a metric space X is compact iff whenever $\mathcal{E}$ is a non-empty collection of closed subsets of X with the finite intersection property, then $\bigcap \mathcal{E} \neq \emptyset$. [L]

53. What is the *dual* of a normed space? Show that the dual of $\mathbf{c}_0$ may be identified with ℓ^1 and the dual of ℓ^1 may be identified with ℓ^∞. [LA]

54. (a) What is the *bidual* of a normed space? Explain (without proof) how any normed space can be identified with a subspace of its bidual.

(b) Let X and Y be normed spaces and $T : X \to Y$ a bounded linear operator. Explain what is meant by the *adjoint* $T^\times$ of T. Now suppose that $x \in X$ and that $\hat{x}$ is the corresponding member of the bidual X^{**}, as described in (a). What can you say about $(T^\times)^\times(\hat{x})$? [LB]

55. State the Hahn-Banach theorem. Use it to show that ℓ^1 is not reflexive. [LC]

56. Writing $\mathbf{c}$ for the space of all convergent real sequences, $\mathbf{c}_0$ for the space of sequences converging to 0, show that there is a bounded linear operator $T : \mathbf{c} \to \mathbf{c}_0$ such that $Tx = x$ for every $x \in \mathbf{c}_0$. [LD]

57. (a) Define the (real) space ℓ^2, showing that it is a linear subspace of $\mathbb{R}^{\mathbb{N}}$.
(b) Define its norm $\| \|_2$, showing that it is a norm.
(c) Show that ℓ^2 is a Banach space.
(d) Explain how the dual of ℓ^2 may be identified, as normed linear space, with ℓ^2. [LE]

58. Give an example of an open cover of $\mathbb{Q} \cap [0,1]$, regarded as a subset of $\mathbb{R}$ with its usual metric, which has no finite subcover. [M]

59. State and prove the Uniform Boundedness Theorem. Explain how it may be used to provide characterizations of bounded sets in Banach spaces and their duals in terms of boundedness of certain linear functionals.

Give an example of a normed space X and a sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ in X^* such that $\lim_{n\to\infty} f_n(x) = 0$ for every $x \in X$ but $\{f_n : n \in \mathbb{N}\}$ is not bounded. [MA]

60. For normed spaces X and Y describe (without proof) the linear structure of $X \times Y$ and a suitable norm on $X \times Y$. Show that if $T : X \to Y$ is a linear operator its graph is a linear subspace of $X \times Y$, and that if moreover T is bounded then its graph is closed. Give an example in which the graph of T is closed but T is not bounded. [MB]

61. State *Baire's theorem.* State and prove the Open Mapping Theorem and the Closed Graph Theorem. [MC]

62. State *Baire's theorem.* State and prove the Uniform Boundedness Theorem. Show that if X is a normed space and $A \subseteq X$ is a (non-empty) set such that $\sup_{x\in A} |f(x)| < \infty$ for every f in the dual X^* of X, then A is bounded. Hence show that if Y is another normed space and $T : Y \to X$ is a linear operator such that $f \circ T$ is continuous for every $f \in X^*$, then T is continuous. [MD]

63. Show that compact subsets of $\mathbb{R}^n$, with the Euclidean metric, are bounded. [N]

64. If (X, ρ) is a metric space, $A \subseteq X$ a non-empty set, and $x \in X$, write $\rho(x, A) = \inf_{y\in A} \rho(x, y)$. If A, B are non-empty subsets of X, write $\rho(A, B) = \inf_{x\in A, y\in B} \rho(x, y)$.
(a) Show that $x \in \overline{A}$ iff $\rho(x, A) = 0$.
(b) Show that, for any x, $y \in X$ and any non-empty $A \subseteq X$, $|\rho(x, A) - \rho(y, A)| \le \rho(x, y)$. (*Hint:* show first that $\rho(x, A) \le \rho(x, y) + \rho(y, A) + \epsilon$ for every $\epsilon > 0$.)
(c) Show that if $K \subseteq X$ is compact, $F \subseteq X$ is closed, and $K \cap F = \emptyset$, then $\rho(K, F) > 0$ (assuming, of course, that K, F are both non-empty).
(d) Give an example of two non-empty closed sets E, $F \subseteq \mathbb{R}^2$ such that $E \cap F = \emptyset$ but $\rho_2(E, F) = 0$, where ρ_2 is the Euclidean metric on $\mathbb{R}^2$. [O]

65. Let (X, ρ) and (Y, σ) be metric spaces, $f : X \to Y$ a function and x^* a point of X. Show that the following conditions are equivalent:
(i) f is continuous at x^*;
(ii) $\lim_{n\to\infty} f(x_n) = f(x^*)$ whenever $\langle x_n \rangle_{n\in\mathbb{N}}$ is a sequence in X converging to x^*;
(iii) $f(x^*) \in \overline{f[A]}$ whenever $A \subseteq X$ and $x^* \in \overline{A}$. [P]

66. Let (X, ρ) be a metric space.
(a) Show that any convergent sequence in X must be Cauchy.
(b) Let A be a subset of X, and ρ_A the induced metric on A. Show that if A is complete for ρ_A then A must be a closed subset of X.
(c) Now suppose that X is complete for ρ, and that F is a closed subset of X. Show that F is complete for its induced metric ρ_F. [Q]

