Normed Spaces Answers

1.

(a) F	(b) T	(c) T	(d) T
(e) F	(f) T	(g) F	(h) F
(i) T	(j) T	(k) T	

2. (Not available.)

3. (Not available.)

4. (Not available.)

5. (Not available.)

6. (a) Take any x, $y \in \mathbb{R}^n$. If $j \le n$, then

$$|\xi_j - \eta_j| \le \sum_{i=1}^n |\xi_1 - \eta_i| = \rho_1(x,y),$$

so $\rho_\infty(x,y) \le \rho_1(x,y)$; also

$$|\xi_j - \eta_j|^2 \le \sum_{i=1}^n |\xi_1 - \eta_i|^2 = \rho_2(x,y)^2,$$

so $|\xi_j - \eta_j| \le \rho_2(x,y)$ for every j, and $\rho_\infty(x,y) \le \rho_2(x,y)$.

Next, $|\xi_i - \eta_i| \le \rho_\infty(x,y)$, $|\xi_1 - \eta_i|^2 \le \rho_\infty(x,y)^2$ for every i, so

$$\rho_1(x,y) = \sum_{i=1}^n |\xi_i - \eta_i| \le \sum_{i=1}^n \rho_\infty(x,y) = n\rho_\infty(x,y),$$

$$\rho_2(x,y)^2 = \sum_{i=1}^n |\xi_i - \eta_i|^2 \le \sum_{i=1}^n \rho_\infty(x,y)^2 = n\rho_\infty(x,y)^2,$$

so $\rho_2(x,y) \le \sqrt{n}\rho_\infty(x,y)$.

(b) Now, given $\epsilon > 0$, $x \in \mathbb{R}^n$ set $\delta = \epsilon/n > 0$; then

$$\rho_1(x,y) \le \delta \Rightarrow \rho_\infty(x,y) \le \rho_1(x,y) \le \delta \le \epsilon,$$

$$\rho_\infty(x,y) \le \delta \Rightarrow \rho_1(x,y) \le n\rho_\infty(x,y) \le n\delta = \epsilon,$$

$$\rho_2(x,y) \le \delta \Rightarrow \rho_\infty(x,y) \le \rho_2(x,y) \le \delta \le \epsilon,$$

$$\rho_\infty(x,y) \le \delta \Rightarrow \rho_2(x,y) \le \sqrt{n}\rho_\infty(x,y) \le \sqrt{n}\delta \le n\delta = \epsilon.$$

As x, ϵ are arbitrary, ρ_1 and ρ_2 are both equivalent to ρ_∞. Because equivalence-of-metrics is an equivalence relation, ρ_1 and ρ_2 are also equivalent to each other.

7. (i) ρ_∞ takes values in $[0,\infty[$ because $\{|f(\alpha) - g(\alpha)| : \alpha \in [0,1]\}$ is always a non-empty set, bounded below by 0, and also bounded above because f, g and $f-g$ are continuous functions and $[0,1]$ is a closed bounded interval.

Now

(i) $\rho_\infty(f,f) = \sup_{\alpha\in[0,1]} 0 = 0$,

(ii) $\rho_\infty(f,g) = \sup_{\alpha\in[0,1]} |f(\alpha) - g(\alpha)| = \sup_{\alpha\in[0,1]} |g(\alpha) - f(\alpha)| = \rho_\infty(g,f)$,

(iii) if $\alpha \in [0,1]$, then

$$|f(\alpha) - h(\alpha)| \le |f(\alpha) - g(\alpha)| + |g(\alpha) - h(\alpha)| \le \rho_\infty(f,g) + \rho_\infty(g,h),$$

for all $\alpha \in [0,1]$, so $\rho_\infty(f,h) \le \rho_\infty(f,g) + \rho_\infty(g,h)$,

(iv) if $\rho_\infty(f,g) = 0$ then $|f(\alpha) - g(\alpha)| \le 0$ for every $\alpha \in [0,1]$, so $f(\alpha) = g(\alpha)$ for every $\alpha \in [0,1]$ and $f = g$

for all f, g, $h \in X$. Thus ρ_∞ is a metric.

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8. (a) σ_1 takes values in $[0,\infty[$ because ρ does.
If $x,\, y,\, z \in X$, then

$$\sigma_1(x,x) = \min(1, \rho(x,x)) = \min(1,0) = 0,$$

$$\sigma_1(x,y) = \min(1, \rho(x,y)) = \min(1, \rho(y,x)) = \sigma_1(y,x),$$

$$\begin{aligned}\sigma_1(x,z) &= \min(1, \rho(x,z)) \\ &\le \min(1, \rho(x,y) + \rho(y,z)) \\ &\le \min(2, 1 + \rho(x,y), 1 + \rho(y,z), \rho(x,y) + \rho(y,z)) \\ &= \min(1, \rho(x,y)) + \min(1, \rho(y,z)) \\ &= \sigma_1(x,y) + \sigma_2(y,z)),\end{aligned}$$

$$\sigma_1(x,y) = 0 \Rightarrow \min(1, \rho(x,y)) = 0 \Rightarrow \rho(x,y) = 0 \Rightarrow x = y.$$

So σ_1 is a metric.

(b) σ_2 takes values in $[0,\infty[$ because ρ does.
If $x,\, y,\, z \in X$, then

$$\sigma_2(x,x) = \frac{\rho(x,x)}{1+\rho(x,x)} = \frac{0}{1+0} = 0,$$

$$\sigma_2(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)} = \frac{\rho(y,x)}{1+\rho(y,x)} = \sigma_2(y,x),$$

$$\begin{aligned}\sigma_2(x,z) &= \frac{\rho(x,z)}{1+\rho(x,z)} \\ &= 1 - \frac{1}{1+\rho(x,z)} \\ &\le 1 - \frac{1}{1+\rho(x,y)+\rho(y,z)} \\ &= \frac{\rho(x,y)+\rho(y,z)}{1+\rho(x,y)+\rho(y,z)} \\ &= \frac{\rho(x,y)}{1+\rho(x,y)+\rho(y,z)} + \frac{\rho(y,z)}{1+\rho(x,y)+\rho(y,z)} \\ &\le \frac{\rho(x,y)}{1+\rho(x,y)} + \frac{\rho(y,z)}{1+\rho(y,z)} \\ &= \sigma_2(x,y) + \sigma_2(y,z),\end{aligned}$$

$$\sigma_1(x,y) = 0 \Rightarrow \frac{\rho(x,y)}{1+\rho(x,y)} = 0 \Rightarrow \rho(x,y) = 0 \Rightarrow x = y.$$

So σ_2 is a metric.

(c) Given $x \in X$, $\epsilon > 0$, set $\delta = \frac{\epsilon}{1+\epsilon}$. Note that $0 < \delta < 1$. Now

$$\rho(x,y) \le \delta \Rightarrow \sigma_1(x,y) = \min(1, \rho(x,y)) \le \min(1,\delta) = \delta \le \epsilon,$$

$$\sigma_1(x,y) \le \delta \Rightarrow \min(1, \rho(x,y)) \le \delta \Rightarrow \rho(x,y) \le \delta \le \epsilon,$$

$$\rho(x,y) \le \delta \Rightarrow \sigma_2(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)} \le \rho(x,y) \le \delta \le \epsilon,$$

$$\begin{aligned}\sigma_2(x,y) \le \delta &\Rightarrow\ 1 - \frac{1}{1+\rho(x,y)} = \sigma_2(x,y) \le \delta = 1 - \frac{1}{1+\epsilon} \\ &\Rightarrow\ \rho(x,y) \le \epsilon.\end{aligned}$$

As x, ϵ are arbitrary, σ_1 and σ_2 are both equivalent to ρ.

9. (a)

$$\overline{[0,1]} = \overline{]0,1[} = \overline{]0,1]} = [0,1],$$

$$\overline{\mathbb{Q}} = \overline{\mathbb{R}\setminus\mathbb{Q}} = \mathbb{R},$$

$$\overline{\mathbb{N}} = \mathbb{N}, \quad \overline{\mathbb{Z}} = \mathbb{Z},$$

$$\overline{\{\tfrac{1}{n} : n \in \mathbb{N}\}} = \{\tfrac{1}{n} : n \in \mathbb{N}\} \cup \{0\}.$$

(b)

$$\overline{\{(x,0) : x \in \mathbb{R}\}} = \{(x,0) : x \in \mathbb{R}\}$$

$$\overline{\{(x, \sin\tfrac{1}{x}) : x > 0\}} = \{(x, \sin\tfrac{1}{x}) : x > 0\} \cup \{(0,y) : y \in [-1,1]\}$$

$$\overline{\{(x,y) : 0 < x < 1, 0 < y < 1\}} = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$$

$$\overline{\{(x,y) : x+y \in \mathbb{Q}\}} = \mathbb{R}^2$$

$$\overline{\{(x,y) : \exists\, n \in \mathbb{N},\ n(x^2+y^2) = 1\}} = \{(x,y) : \exists\, n \in \mathbb{N},\ n(x^2+y^2) = 1\} \cup \{(0,0)\}$$

$$\overline{\{(x,y) : x^2+y^2 \in \mathbb{N}\}} = \{(x,y) : x^2+y^2 \in \mathbb{N}\}.$$

Only the sets $]0,1[\subseteq \mathbb{R}$ and $\{(x,y) : 0 < x < 1, 0 < y < 1\} \subseteq \mathbb{R}^2$ are open.

10. **?** Suppose, if possible, otherwise; that $\langle x_n \rangle_{n\in\mathbb{N}}$ converges simultaneously to two distinct points x, y. Set $\epsilon = \frac{1}{3}\rho(x,y) > 0$. Then there are n_1, $n_2 \in \mathbb{N}$ such that

$$\rho(x_n, x) \le \epsilon \text{ for every } n \ge n_1,$$

$$\rho(x_n, y) \le \epsilon \text{ for every } n \ge n_2.$$

Set $n = \max(n_1, n_2)$; then

$$\rho(x,y) \le \rho(x_n,x) + \rho(x_n,y) \le 2\epsilon = \frac{2}{3}\rho(x,y) < \rho(x,y),$$

which is absurd. **X**

11. (Not available.)