67. On $\mathbb{N}$, define ρ_1, ρ_2 by setting $\rho_1(m, n) = |m - n|$, $\rho_2(m, n) = |\frac{1}{m} - \frac{1}{n}|$.
(a) Show that ρ_2 is a metric, and that ρ_1, ρ_2 are equivalent.
(b) Describe the Cauchy sequences in $\mathbb{N}$ for each of these metrics.
(c) Is $\mathbb{N}$ (α) complete (β) totally bounded (γ) compact under (i) ρ_1 (ii) ρ_2? [R]

68. Let (X, ρ) be a metric space, $\langle x_n \rangle_{n \in \mathbb{N}}$ a sequence in X and x a point of X. Show that the following are equivalent: (i) there is a subsequence of $\langle x_n \rangle_{n \in \mathbb{N}}$ converging to x (ii) $x \in \overline{\{x_i : i \geq n\}}$ for every $n \in \mathbb{N}$.
(In this case, x is called a **cluster point** of the sequence $\langle x_n \rangle_{n \in \mathbb{N}}$.) [S]

69. Let (X, ρ) be a metric space and $\langle x_n \rangle_{n \in \mathbb{N}}$ a sequence in X. Show that the set of limits of subsequences of $\langle x_n \rangle_{n \in \mathbb{N}}$ is a closed subset of X. [T]

70. Let (X, ρ) be a metric space and $\langle x_n \rangle_{n \in \mathbb{N}}$ a Cauchy sequence in X. Show that if x is a cluster point of $\langle x_n \rangle_{n \in \mathbb{N}}$ then $\langle x_n \rangle_{n \in \mathbb{N}}$ converges to x. [U]

71. Show that the space $C([0,1])$ of continuous real-valued functions on $[0,1]$ is complete under its metric ρ_∞. [V]

72. For subsets A, B of $\mathbb{R}$, set $A + B = \{x + y : x \in A,\ y \in B\}$.
(a) Show that if $K \subseteq \mathbb{R}$ is compact and $F \subseteq \mathbb{R}$ is closed, then $K + F$ is closed.
(b) Show that if K, $L \subseteq \mathbb{R}$ are compact, then $K + L$ is compact.
(c) Find an example of closed sets E, $F \subseteq \mathbb{R}$ such that $E + F$ is not closed. [W]

73. Let (X, ρ) be a metric space. Let ρ^* be the product metric on $X \times X$. Show that ρ, regarded as a function from $X \times X$ to $\mathbb{R}$, is uniformly continuous for ρ^* and the usual metric on $\mathbb{R}$. [X]

74. Let (X_1, ρ_1), (X_2, ρ_2) be two metric spaces with product (X, ρ). Show that a sequence $\langle x^{(n)} \rangle_{n \in \mathbb{N}}$ in X converges in X iff both sequences $\langle \xi_1^{(n)} \rangle_{n \in \mathbb{N}}$, $\langle \xi_2^{(n)} \rangle_{n \in \mathbb{N}}$ converge, writing $x^{(n)} = (\xi_1^{(n)}, \xi_2^{(n)})$ for each n. Hence show directly that if each X_j is sequentially compact (i.e., every sequence in X_j has a convergent subsequence), so is X. [Y]

75. Let $\langle (X_n, \rho_n) \rangle_{n \in \mathbb{N}}$ be a sequence of metric spaces. Set $X = \prod_{n \in \mathbb{N}} X_n$, the set of all sequences $x = \langle \xi_n \rangle_{n \in \mathbb{N}}$ such that $\xi_n \in X_n$ for every $n \in \mathbb{N}$. Define $\rho : X \times X \to \mathbb{R}$ by writing

$$\rho(x, y) = \sum_{n=1}^{\infty} \min(2^{-n}, \rho_n(\xi_n, \eta_n))$$

whenever $x = \langle \xi_n \rangle_{n \in \mathbb{N}}$, $y = \langle \eta_n \rangle_{n \in \mathbb{N}} \in X$.
(a) Show that ρ is a metric on X.
(b) Show that a sequence $\langle x^{(r)} \rangle_{r \in \mathbb{N}}$ in X converges to $x \in X$ iff $\xi_n = \lim_{r \to \infty} \xi_n^{(r)}$ for every $n \in \mathbb{N}$, writing $x = \langle \xi_n \rangle_{n \in \mathbb{N}}$, $x^{(r)} = \langle \xi_n^{(r)} \rangle_{n \in \mathbb{N}}$ for each r.
(c) Show that a sequence $\langle x^{(r)} \rangle_{r \in \mathbb{N}}$ in X is Cauchy in X iff $\langle \xi_n^{(r)} \rangle_{r \in \mathbb{N}}$ is Cauchy in X_n for every $n \in \mathbb{N}$.
(d) Show that if every X_n is complete, so is X.
(e) Show that if every X_n is totally bounded, so is X.
(f) Show that if every X_n is compact, so is X.
(g) Let (Y, σ) be another metric space, $f : Y \to X$ a function, and y_0 a point of Y. Show that f is continuous at y_0 iff every f_n is continuous at y_0, setting $f_n(y) = \xi_n$ if $f(y) = (\xi_1, \xi_2, \ldots)$. [Z]