12.

	non-empty?	bounded above?	bounded below?	sup	inf	contains
A	Y	Y	Y	9	-9	both
B	Y	N	N	—	—	—
C	Y	Y	Y	π	0	both
D	Y	Y	Y	π	0	inf only
E	Y	N	Y	—	1	inf only
F	Y	Y	N	-1	—	sup only
G	Y	Y	Y	1	0	sup only
H	Y	Y	Y	π	0	neither
I	Y	Y	Y	1	0	neither

13. $A+B=\mathbb{R}$, $C+C=[0,2\pi]$, $C-C=[-\pi,\pi]$, $C+H=\,]0,2\pi[$, $E+H=\,]1,\infty[$.

Set $a=\sup X$ and $b=\sup Y$. Then $x\le a$ and $y\le b$ for every $x\in X$, $y\in Y$, so $x+y\le a+b$ whenever $x\in X$ and $y\in Y$, and $a+b$ is an upper bound for $X+Y$.

To see that $a+b$ is the least upper bound for $X+Y$, suppose that c is any upper bound for $X+Y$. Fix $x\in X$ for the moment. Then $x+y\le c$, that is, $y\le c-x$ for every $y\in Y$; because $b=\sup Y$, $b\le c-x$, that is, $x\le c-b$. But this is true for every $x\in X$, and $\sup X=a$, so $a\le c-b$, that is, $a+b\le c$. As c is arbitrary, $a+b$ is the least upper bound for $X+Y$, as claimed.

Similarly,

$\inf(X+Y)=\inf X+\inf Y$ if $\inf X$ and $\inf Y$ are defined,
$\sup(X-Y)=\sup X-\inf Y$ if $\sup X$ and $\inf Y$ are defined,
$\inf(X-Y)=\inf X-\sup Y$ if $\inf X$ and $\sup Y$ are defined.

14. $\lim_{n\to\infty}x_n=x$ means that for every $\epsilon>0$ there is an $n_0\in\mathbb{N}$ such that $|x_n-x|\le\epsilon$ for every $n\ge n_0$.

(i) **?** Suppose, if possible, that $a\not\le x$, that is, that $x<a$. Then set $\epsilon=\frac{1}{2}(a-x)>0$. Let $n_0\in\mathbb{N}$ be such that $|x_n-x|\le\epsilon$ for every $n\ge n_0$. Then, in particular, $|x_{n_0}-x|\le\epsilon$, so that $x_{n_0}\le x+\epsilon<a$; but this is impossible. **X**

Similarly, **?** if $x>b$, take $\epsilon=\frac{1}{2}(x-b)>0$; take n_0 such that $|x_{n_0}-x|\le\epsilon$ and see that $x_{n_0}\ge x-\epsilon>b$. **X**

(ii) **?** Suppose, if possible, that $x>y$. Take $\epsilon=\frac{1}{3}(x-y)>0$. Then there are n_1, $n_2\in\mathbb{N}$ such that $|x_n-x|\le\epsilon$ for every $n\ge n_1$, $|y_n-y|\le\epsilon$ for every $n\ge n_2$. Set $n=\max(n_1,n_2)$. Then

$$x_n\ge x-\epsilon,\ y_n\le y+\epsilon,$$

so

$$x_n-y_n\ge x-y-2\epsilon=\epsilon>0;$$

but we were assuming that $x_n\le y_n$. **X**

(iii) It is false. Consider $x_n=\frac{1}{n}$ for $n\in\mathbb{N}$. Then $0<x_n<2$ for every n. But $\lim_{n\to\infty}x_n=0$. (For given $\epsilon>0$, we may take $n_0\ge\frac{1}{\epsilon}$; then $|x_n-0|=\frac{1}{n}\le\frac{1}{n_0}\le\epsilon$ for every $n\ge n_0$.) And $0\not<0$.

15.

a_n	$\nearrow$	yes, sup $=1$, inf $=0$	convergent to 1	$n_0=99$.
b_n	not monotonic	below only, inf $=0$	not convergent	$-$
c_n	$\searrow$	yes, sup $=10^{-5}$, inf $=0$	convergent to 0	$n_0=0$
d_n	$\searrow$	yes, sup $=0$, inf $=-\frac{1}{2}$	convergent to $-\frac{1}{2}$	$n_0=13$
e_n	not monotonic	yes, sup $=1$, inf $=-1$	not convergent	$-$
f_n	not monotonic	yes, sup $=2$, inf $=1$	convergent to $\frac{1+\sqrt{5}}{2}$	$n_0=6$.

16. (Not available.)

17. (a) **Contraction Mapping Theorem**: Let (X, ρ) be a non-empty complete metric space and $f : X \to X$ a *contraction*, that is, a function such that, for some $\gamma < 1$, $\rho(f(x), f(y)) \leq \rho(x, y)$ for all x, $y \in X$. Then f has a unique fixed point.

proof Because X is non-empty, there is an $x_0 \in X$. Set $x_1 = f(x_0)$, $x_2 = f(x_1)$, $\ldots$. Then for each $n \geq 1$,

$$\rho(x_{n+1}, x_n) = \rho(f(x_n), f(x_{n-1})) \leq \alpha\rho(x_n, x_{n-1}).$$

So $\rho(x_2, x_1) \leq \alpha\rho(x_1, x_0)$, $\rho(x_3, x_2) \leq \alpha\rho(x_2, x_1) \leq \alpha^2\rho(x_1, x_0)$, and in general $\rho(x_{n+1}, x_n) \leq \alpha^n\rho(x_1, x_0)$. It follows that $\langle x_n\rangle_{n\in\mathbb{N}}$ is Cauchy. **P** If $m < n$, then

$$\rho(x_n, x_m) \leq \sum_{k=m}^{n-1} \rho(x_{k+1}, x_k) \leq \sum_{k=m}^{n-1} \alpha^k\rho(x_1, x_0) = \tfrac{\alpha^m - \alpha^n}{1-\alpha}\rho(x_1, x_0) \leq \tfrac{\alpha^m}{1-\alpha}\rho(x_1, x_0).$$

Similarly, if $n < m$, then $\rho(x_n, x_m) \leq \frac{\alpha^n}{1-\alpha}$, while if $m = n$ then of course $\rho(x_n, x_m) = 0$; so we always have

$$\rho(x_n, x_m) \leq \frac{\alpha^{\min(m,n)}}{1-\alpha} \to 0 \text{ as } m, n \to \infty. \textbf{ Q}$$

Because X is complete, $z_0 = \lim_{n\to\infty} x_n$ is defined in X. Now

$$f(z_0) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = z_0,$$

so z_0 is a fixed point of f.

To see that z_0 is the unique fixed point of f, observe that if $f(z) = f(z_0)$ then

$$\rho(z, z_0) = \rho(f(z), f(z_0)) \leq \alpha\rho(z, z_0);$$

because $\alpha < 1$, we must have $\rho(z, z_0) \leq 0$ and $z = z_0$.

(b)(i) $f(x) = \frac{\cos x}{2+x^4}$ works. To see that this is a contraction, consider

$$\begin{aligned}|f'(x)| &= \left|\tfrac{-(2+x^4)\sin x - 4x^3\cos x}{(2+x^4)^2}\right| \\ &\leq \tfrac{\sqrt{(2+x^4)^2+(4x^3)^2}}{(2+x^4)^2}\end{aligned}$$

because $|\alpha\cos x + \beta\sin x| = |r\cos(x+\theta)| \leq r$ where $\alpha = r\cos\theta$, $-\beta = r\sin\theta$, $r = \sqrt{\alpha^2+\beta^2}$). Now

$$\begin{aligned}\tfrac{(2+x^4)^2+(4x^3)^2}{(2+x^4)^2} &= \tfrac{4+4x^4+16x^6+x^8}{16+32x^4+24x^8+8x^12+x^16} \\ &\leq \tfrac{4+12x^4+9x^8}{16+32x^4+24x^8}\end{aligned}$$

because $(1-x^2)^2 \geq 0$, so $2x^2 \leq 1 + x^4$, so

$$2x^6 \leq x^4 + x^8 \leq \tfrac{8+16x^4+12x^8}{16+32x^4+24x^8} = \tfrac{1}{2}$$

and

$$|f'(x)| \leq \tfrac{\sqrt{(2+x^4)^2+(4x^3)^2}}{(2+x^4)^2} \leq \tfrac{1}{\sqrt{2}} < 1$$

for every x, and f is a contraction.

(ii) Now if we start with $x_0 = 0$ and look at $x_1 = f(x_0)$, $x_2 = f(x_1)$ etc., we get

$$0.0, 0.5000, 0.4254, 0.4481, 0.4417, 0.4436, 0.4430, 0.4432, 0.4431,$$

giving a solution 0.443 to three decimal places.

18. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and $\alpha \geq 0$.

(i) If $|f(x) - f(y)| \leq \alpha|x - y|$ for all x, $y \in \mathbb{R}$ then $\frac{|f(x)-f(y)|}{|x-y|} \leq \alpha$ for all $x \neq y$, so for any x

$$|f'(x)| = \lim_{y\to x} |\tfrac{f(y)-f(x)}{y-x}| = \lim_{y\to x} \tfrac{|f(y)-f(x)|}{|y-x|} \leq \alpha.$$

(ii) If $|f'(x)| \leq \alpha$ for all $x \in \mathbb{R}$, then whenever $x > y$ there is a $c \in \,]y, x[$ such that $\frac{f(x)-f(y)}{x-y} = f'(c)$, so that

$$\tfrac{|f(x)-f(y)|}{|x-y|} = |f'(c)| \leq \alpha$$

and $|f(x) - f(y)| \leq \alpha|y - x|$. If $x < y$ then

$$|f(x) - f(y)| = |f(y) - f(x)| \leq \alpha|y - x| = \alpha|x - y|.$$

If $x = y$ then

$$|f(x) - f(y)| = 0 = \alpha|x - y|.$$

Thus we have $|f(x) - f(y)| \leq \alpha|x - y|$ in all cases, as required.

19. **(a)** Start from any $z_0 \in X$ and set $z_1 = f(z_0)$, $z_2 = f(z_1)$ and so on. Set $\gamma_n = \rho(z_n, z_{n+1})$ for each n; then

$$\gamma_{n+1} = \rho(z_{n+1}, z_{n+2}) = \rho(f(z_n), f(z_{n+1})) = \rho(f(z_n), f^2(z_n)) \leq \alpha\rho(z_n, f(z_n)) = \alpha\gamma_n$$

for every n, and $\gamma_n \leq \alpha^n\gamma_0$ for every n. If $m < n$, then

$$\begin{aligned}\rho(z_m, z_n) &\leq \rho(z_m, z_{m+1}) + \rho(z_{m+1}, z_{m+2}) + \ldots + \rho(z_{n-1}, z_n)\\ &= \gamma_m + \gamma_{m+1} + \ldots + \gamma_{n-1}\\ &\leq \alpha^m\gamma_0 + \alpha^{m+1}\gamma_0 + \ldots + \alpha^{n-1}\gamma_0\\ &= \frac{\alpha^m - \alpha^n}{1-\alpha}\gamma_0\\ &\leq \frac{\alpha^m\gamma_0}{1-\alpha}\\ &\to 0 \text{ as } m \to \infty.\end{aligned}$$

Thus $\langle z_n\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in X and has a limit x_0 say, because X is complete. Now, because f is continuous,

$$f(x_0) = \lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} z_{n+1} = x_0,$$

and x_0 is a fixed point of f.

(b) f can have many fixed points; for instance, set $X = \mathbb{R}$ and $f(x) = x$ for every x and $\alpha = 0$; then $\rho(f(x), f^2(x)) = 0 = \alpha\rho(x, f(x))$ for every x, while every point of $\mathbb{R}$ is a fixed point of f.

(b) Let x_0 be the fixed point of f^n. Then $f^n(f(x_0)) = f(f^n(x_0)) = f(x_0)$, so $f(x_0)$ is also a fixed point of f^n, and must be equal to x_0. Thus x_0 is a fixed point of f.

On the other hand, if x' is any fixed point of f, then

$$x' = f(x') = f(f(x')) = f(f(f(x'))) = \ldots = f^n(x')$$

and $x' = x_0$. Thus x_0 is the unique fixed point of f.

20. **(a) Baire's theorem:** Let (X,ρ) be a complete metric space. If $\langle G_n\rangle_{n\in\mathbb{N}}$ is a sequence of dense open subsets of X, then $\bigcap_{n\in\mathbb{N}} G_n$ is dense in X.

(b) Set $H_n = G_n \cup (X\setminus\overline{G_n})$ for each n. Then each H_n is open, and it is also dense because

$$\overline{H_n} \supseteq \overline{G_n} \cup (X\setminus\overline{G_n}) = X.$$

Consequently $\bigcap_{n\in\mathbb{N}} H_n$ is dense. **?** Suppose, if possible, that U is a non-empty open set such that $U \subseteq \bigcap_{n\in\mathbb{N}} \overline{G_n}$. In this case $U\cap\bigcap_{n\in\mathbb{N}} H_n \neq \emptyset$; take $x \in U\cap\bigcap_{n\in\mathbb{N}} H_n$. Because $\bigcap_{n\in\mathbb{N}} G_n = \emptyset$, there is some n such that $x \notin G_n$. But $x \in H_n$, so $x \in X\setminus\overline{G_n}$. On the other hand, x also belongs to U, so $x \in \overline{G_n}$. **X**

21. H_q is the union of two open sets, so is open; also

$$\overline{H_q} \supseteq \overline{G_q} \cup (X\setminus\overline{G_q}) = X,$$

so H_q is dense for each $q\in\mathbb{Q}$. As $\mathbb{Q}$ is countable, Baire's theorem tells us that $E=\bigcap_{q\in\mathbb{Q}} H_q$ is dense.

Let x be any point of E, and $\epsilon>0$. Then there is a $q\in\mathbb{Q}$ such that $f(x)-\epsilon\le q\le f(x)$. Now $x\in H_q$, but $x\notin G_q$ so $x\in X\setminus\overline{G_q}$, and there is a $\delta_1>0$ such that $N_{\delta_1}(x)\cap G_q=\emptyset$, that is, $f(y)\ge q\ge f(x)-\epsilon$ for every $y\in N_{\delta_1}(x)$. Next, $G_{f(x)+\epsilon}$ is open and contains x, so there is a $\delta_2>0$ such that $N_{\delta_2}(x)\subseteq G_{f(x)+\epsilon}$, that is, $f(y)<f(x)+\epsilon$ whenever $y\in N_{\delta_2}(x)$. Set $\delta=\min(\delta_1,\delta_2)>0$; if $\rho(y,x)\le\delta$, then $f(x)-\epsilon\le f(y)<f(x)+\epsilon$, and $|f(y)-f(x)|\le\epsilon$. As ϵ is arbitrary, f is continuous at x. As x is arbitrary, $E\subseteq\{x : f \text{ is continuous at } x\}$. Because E is dense, so is $\{x : f \text{ is continuous at } x\}$.

22. **(a)** Set $G_k=\bigcup\{H : H\subseteq X \text{ is open}, \sigma(f(x),f(y))\le\frac{1}{k} \text{ for all } x,\,y\in H\}$ for each $k\in\mathbb{N}$; then G_k is a union of open sets, so is open, and $E'=\bigcap_{k\in\mathbb{N}} G_k$ is a G_δ set. Set $E=\{x : f \text{ is continuous at } x\}$.

If $x_0\in E$, then for every $k\in\mathbb{N}$ there is a $\delta>0$ such that $\sigma(f(x),f(x_0))\le\frac{1}{2k}$ whenever $\rho(x,x_0)\le\delta$. Now

$$\sigma(f(x),f(y))\le\sigma(f(x),f(x_0))+\sigma(f(y),f(x_0))\le\frac{1}{2k}+\frac{1}{2k}=\frac{1}{k}$$

for all $x,\,y\in U_\delta(x_0)$; since $U_\delta(x_0)$ is open, it is included in G_k, and $x_0\in G_k$. As k is arbitrary, $x_0\in E'$. As x_0 is arbitrary, $E\subseteq E'$.

If $x_0\in E'$ and $\epsilon>0$, then there is a $k\in\mathbb{N}$ such that $\frac{1}{k}\le\epsilon$. Since $x_0\in G_k$, there is an open set H such that $x_0\in H$ and $\sigma(f(x),f(y))\le\frac{1}{k}$ for all $x,\,y\in H$. Now there is a $\delta>0$ such that $N_\delta(x_0)\subseteq H$. If $\rho(x,x_0)\le\delta$, then $x,\,x_0$ both belong to H, so $\sigma(f(x),f(x_0))\le\frac{1}{k}\le\epsilon$. As ϵ is arbitrary, f is continuous at x_0; as x_0 is arbitrary, $E'\subseteq E$.

Thus $E=E'$ is a G_δ set.

(b) Let (X,ρ) be a complete metric space and E, F two dense G_δ subsets of X. Then we can express E as $\bigcap_{n\in\mathbb{N}} G_n$, F as $\bigcap_{n\in\mathbb{N}} H_n$ where all the G_n, H_n are open. For each n,

$$\overline{G}_n\supseteq\overline{E}=X,\quad \overline{H}_n\supseteq\overline{F}=X$$

so every G_n, H_n is dense. Now set

$$V_{2n-1}=G_n,\quad V_{2n}=H_n$$

for every $n\ge 1$; then $\langle V_n\rangle_{n\in\mathbb{N}}$ is a sequence of dense open sets so (by Baire's theorem) $\bigcap_{n\in\mathbb{N}} V_n$ is dense. But

$$\bigcap_{n\in\mathbb{N}} V_n = G_1\cap H_1\cap G_2\cap H_2\cap\ldots = E\cap F.$$

(c) Set $E=\{x : f \text{ is continuous at } x\}$, $F=\{x : g \text{ is continuous at } x\}$. By (a), both E and F are G_δ sets. By (b), $E\cap F$ is dense. But $f+g$ is continuous at every point of $E\cap F$.

23.

Let $\epsilon > 0$. Set

$$\eta = \min(\tfrac{1}{2}|y|, \tfrac{\epsilon|y|^2}{2(|x|+|y|)}) > 0.$$

Because $\lim_{n\to\infty} x_n = x$, there is an $n_1 \in \mathbb{N}$ such that $|x_n - x| \le \eta$ for every $n \ge n_1$. Because $\lim_{n\to\infty} y_n = y$, there is an $n_2 \in \mathbb{N}$ such that $|y_n - y| \le \eta$ for every $n \ge n_2$. Set $n_0 = \max(n_1, n_2) \in \mathbb{N}$. Then if $n \ge n_0$ we have $n \ge n_1$ and $n \ge n_2$, so that $|x_n - x| \le \eta$ and $|y_n - y| \le \eta$ and, in the first place,

$$|y_n| \ge |y| - |y_n - y| \ge |y| - \eta \ge |y| - \tfrac{1}{2}|y| = \tfrac{1}{2}|y| > 0;$$

in particular, $y_n \neq 0$ and x_n/y_n is defined. Next, still taking $n \ge n_0$,

$$\begin{aligned}
|\frac{x_n}{y_n} - \frac{x}{y}| = |\frac{x_n y - x y_n}{y_n y}| &= \frac{|x_n y - x y_n|}{|y_n||y|} \\
&\le \frac{|x_n y - x y_n|}{\frac{1}{2}|y|^2} = \frac{2}{|y|^2}|x_n y - x y_n| \\
&= \frac{2}{|y|^2}|(x_n - x)y + x(y - y_n)| \le \frac{2}{|y|^2}(|(x_n - x)y| + |x(y - y_n)|) \\
&= \frac{2}{|y|^2}(|x_n - x||y| + |x||y - y_n|) \le \frac{2}{|y|^2}(\eta|y| + |x|\eta) \\
&= \frac{2(|x| + |y|)}{|y|^2}\eta \le \epsilon.
\end{aligned}$$

This is true for every $n \ge n_0$. Because ϵ is arbitrary, $\lim_{n\to\infty} x_n/y_n$ exists $= x/y$.

24. (a) A linear space ofver $\mathbb{R}$ is a set X together with operations $+ : X \times X \to X$, $- : X \to X$, $. : \mathbb{R} \times X \to X$ and a distinguished element 0 of X such that

$(X, +)$ is a commutative group with inverse $-$ and identity 0, that is,

$x + (y + z) = x + (y + z)$, $x + 0 = 0 + x = x$, $x + (-x) = (-x) + x = 0$, $x + y = y + x$

for all x, $y \in X$;

$\alpha(x + y) = \alpha x + \alpha y$, $(\alpha + \beta)x = \alpha x + \beta x$, $(\alpha\beta)x = \alpha(\beta x)$, $1x = x$

for all α, $\beta \in \mathbb{R}$, x, $y \in X$.

(b) ℓ^2 is the space of all real sequences $x = (\xi_1, \xi_2, \dots)$ such that $\sum_{i=1}^{\infty} |\xi_i|^2 < \infty$.

ℓ^2 is a linear subspace of $\mathbb{R}^{\mathbb{N}}$. **P** (i) If x, $y \in \ell^2$ then

$$\begin{aligned}
\sum_{i=1}^{\infty} |\xi_i + \eta_i|^2 &\le \sum_{i=1}^{\infty} (|\xi_i| + |\eta_i|)^2 \\
&\le \sum_{i=1}^{\infty} 4(|\xi_i|^2 + |\eta_i|^2) < \infty.
\end{aligned}$$

So $x + y \in \ell^2$. (ii) If $x \in \ell^2$, $\alpha \in \mathbb{R}$ then

$$\textstyle\sum_{i=1}^{\infty} |\alpha\xi_i|^2 = |\alpha|^2 \sum_{i=0}^{\infty} |\xi_i|^2 < \infty.$$

So $\alpha x \in \ell^2$. (iii) If $x = 0$ then $\xi_i = 0$ for every i so $\sum_{i=1}^{\infty} |\xi_i|^2 = 0 < \infty$ and $x \in \ell^2$. **Q**

Being a linear subspace of the linear space $\mathbb{R}^{\mathbb{N}}$, ℓ^2 is a linear space in its own right.

25. (a) If X is a real linear space, a **norm** on X is a functional $\| \, \| : X \to [0, \infty[$ such that

$$\|x+y\| \le \|x\| + \|y\|, \quad \|\alpha x\| = |\alpha| \|x\|$$

for all x, $y \in X$ and $\alpha \in \mathbb{R}$, and

$$\|x\| = 0 \Rightarrow x = 0.$$

(b)(i) For $x = (\xi_1, \dots) \in \ell^1$, set $\|x\|_1 = \sum_{i=1}^{\infty} |\xi_i|$. By the definition of ℓ^1, this is finite.
If x, $y \in \ell^1$ then

$$\begin{aligned} \|x+y\|_1 &= \sum_{i=1}^{\infty} |\xi_i + \eta_i| \le \sum_{i=1}^{\infty} |\xi_i| + |\eta_i| \\ &= \sum_{i=1}^{\infty} |\xi_i| + \sum_{i=1}^{\infty} |\eta_i| = \|x\|_1 + \|y\|_1 < \infty. \end{aligned}$$

If $x \in \ell^1$ and $\alpha \in \mathbb{R}$ then

$$\|\alpha x\|_1 = \textstyle\sum_{i=1}^{\infty} |\alpha \xi_i| = |\alpha| \sum_{i=1}^{\infty} |\xi_i| = |\alpha| \|x\|.$$

If $x \in \ell^1$ and $\|x\|_1 = 0$ then $|\xi_i| = 0$ for every i so $x = 0$.
Thus $\| \, \|_1$ is a norm on ℓ^1.

(ii) For $x \in C([a,b])$ set $\|x\|_\infty = \sup_{t \in [a,b]} |x(t)|$; this is finite because $[a,b]$ is compact and continuous functions on compact sets are bounded.
If x, $y \in C([a,b])$ then, for any $t \in [a,b]$,

$$|(x+y)(t)| = |x(t) + y(t)| \le |x(t)| + |y(t)| \le \|x\|_\infty + \|y\|_\infty.$$

As t is arbitrary, $\|x+y\|_\infty \le \|x\|_\infty + \|y\|_\infty$.
If $x \in C([a,b])$, $\alpha \in \mathbb{R}$ then, for any $t \in [a,b]$,

$$|(\alpha x)(t)| = |\alpha||x(t)| \le |\alpha| \|x\|_\infty.$$

So $\|\alpha x\|_\infty \le |\alpha| \|x\|_\infty$. If $\alpha = 0$, we surely have $\|\alpha x|_\infty = |\alpha| \|x\|_\infty$. Otherwise,

$$|\alpha| \|x\|_\infty = |\alpha| \|\alpha^{-1} \alpha x\|_\infty \le |\alpha| |\alpha^{-1}| \|\alpha x\|_\infty = \|\alpha x\|_\infty,$$

so again we have equality.
If $\|x\|_\infty = 0$ then $|x(t)| = 0$ for every $t \in [a,b]$, so $x(t) = 0$ for every $t \in [a,b]$ and $x = 0$.

26. (a) If X is a real linear space, a **norm** on X is a functional $\| \,\| : X \to [0, \infty[$ such that

$$\|x + y\| \leq \|x\| + \|y\|, \quad \|\alpha x\| = |\alpha|\|x\|$$

for all x, $y \in X$ and $\alpha \in \mathbb{R}$, and

$$\|x\| = 0 \Rightarrow x = 0.$$

A **metric** on a set X is a function $\rho : X \times X \to [0, \infty[$ such that

$$\rho(x, y) = \rho(y, x), \quad \rho(x, z) \leq \rho(x, y) + \rho(y, z)$$

for all x, y, $z \in X$, and $\rho(x, y) = 0 \iff x = y$.

A sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in a metric space (X, ρ) is **convergent** to $x \in X$ if $\lim_{n\to\infty} \rho(x_n, x) = 0$.

A **Cauchy sequence** in a metric space (X, ρ) is a sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ such that $\lim_{m,n\to\infty} \rho(x_m, x_n) = 0$, that is, for every $\epsilon > 0$ there is an n_0 such that $\rho(x_m, x_n) \leq \epsilon$ for all m, $n \geq n_0$.

(b) If $\| \,\|$ is a norm on a linear space X, set $\rho(x, y) = \|x - y\|$ for all x, $y \in X$.

ρ takes values in $[0, \infty[$ because $\| \,\|$ does.

If x, y, $z \in X$ then

$$\rho(y, x) = \|y - x\| = \|(-1)(x - y)\| = |-1|\|x - y\| = \rho(x, y),$$

$$\rho(x, z) = \|x - z\| = \|(x - y) + (y - z)\| \leq \|x - y\| + \|y - z\| = \rho(x, y) + \rho(y, z).$$

If $x = y$ then

$$\rho(x, x) = \|x - x\| = \|\mathbf{0}\| = \|0.\mathbf{0}\| = |0|\|\mathbf{0}\| = 0.$$

If $\rho(x, y) = 0$ then $\|x - y\| = 0$ so $x - y = \mathbf{0}$ and $x = y$.

Thus ρ is a metric.

(c) Let S be the set of all real sequences $x = (\xi_1, \dots)$ such that $\{i : \xi_i \neq 0\}$ is finite. Then S is a linear subspace of ℓ^1. Give it the norm $\| \,\|_1$. If we set

$$x_n = (1, \tfrac{1}{2}, \tfrac{1}{4}, \dots, \tfrac{1}{2^n}, 0, 0, 0, \dots)$$

for $n \in \mathbb{N}$, then

$$\rho(x_m, x_n) = \textstyle\sum_{i=\min(m,n)+1}^{\max(m,n)} 2^{-i} \leq 2^{-\min(m,n)} \to 0$$

as m, $n \to \infty$, so the sequence is Cauchy. **?** But if $x = (\xi_1, \dots) \in S$ is such that $\langle x_n \rangle_{n \in \mathbb{N}} \to x$, then $|\xi_i - 2^{-i}| \leq \rho(x, x_n)$ for any $n \geq i$, so $\xi_i = 2^{-i}$ for every i and $x \notin S$. **X**

Thus $\langle x_n \rangle_{n \in \mathbb{N}}$ is a Cauchy sequence in S which is not convergent in S.

27. For $x \in \mathbb{R}^n$, we have

$$\begin{aligned}\|Ax\|_2^2 &= \sum_{i=1}^{m}(\sum_{j=1}^{n}\alpha_{ij}\xi_j)^2\\ &= \sum_{i=1}^{m}(\sum_{j=1}^{n}\alpha_{ij}\xi_j)(\sum_{k=1}^{n}\alpha_{ik}\xi_k)\\ &= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n}\alpha_{ij}\xi_j\alpha_{ik}\xi_k\\ &= \sum_{j=1}^{n}\sum_{k=1}^{n}\xi_j\xi_k\sum_{i=1}^{n}\alpha_{ij}\alpha_{ik}\\ &\le \sum_{j=1}^{n}\sum_{k=1}^{n}|\xi_j||\xi_k|\sum_{i=1}^{n}|\alpha_{ij}\alpha_{ik}|\\ &\le \sum_{j=1}^{n}\sum_{k=1}^{n}|\xi_j||\xi_k|\sqrt{\sum_{i=1}^{n}\alpha_{ij}^2}\sqrt{\sum_{i=1}^{n}\alpha_{ik}^2}\end{aligned}$$

(by Schwartz' inequality)

$$\begin{aligned}&= \sum_{j=1}^{n}|\xi_j|\sqrt{\sum_{i=1}^{n}\alpha_{ij}^2}\sum_{k=1}^{n}|\xi_k|\sqrt{\sum_{i=1}^{n}\alpha_{ik}^2}\\ &= (\sum_{j=1}^{n}|\xi_j|\sqrt{\sum_{i=1}^{n}\alpha_{ij}^2})^2,\end{aligned}$$

so

$$\begin{aligned}\|Ax\|_2 &\le \sum_{j=1}^{n}|\xi_j|\sqrt{\sum_{i=1}^{n}\alpha_{ij}^2}\\ &\le \sqrt{\sum_{j=1}^{n}\xi_j^2}\sqrt{\sum_{j=1}^{n}(\sqrt{\sum_{i=1}^{n}\alpha_{ij}^2})^2}\end{aligned}$$

(by Schwartz' inequality again)

$$\begin{aligned}&= \|x\|_2\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\alpha_{ij}^2}\\ &= \|\mathbf{A}\|_{\mathrm{HS}}\|x\|.\end{aligned}$$

28. (a) $J \not\subseteq \bigcup\mathcal{A}$; $0 \in J \setminus \bigcup\mathcal{A}$. Consequently there is surely no finite subfamily of $\mathcal{A}$ covering J.

(b) $J \subseteq \bigcup\mathcal{A}$, but there is no finite subfamily of $\mathcal{A}$ covering J.

(c) $J \subseteq \bigcup\mathcal{A}$; indeed $J \subseteq I_0 \cup \ldots \cup I_n$, writing $n = 10^{25}$ and $I_j = \left]\frac{1}{n+1}, 2\right[$ for each $j \le n$.

(d) $J \subseteq \bigcup\mathcal{A}$ but there is no finite subfamily of $\mathcal{A}$ covering J.

29. **(a)** If X is a linear space over $\mathbb{R}$, two norms $\| \|$, $\| \|'$ on X are **equivalent** if there is an $M \ge 0$ such that $\|x\| \le M\|x\|'$ and $\|x\|' \le M\|x\|$ for every $x \in X$.

(b) If $x = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ then

$$\|x\|_\infty = \max_{i \le n} |\xi_i| \le \sum_{i=1}^n |\xi_i| = \|x\|_1,$$

$$\|x\|_1 = \sum_{i=1}^n |\xi_i| \le n \max_{i \le n} |\xi_i| = n\|x\|_\infty,$$

$$\|x\|_\infty = \max_{i \le n} |\xi_i| \le \sqrt{\sum_{i=1}^n |\xi_i|^2} = \|x\|_2,$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^n |\xi_i|^2} \le \sqrt{n} \max_{i \le n} |\xi_i| = \sqrt{n}\|x\|_\infty,$$

$$\|x\|_1 \le n\|x\|_\infty \le n\|x\|_2,$$

$$\|x\|_2 \le \sqrt{n}\|x\|_\infty \le \sqrt{n}\|x\|_1.$$

(c) Set $x_n(t) = t^n$ for $n \in \mathbb{N}$, $t \in [0,1]$. Then

$$\|x_n\|_\infty = 1,\ \|x_n\|_1 = \tfrac{1}{n+1},\ \|x_n\|_2 = \tfrac{1}{\sqrt{2n+1}}$$

so

$$\lim_{n\to\infty} \tfrac{\|x_n\|_\infty}{\|x_n\|_1} = \lim_{n\to\infty} \tfrac{\|x_n\|_\infty}{\|x_n\|_2} = \lim_{n\to\infty} \tfrac{\|x_n\|_2}{\|x_n\|_1} = \infty$$

and

there is no M such that $\|x\|_\infty \le M\|x\|_1$ for every x
there is no M such that $\|x\|_\infty \le M\|x\|_2$ for every x
there is no M such that $\|x\|_1 \le M\|x\|_2$ for every x.

30. **(a)** Let (X, ρ) be a metric space.
first definition A subset F of X is **closed** if $X \setminus F$ is open.
second definition A subset F of X is **closed** if every point of X which is the limit of a sequence in F itself belongs to F.

(b)(i) I note first that $\boldsymbol{c} \subseteq \ell^\infty$, because convergent sequences are bounded.
Suppose that $\langle x^{(n)} \rangle_{n\in\mathbb{N}}$ is a sequence in $\boldsymbol{c}$ converging to $x \in \ell^\infty$. Write $x = (\xi_1, \xi_2, \dots)$.
Let $\epsilon > 0$. Then there is an $n \in \mathbb{N}$ such that $\rho(x, x^{(m)}) \le \frac{1}{3}\epsilon$ for every $m \ge n$; in particular, $\rho(x, x^{(n)}) \le \frac{1}{3}\epsilon$. Write $x^{(n)} = (\xi_1^{(n)}, \xi_2^{(n)}, \dots)$. Because $\langle \xi_r^{(n)} \rangle_{r\in\mathbb{N}}$ is convergent, it is Cauchy; let r_0 be such that $|\xi_r^{(n)} - \xi_s^{(n)}| \le \frac{1}{3}\epsilon$ for all r, $s \ge r_0$. Then for r, $s \ge r_0$,

$$\begin{aligned} |\xi_r - \xi_s| &\le |\xi_r - \xi_r^{(n)}| + |\xi_r^{(n)} + \xi_s^{(n)}| + |\xi_s^{(n)} - \xi_s| \\ &\le \rho(x, x^{(n)}) + \frac{1}{3}\epsilon + \rho(x^{(n)}, x) \\ &\le \frac{1}{3}\epsilon + \frac{1}{3}\epsilon + \frac{1}{3}\epsilon \\ &= \epsilon. \end{aligned}$$

As $\epsilon > 0$, x is Cauchy, therefore convergent, and belongs to $\boldsymbol{c}$. As $\langle x^{(n)} \rangle_{n\in\mathbb{N}}$ is Cauchy, $\boldsymbol{c}$ is closed.

(c) Try $X = \ell^\infty$, Y the set of eventually-zero sequences.

31. **(a)** For x, $y \in X$ set $\rho(x,y) = \|x-y\|$. Then ρ is a metric on X. **P** ρ takes values in $[0,\infty[$ because $\| \|$ does. For any x, y, $z \in X$,

$$\rho(x,z) = \|x-z\| = \|(x-y)+(y-z)\| \le \|x-y\| + \|y-z\| = \rho(x,y) + \rho(y,z),$$

$$\rho(y,x) = \|y-x\| = \|(-1)(x-y)\| = |-1|\|x-y\| = \|x-y\| = \rho(x,y),$$

$$\rho(x,y) = 0 \iff \|x-y\| = 0 \iff x-y = \mathbf{0} \iff x = y. \text{ **Q**}$$

(b) Let $\tilde\rho$ be the product metric on $X \times X$, so that

$$\tilde\rho((x,y),(x',y')) = \max(\rho(x,x'),\rho(y,y')) = \max(\|x-x'\|,\|y-y'\|).$$

Given $\epsilon > 0$, set $\delta = \frac{1}{2}\epsilon > 0$. Then if (x,y), $(x',y') \in X \times X$ and $\tilde\rho((x,y),(x',y')) \le \delta$,

$$\begin{aligned}\rho(x+y,x'+y') &= \|x+y-x'-y'\| \le \|x-x'\| + \|y-y'\| \\ &\le 2\max(\|x-x'\|,\|y-y'\|) \le 2\delta = \epsilon.\end{aligned}$$

As ϵ is arbitrary, $(x,y) \mapsto x+y$ is uniformly continuous, therefore continuous.

(c) Let $\tilde\sigma$ be the product metric on $\mathbb{R} \times X$, so that $\tilde\sigma((\alpha,x),(\beta,y)) = \max(|\alpha-\beta|,\|x-y\|)$. Given $(\alpha_0,x_0) \in \mathbb{R} \times X$ and $\epsilon > 0$, set

$$\delta = \min(1, \tfrac{\epsilon}{1+|\alpha_0|+\|x_0\|}) > 0.$$

Then if $\tilde\sigma((\alpha,x),(\alpha_0,x_0)) \le \delta$,

$$\begin{aligned}\rho(\alpha x,\alpha_0 x_0) &= \|\alpha x - \alpha_0 x_0\| \\ &= \|(\alpha-\alpha_0)(x-x_0) + \alpha_0(x-x_0) + (\alpha-\alpha_0)x_0\| \\ &\le \|(\alpha-\alpha_0)(x-x_0)\| + \|\alpha_0(x-x_0)\| + \|(\alpha-\alpha_0)x_0\| \\ &= |\alpha-\alpha_0|\|x-x_0\| + |\alpha_0|\|x-x_0\| + |\alpha-\alpha_0|\|x_0\| \\ &\le \delta^2 + |\alpha_0|\delta + \delta\|x_0\| \\ &= \delta(\delta + |\alpha_0| + \|x_0\|) \\ &\le \delta(1 + |\alpha_0| + \|x_0\|) \\ &\le \epsilon.\end{aligned}$$

As ϵ is arbitrary, $(\alpha,x) \mapsto \alpha x$ is continuous at (α_0,x_0); as α_0 and x_0 is arbitrary, it is continuous.

It is not uniformly continuous (unless X is zero-dimensional). **P** If there is some $x_0 \in X \setminus 0$, set $x_1 = \|x_0\|^{-1}x_0$ so that $\|x_1\| = 1$. In this case, for any $\delta > 0$ we have

$$\tilde\sigma((\tfrac{1}{\delta},\tfrac{1}{\delta}x_1),(\delta+\tfrac{1}{\delta},(\delta+\tfrac{1}{\delta})x_1)) = \delta,$$

while

$$\rho(\tfrac{1}{\delta^2}x_1,(\delta+\tfrac{1}{\delta})^2x_1) = 2+\delta^2 > 1. \text{ **Q**}$$

32. **(a)** If $(X, \| \|)$ is a normed space, a subset A of X is **bounded** if there is some $M \geq 0$ such that $\|x\| \leq M$ for every $x \in A$.

Two norms $\| \|$ and $\| \|$ are **equivalent** if there is an $M \geq 0$ such that $\|x\| \leq M\|x\|'$ and $\|x\|' \leq M\|x\|$ for every $x \in X$.

(b) part 1 Suppose that $\| \|$ and $\| \|'$ are equivalent. Take M such that $\|x\| \leq M\|x\|'$ and $\|x\|' \leq M\|x\|$ for every $x \in X$.

If $A \subseteq X$ is $\| \|$-bounded, there is some $M' \geq 0$ such that $\|x\| \leq M'$ for every $x \in A$; now $\|x\|' \leq M\|x\| \leq MM'$ for every $x \in A$, and A is $\| \|'$-bounded. Similarly, if A is $\| \|'$-bounded it will be $\| \|$-bounded. So (ii) is true.

If $x_0 \in X$ and $\epsilon > 0$, set $\delta = \frac{\epsilon}{M+1}$. Then

$$\rho(x, x_0) \leq \delta \Rightarrow \rho'(x, x_0) = \|x - x_0\|' \leq M\|x - x_0\| = M\rho(x, x_0) \leq M\delta \leq \epsilon,$$

$$\rho'(x, x_0) \leq \delta \Rightarrow \rho(x, x_0) = \|x - x_0\| \leq M\|x - x_0\|' = M\rho'(x, x_0) \leq M\delta \leq \epsilon;$$

as x_0 and ϵ are arbitrary, ρ and ρ' are equivalent, and (iii) is true.

Equivalent norms must give rise to the same convergent sequences, so (iv) is true.

If $\langle x_n \rangle_{n \in \mathbb{N}}$ is ρ-Cauchy, then

$$\begin{aligned}\lim_{m,n\to\infty} \rho'(x_m, x_n) &= \lim_{m,n\to\infty} \|x_m - x_n\|' \leq M \lim_{m,n\to\infty} \|x_m - x_n\| \\ &= M \lim_{m,n\to\infty} \rho(x_m, x_n) = 0,\end{aligned}$$

so $\langle x_n \rangle_{n \in \mathbb{N}}$ is ρ'-Cauchy. Similarly, every ρ'-Cauchy sequence is ρ-Cauchy. Thus (v) is true.

If $A \subseteq X$ is ρ-totally bounded it is ρ'-totally bounded. **P** If $A = \emptyset$ it is surely ρ'-totally bounded. Otherwise, given $\epsilon > 0$, set $\delta = \frac{\epsilon}{M+1} > 0$. There is a finite set $x_0, \ldots, x_n$ of members of A such that $A \subseteq \bigcup_{i \leq n} N_\delta^{(\rho)}(x_i)$. But

$$N_\delta^{(\rho)}(x_i) = \{y : \|y - x_i\| \leq \delta\} \subseteq \{y : \|y - x_i\|' \leq M\delta\} \subseteq \{y : \|y - x_i\|' \leq \epsilon\} = N_\epsilon^{(\rho')}(x_i),$$

so $A \subseteq \bigcup_{i \leq n} N_\epsilon^{(\rho')}(x_i)$. As ϵ is arbitrary, A is ρ'-totally bounded. **Q** Similarly, every ρ'-totally bounded set is ρ-totally bounded, and (vi) is true.

Because ρ and ρ' have the same Cauchy sequences and the same convergent sequences, they give rise to the same complete sets; thus (vii) is true.

Because ρ and ρ' give rise to the same totally bounded sets and the same complete sets, they give rise to the same compact sets, and (viii) is true.

Thus condition (i) implies all the others.

part 2 Suppose now that $\| \|$ and $\| \|'$ are two norms on X which are *not* equivalent, with associated metrics ρ, ρ'.

case (α) Suppose that for every $n \in \mathbb{N}$ there is an $x_n \in X$ such that $\|x_n\|' > n\|x_n\|$. Then no x_n can be $\mathbf{0}$ (since $\|x_n\|' > 0$), so no $\|x_n\|$ is zero, and we can set

$$y_n = \frac{1}{\sqrt{n}\|x_n\|} x_n$$

for each n to obtain a sequence $\langle y_n \rangle_{n \in \mathbb{N}}$ such that $\|y_n\| = \frac{1}{\sqrt{n}}$ and $\|y_n\|' \geq \sqrt{n}$ for each n.

Now we see that

$$A = \{y_n : n \in \mathbb{N}\}$$

is bounded for $\| \|$ but not for $\| \|'$, and (ii) is false. Next, for every $\delta > 0$ there is an n such that $\rho(y_n, \mathbf{0}) \leq \delta$ but $\rho'(y_n, \mathbf{0}) \geq 1$, so ρ and ρ' are not equivalent, and (iii) is false. $\langle y_n \rangle_{n \in \mathbb{N}}$ is convergent (therefore Cauchy) for $\| \|$, but is unbounded, therefore neither Cauchy nor convergent, for $\| \|'$; thus (iv) and (v) are false. The set $K = A \cup \{\mathbf{0}\}$ is compact, therefore totally bounded, for $\| \|$, but is unbounded, therefore not totally bounded nor compact, for $\| \|'$; thus (vi) and (viii) are false. Finally, A is not complete for $\| \|$ (the sequence $\langle y_n \rangle_{n \in \mathbb{N}}$ in A is a $\| \|$-Cauchy sequence in A, but is not $\| \|$-convergent to any point of A, since no y_n is zero) while it is complete for $\| \|'$ (any $\| \|'$-Cauchy sequence in A must be $\| \|'$-bounded, therefore takes only finitely many values, therefore is eventually constant, therefore is $\| \|'$-convergent), and (vii) is false.

case (β) Otherwise, there is an M such that $\|x\|' \leq M\|x\|$ for every $x \in X$. But now, because the norms are not equivalent, there must be for every $n \in \mathbb{N}$ an $x_n \in X$ such that $\|x_n\| > n\|x_n\|'$. It follows, exactly as in case (α), that (ii)-(viii) are all false.

Thus any of the conditions (ii)-(viii) implies (i).

33. Yes. The point is that if $x,\, y \in X$ then

$$\rho(x,x_0) \le \rho(x,y) + \rho(y,x_0) \text{ so } \rho(x,x_0) - \rho(y,x_0) \le \rho(x,y),$$

$$\rho(y,x_0) \le \rho(x,y) + \rho(x,x_0) \text{ so } \rho(y,x_0) - \rho(x,x_0) \le \rho(x,y);$$

putting these together, $|\rho(x,x_0) - \rho(y,x_0)| \le \rho(x,y)$. Now, given $y \in X$ and $\epsilon > 0$, set $\delta = \epsilon > 0$; then

$$\rho(x,y) \le \delta \Rightarrow |\rho(x,x_0) - \rho(y,x_0)| \le \rho(x,y) \le \delta = \epsilon.$$

As ϵ is arbitrary, the function $x \mapsto \rho(x,x_0)$ is continuous at y; as y is arbitrary, it is continuous.

34. **(a)** A metric space (X,ρ) is **complete** if every Cauchy sequence in X has a limit in X. A **Banach space** is a normed space $(X,\| \|)$ such that X is complete under the metric $\rho(x,y)=\|x-y\|$.

(b) To see that X is a linear subspace of $\mathbb{R}^{\mathbb{N}}$ and that $\| \|$ is a norm on X:

(i) If $x=(\xi_1,\dots)$ and $y=(\eta_1,\dots)\in X$, then

$$\begin{aligned}\sup_{n\in\mathbb{N}}\|P_n(x+y)\| &= \sup_{n\in\mathbb{N}}\|P_nx+P_ny\|\\ &\le \sup_{n\in\mathbb{N}}\|P_nx\|+\|P_ny\|\\ &\le \sup_{n\in\mathbb{N}}\|P_nx\|+\sup_{n\in\mathbb{N}}\|P_ny\|\\ &= \|x\|+\|y\|.\end{aligned}$$

So $x+y\in X$ and $\|x+y\|\le\|x\|+\|y\|$.

(ii) If $x=(\xi_1,\dots)\in X$ and $\alpha\in\mathbb{R}$, then

$$\begin{aligned}\sup_{n\in\mathbb{N}}\|P_n(\alpha x)\| &= \sup_{n\in\mathbb{N}}\|\alpha P_nx\|\\ &\le \sup_{n\in\mathbb{N}}|\alpha|\|P_nx\|\\ &= |\alpha|\sup_{n\in\mathbb{N}}\|P_nx\|\\ &= |\alpha|\|x\|.\end{aligned}$$

So $\alpha x\in X$ and $\|\alpha x\|=|\alpha|\|x\|$.

(iii) The zero sequence $\mathbf{0}$ belongs to X because

$$\sup_{n\in\mathbb{N}}\|P_n(\mathbf{0})\|=\sup_{n\in\mathbb{N}}\|\mathbf{0}\|=0<\infty.$$

(iv) Thus X is a linear subspace of $\mathbb{R}^{\mathbb{N}}$. The only remaining point to check about the functional $\| \|:X\to[0,\infty[$ is that

$$\|x\|=0\Rightarrow\|P_nx\|=0\ \forall\ n\Rightarrow P_nx=\mathbf{0}\ \forall\ n\Rightarrow x=\mathbf{0}.$$

(c) Now suppose that $\langle x^{(r)}\rangle_{r\in\mathbb{N}}$ is a Cauchy sequence in X. For each r express $x^{(r)}$ as $(\xi_1^{(r)},\dots)$.

(i) For each $n\in\mathbb{N}$ we have

$$|\xi_n^{(r)}-\xi_n^{(s)}|\le\|P_nx^{(r)}-P_nx^{(s)}\|_\infty\le\alpha_n^{-1}\|P_nx^{(r)}-P_nx^{(s)}\|\le\alpha_n\|x^{(r)}-x^{(s)}\|\to0$$

as $r,\,s\to\infty$; that is, $\langle\xi_n^{(r)}\rangle_{r\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Because $\mathbb{R}$ is complete, it has a limit ξ_n say. Set $x=(\xi_1,\dots)\in\mathbb{R}^{\mathbb{N}}$.

(ii) Because $\langle x^{(r)}\rangle_{r\in\mathbb{N}}$ is a Cauchy sequence in X, it is bounded; say $\|x^{(r)}\|\le M$ for every r. Now, for each $n,\,r\in\mathbb{N}$, we have

$$\begin{aligned}\|P_nx\| &\le \|P_nx-P_nx^{(r)}\|+\|P_nx^{(r)}\|\\ &\le \beta_n\|P_nx-P_nx^{(r)}\|_\infty+\|x^{(r)}\|\\ &\le \beta_n\sum_{i=1}^n|\xi_i-\xi_i^{(r)}|+M\\ &\to M\end{aligned}$$

as $r\to\infty$. Thus $\|P_nx\|\le M$ for every $n\in\mathbb{N}$, and $x\in X$.

(iii) Now let $\epsilon>0$. Then there is an r_0 such that $\|x^{(r)}-x^{(s)}\|\le\epsilon$ for every $r,\,s\ge r_0$. Now, for each $n\in\mathbb{N}$, $r,\,s\ge r_0$ we have

$$\begin{aligned}\|P_nx-P_nx^{(s)}\| &\le \|P_nx-P_nx^{(r)}\|+\|P_nx^{(r)}-P_nx^{(s)}\|\\ &\le \beta_n\|P_nx-P_nx^{(r)}\|_\infty+\|x^{(r)}-x^{(s)}\|\\ &\le \beta_n\sum_{i=1}^n|\xi_i-\xi_i^{(r)}|+\epsilon\\ &\to \epsilon\end{aligned}$$

as $r\to\infty$. Thus $\|P_nx-P_nx^{(s)}\|\le\epsilon$ for every $n\in\mathbb{N}$, $s\ge r_0$. As n is arbitrary, $\|x-x^{(s)}\|\le\epsilon$ for every $s\ge r_0$. As ϵ is arbitrary, $x=\lim_{s\to\infty}x^{(s)}$ in X. As $\langle x^{(r)}\rangle_{r\in\mathbb{N}}$ is arbitrary, X is complete.

35. **(a)** We give ℓ^1 its usual norm $\| \|_1$, where $\|x\|_1 = \sum_{i=1}^\infty |\xi_i|$ if $x = (\xi_1, \dots)$.

Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be a Cauchy sequence in ℓ^1. For each $n \in \mathbb{N}$, express x_n as $(\xi_1^{(n)}, \dots)$.

If $i \in \mathbb{N}$, then

$$|\xi_i^{(m)} - \xi_i^{(n)}| \le \|x_m - x_n\|_1 \to 0$$

as $m, n \to \infty$, so $\langle \xi_i^{(n)}\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence and has a limit ξ_i say. Set $x = (\xi_1, \dots) \in \mathbb{R}^{\mathbb{N}}$.

Because $\langle x_n\rangle_{n\in\mathbb{N}}$ is Cauchy, it is bounded; say $\|x_n\|_1 \le M$ for every n. Now we have

$$\sum_{i=1}^k |\xi_i^{(n)}| \le \sum_{i=1}^\infty |\xi_i^{(n)}| = \|x_n\|_1 \le M \text{ for all } k, n \in \mathbb{N}.$$

Letting $n \to \infty$,

$$\sum_{i=1}^k |\xi_i| = \lim_{n\to\infty} \sum_{i=1}^k |\xi_i^{(n)}| \le M \text{ for every } k \in \mathbb{N};$$

letting $k \to \infty$,

$$\sum_{i=1}^\infty |\xi_i| = \lim_{k\to\infty} \sum_{i=1}^k |\xi_i| \le M$$

and $x \in \ell^1$.

Let $\epsilon > 0$. Then there is an $n_0 \in \mathbb{N}$ such that $\|x_m - x_n\|_1 \le \epsilon$ for every $m, n \ge n_0$, so that

$$\sum_{i=1}^k |\xi_i^{(m)} - \xi_i^{(n)}| \le \sum_{i=1}^\infty |\xi_i^{(m)} - \xi_i^{(n)}| = \|x_m - x_n\|_1 \le \epsilon \text{ for all } k \in \mathbb{N},\ m, n \ge n_0.$$

Letting $n \to \infty$,

$$\sum_{i=1}^k |\xi_i^{(m)} - \xi_i| = \lim_{n\to\infty} \sum_{i=1}^k |\xi_i^{(m)} - \xi_i^{(n)}| \le \epsilon \text{ for every } k \in \mathbb{N},\ m \ge n_0;$$

letting $k \to \infty$,

$$\|x_m - x\|_1 = \sum_{i=1}^\infty |\xi_i^{(m)} - \xi_i| = \lim_{k\to\infty} \sum_{i=1}^k |\xi_i^{(m)} - \xi_i| \le \epsilon \text{ for every } m \ge n_0.$$

As ϵ is arbitrary, $x = \lim_{m\to\infty} x_m$ in ℓ^1. As $\langle x_n\rangle_{n\in\mathbb{N}}$ is arbitrary, ℓ^1 is complete.

(b) We give $C[a, b]$ the norm $\| \|_\infty$ where $\|x\|_\infty = \sup_{t\in[a,b]} |x(t)|$.

Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be a Cauchy sequence in $C[a, b]$. For each $t \in [a, b]$,

$$|x_m(t) - x_n(t)| \le \|x_m - x_n\|_\infty \to 0 \text{ as } m, n \to \infty,$$

so that $\langle x_n(t)\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ and has a limit $x(t)$ say. This defines a function $x : [a, b] \to \mathbb{R}$.

Given $\epsilon > 0$, there is an $n_0 \in \mathbb{N}$ such that $\|x_m - x_n\|_\infty \le \epsilon$ for all $m, n \ge n_0$, that is, $|x_m(t) - x_n(t)| \le \epsilon$ for every $m, n \ge n_0$ and every $t \in [a, b]$. Letting $n \to \infty$,

$$|x_m(t) - x(t)| \le \epsilon \text{ for every } m \ge n_0,\ t \in [a, b].$$

But this says just that $x_m \to x$ uniformly on $[a, b]$, so x must be continuous.

Given $\epsilon > 0$, there is an $n_0 \in \mathbb{N}$ such that $\|x_m - x_n\|_\infty \le \epsilon$ for all $m, n \ge n_0$, that is, $|x_m(t) - x_n(t)| \le \epsilon$ for every $m, n \ge n_0$ and every $t \in [a, b]$. Letting $n \to \infty$,

$$|x_m(t) - x(t)| \le \epsilon \text{ for every } m \ge n_0,\ t \in [a, b];$$

as t is arbitrary,

$$\|x_m - x\|_\infty \le \epsilon \text{ for every } m \ge n_0.$$

As ϵ is arbitrary, $x_n \to x$ in $C[a, b]$. As $\langle x_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $C[a, b]$ is complete.

36. (a) For any $x, y \in \ell^\infty_I$, $i \in I$, $\alpha \in \mathbb{R}$ we have

$$|(x+y)(i)| = |x(i)+y(i)| \le |x(i)| + |y(i)| \le \|x\|_\infty + \|y\|_\infty,$$

$$|(\alpha x)(i)| = |\alpha x(i)| = |\alpha||x(i)| \le |\alpha|\|x\|_\infty.$$

As i is arbitrary,

$$\|x+y\|_\infty \le \|x\|_\infty + \|y\|_\infty, \quad \|\alpha x\|_\infty \le |\alpha|\|x\|_\infty.$$

If $\|x\|_\infty = 0$ then

$$0 \le |x(i)| \le \|x\|_\infty = 0$$

so $x(i) = 0$ for every i and $x = 0$. This shows that $\| \|_\infty$ is a norm.

(b) Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be a Cauchy sequence in ℓ^∞.
If $i \in I$, then

$$|x_m(i) - x_n(i)| \le \|x_m - x_n\|_\infty \to 0$$

as $m, n \to \infty$, so $\langle x_n(i)\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence and has a limit $x(i)$ say. This defines a function $x : I \to \mathbb{R}$.
Because $\langle x_n\rangle_{n\in\mathbb{N}}$ is Cauchy, it is bounded; say $\|x_n\|_\infty \le M$ for every n. Now we have

$$|x_n(i)| \le \|x_n\|_\infty \le M \text{ for all } n \in \mathbb{N},\ i \in I.$$

Letting $n \to \infty$,

$$|x(i)| = \lim_{n\to\infty} |x_n(i)| \le M \text{ for every } i \in I;$$

so that $x \in \ell^\infty_I$.
Let $\epsilon > 0$. Then there is an $n_0 \in \mathbb{N}$ such that $\|x_m - x_n\|_\infty \le \epsilon$ for every $m, n \ge n_0$, so that

$$|x_m(i) - x_n(i)| \le \|x_m - x_n\|_\infty \le \epsilon \text{ for all } i \in I,\ m, n \ge n_0.$$

Letting $n \to \infty$,

$$|x_m(i) - x(i)| = \lim_{n\to\infty} |x_m(i) - x(i)\| \le \epsilon \text{ for every } i \in I,\ m \ge n_0;$$

as i is arbitrary,

$$\|x_m - x\|_\infty \le \epsilon \text{ for every } m \ge n_0.$$

As ϵ is arbitrary, $x = \lim_{m\to\infty} x_m$ in ℓ^∞. As $\langle x_n\rangle_{n\in\mathbb{N}}$ is arbitrary, ℓ^∞ is complete, therefore a Banach space.

37. **(a)** For $x = (\xi_1, \dots)$, $y = (\eta_1, \dots) \in \ell^1(X)$, $\alpha \in \mathbb{R}$ set $x + y = (\xi_1 + \eta_1, \xi_2 + \eta_2, \dots)$, $\alpha x = (\alpha\xi_1, \alpha\xi_2, \dots)$; these rules define $x + y$, αx as sequences in X. Similarly, define $-x$ to be $(-\xi_1, \dots)$ and $\mathbf{0}$ to be $(0, 0, \dots)$. For $x = (\xi_1, \dots) \in \ell^1(X)$ set $\|x\|_1 = \sum_{n=1}^{\infty} \|\xi_n\|$.

(b) These definitions render $\ell^1(X)$ a normed space.

P **(i)** If $x = (\xi_1, \dots)$, $y = (\eta_1, \dots) \in \ell^1(X)$, then

$$\sum_{n=1}^{\infty} \|\xi_n + \eta_n\| \le \sum_{n=1}^{\infty} \|\xi_n\| + \|\eta_n\| \le \sum_{n=1}^{\infty} \|\xi_n\| + \sum_{n=1}^{\infty} \|\eta_n\| = \|x\|_1 + \|y\|_1.$$

So $x + y \in \ell^1(X)$ and $\|x + y\|_1 \le \|x\|_1 + \|y\|_1$.

(ii) If $x = (\xi_1, \dots)$, $y = (\eta_1, \dots)$, $z = (\zeta_1, \dots)$ belong to $\ell^1(X)$, then

$$x + (y + z) = (\xi_1 + (\eta_1 + \zeta_1), \dots) = ((\xi_1 + \eta_1) + \zeta_1, \dots) = (x + y) + z.$$

(iii) Of course $0 + 0 + \ldots = 0$ so $\mathbf{0} \in \ell^1(X)$. If $x \in \ell^1(X)$ then

$$x + \mathbf{0} = (\xi_1 + 0, \dots) = (\xi_1, \dots) = x = (0 + \xi_1, \dots) = \mathbf{0} + x.$$

(iv) If $x = (\xi_1, \dots) \in \ell^1(X)$ then

$$\sum_{n=1}^{\infty} \| - \xi_n\| = \sum_{n=1}^{\infty} \|\xi_n\| < \infty,$$

so $-x \in \ell^1(X)$. Now

$$x + (-x) = (\xi_1 + (-\xi_1), \dots) = (0, \dots) = \mathbf{0} = ((-\xi_1) + \xi_1, \dots) = (-x) + x.$$

(v) If $x = (\xi_1, \dots)$, $y = (\eta_1, \dots) \in \ell^1(X)$ then

$$x + y = (\xi_1 + \eta_1, \dots) = (\eta_1 + \xi_1, \dots) = y + x.$$

(vi) If $x = (\xi_1, \dots) \in \ell^1(X)$ and $\alpha \in \mathbb{R}$ then

$$\sum_{n=1}^{\infty} \|\alpha\xi_n\| = \sum_{n=1}^{\infty} |\alpha|\|\xi_n\| = |\alpha| \sum_{n=1}^{\infty} \|\xi_n\| = |\alpha|\|x\|_1 < \infty,$$

so $\alpha x \in \ell^1$ and $\|\alpha x\|_1 = |\alpha|\|x\|_1$.

(vii) If $x = (\xi_1, \dots)$ and $y = (\eta_1, \dots)$ belong to $\ell^1(X)$ and $\alpha \in \mathbb{R}$, then

$$\alpha(x + y) = (\alpha(\xi_1 + \eta_1), \dots) = (\alpha\xi_1 + \alpha\eta_1, \dots) = \alpha x + \alpha y.$$

(viii) If $x = (\xi_1, \dots) \in \ell^1(X)$ and α, $\beta \in \mathbb{R}$ then

$$(\alpha + \beta)x = ((\alpha + \beta)\xi_1, \dots) = (\alpha\xi_1 + \beta\xi_1, \dots) = \alpha x + \beta x.$$

(ix) If $x = (\xi_1, \dots) \in \ell^1(X)$ and α, $\beta \in \mathbb{R}$ then

$$(\alpha\beta)x = ((\alpha\beta)\xi_1, \dots) = ((\alpha\beta)\xi_1, \dots) = (\alpha x\beta)x.$$

(x) If $x = (\xi_1, \dots) \in \ell^1(X)$ then

$$1x = (1\xi_1, \dots) = (\xi_1, \dots) = x.$$

(xi) If $x = (\xi_1, \dots) \in \ell^1(X)$ and $\|x\|_1 = 0$, then $\xi_n = 0$ in X for every $n \in \mathbb{N}$ so $x = 0$ in $\ell^1(X)$.

Thus $\ell^1(X)$ is a linear space and $\| \|_1$ is a norm on $\ell^1(X)$. **Q**

[Remark: The question as written was

Show that $\ell^1(X)$ is a linear space (describing carefully the linear operations)

which could be read as a demand to show from first principles that $\ell^1(X)$ is a linear space, as done above. Of course the work can be greatly shortened if you allow yourself to assume that $X^{\mathbb{N}}$ is a linear space, as proved in lectures, and show only that $\ell^1(X)$ is a linear subspace of $X^{\mathbb{N}}$.]

(c) Now suppose that $\langle x^{(r)} \rangle_{r\in\mathbb{N}}$ is a Cauchy sequence in $\ell^1(X)$. For each r express $x^{(r)}$ as $(\xi_1^{(r)}, \dots)$.

(i) For each $n \in \mathbb{N}$,

$$\|\xi_n^{(r)} - \xi_n^{(s)}\| \le \|x^{(r)} - x^{(s)}\|_1 \to 0 \text{ as } r, s \to \infty,$$

so $\langle \xi_n^{(r)} \rangle_{r\in\mathbb{N}}$ is a Cauchy sequence in X; because X is a Banach space, it has a limit $\xi_n \in X$. Set $x = (\xi_1, \dots)$; then x is a sequence in X.

(ii) Because $\langle x^{(r)} \rangle_{r\in\mathbb{N}}$ is a Cauchy sequence, it must be bounded; say $\|x^{(r)}\|_1 \le M$ for every $r \in \mathbb{N}$. Then for any r, $n \in \mathbb{N}$

$$\begin{aligned}
\sum_{k=1}^{n} \|\xi_k\| &\le \sum_{k=1}^{n} \|\xi_k - \xi_k^{(r)}\| + \sum_{k=1}^{n} \|\xi_k^{(r)}\| \\
&\le \sum_{k=1}^{n} \|\xi_k - \xi_k^{(r)}\| + \|x^{(r)}\|_1 \\
&\le \sum_{k=1}^{n} \|\xi_k - \xi_k^{(r)}\| + M \\
&\to M
\end{aligned}$$

as $r \to \infty$, so $\sum_{k=1}^{n} \|\xi_k\| \le M$. As n is arbitrary, $\sum_{k=1}^{\infty} \|\xi_k\| \le M$ and $x \in \ell^1(X)$.

(iii) Let $\epsilon > 0$. Then there is an r_0 such that $\|x^{(r)} - x^{(s)}\|_1 \le \epsilon$ for every r, $s \ge r_0$. Then for any r, $s \ge r_0$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\sum_{k=1}^{n} \|\xi_k - \xi_k^{(s)}\| &\le \sum_{k=1}^{n} \|\xi_k - \xi_k^{(r)}\| + \sum_{k=1}^{n} \|\xi_k^{(r)} - \xi_k^{(s)}\| \\
&\le \sum_{k=1}^{n} \|\xi_k - \xi_k^{(r)}\| + \|x^{(r)} - \xi^{(s)}\|_1 \\
&\le \sum_{k=1}^{n} \|\xi_k - \xi_k^{(r)}\| + \epsilon \\
&\to \epsilon
\end{aligned}$$

as $r \to \infty$, so $\sum_{k=1}^{n} \|\xi_k - \xi_k^{(s)}\| \le \epsilon$. As n is arbitrary, $\|x - x^{(s)}\| \le \epsilon$, for every $s \ge r_0$. As ϵ is arbitrary, $\langle x^{(r)} \rangle_{r\in\mathbb{N}} \to x$. As $\langle x^{(r)} \rangle_{r\in\mathbb{N}}$ is arbitrary, $\ell^1(X)$ is complete.

38. (a) A linear space over $\mathbb{R}$ is a structure $(X, +, ., 0, -)$ where X is a set, $+ : X\times X \to X$, $. : \mathbb{R}\times X \to X$ and $- : X \to X$ are functions, and 0 is a distinguished element of X, such that

$(X, +, 0, -)$ is an abelian group, that is,

$$(x+y)+z = x+(y+z), \quad x+0 = 0+x = x, \quad x+(-x) = (-x)+x = 0, \quad x+y = y+x$$

for all x, y, $z \in X$,

$(\alpha+\beta)x = \alpha x+\beta x$,
$\alpha(x+y) = \alpha x+\alpha y$,
$(\alpha\beta)x = \alpha(\beta x)$,
$1x = x$

for all x, $y \in X$ and α, $\beta \in \mathbb{R}$.

(b) Suppose that X is any linear space over $\mathbb{R}$ and that I is any set. Write X^I for the set of all functions from I to X. For x, $y \in X^I$ and $\alpha \in \mathbb{R}$ write

$$(x+y)(i) = x(i)+y(i), \quad (-x)(i) = -x(i), \quad (\alpha x)(i) = \alpha x(i)$$

for every $i \in I$; this defines $x+y$, αx and $-x$ as functions from I to X. If we also define $\mathbf{0}(i) = 0$ for every $i \in I$, then $(X^I, +, \mathbf{0}, -, .)$ is a linear space over $\mathbb{R}$. **P** We have to check the following. If x, y, $z \in X^I$ and α, $\beta \in \mathbb{R}$ then

$$\begin{aligned}((x+y)+z)(i) &= (x+y)(i)+h(i)\\ &= (x(i)+y(i))+z(i)\\ &= x(i)+(y(i)+z(i))\\ &= x(i)+(y+z)(i)\\ &= (x+(y+z))(i),\end{aligned}$$

$$\begin{aligned}(x+\mathbf{0})(i) &= x(i)+\mathbf{0}(i)\\ &= x(i)+0\\ &= x(i)\\ &= 0+x(i)\\ &= \mathbf{0}(i)+x(i)\\ &= (\mathbf{0}+x)(i),\end{aligned}$$

$$\begin{aligned}(x+(-x))(i) &= x(i)+(-x)(i)\\ &= x(i)+(-x(i))\\ &= 0\\ &= \mathbf{0}(i)\\ &= (-x(i))+x(i)\\ &= ((-x)+x)(i),\end{aligned}$$

$$(x+y)(i) = x(i)+y(i) = y(i)+x(i) = (y+x)(i),$$

$$(\alpha(x+y))(i) = \alpha(x+y)(i) = \alpha(x(i)+y(i)) = \alpha x(i)+\alpha y(i) = (\alpha x)(i)+(\alpha y)(i) = (\alpha x+\alpha y)(i),$$

$$((\alpha+\beta)x)(i) = (\alpha+\beta)x(i) = \alpha x(i)+\beta x(i) = (\alpha x)(i)+(\beta x)(i) = (\alpha x+\beta x)(i),$$

$$(\alpha(\beta x))(i) = \alpha(\beta x)(i) = \alpha(\beta x(i)) = (\alpha\beta)x(i) = ((\alpha\beta)x)(i),$$

$$(1x)(i) = 1.x(i) = x(i)$$

for every $i \in I$, so that

$$(x+y)+z = x+(y+z),$$

$$x+\mathbf{0} = \mathbf{0}+f = x,$$

$$x+(-x) = \mathbf{0} = (-x)+x,$$

$$x+y = y+x,$$

$$\alpha(x+y) = \alpha x+\alpha y,$$

$$(\alpha+\beta)x = \alpha x+\beta x,$$

$$\alpha(\beta x) = (\alpha\beta)x,$$

$$1x = x$$

for all x, y, $z \in X^I$ and α, $\beta \in \mathbb{R}$.

39. **(a)** If $x = (\xi_1, \dots)$ and $y = (\eta_1, \dots)$ belong to X and $\alpha \in \mathbb{R}$ then

$$\sum_{i=1}^{\infty} \|\xi_i + \eta_i\| \le \sum_{i=0}^{\infty} \|\xi_i\| + \|\eta_i\| = \sum_{i=0}^{\infty} \|\xi_i\| + \sum_{i=0}^{\infty} \|\eta_i\| = \|x\|_1 + \|y\|_1,$$

so $x + y \in \ell^1(X)$ and $\|x + y\|_1 \le \|x\|_1 + \|y\|_1$;

$$\sum_{i=1}^{\infty} \|\alpha\xi_i\| = \sum_{i=1}^{\infty} |\alpha|\|\xi_i\| = |\alpha| \sum_{i=1}^{\infty} \|\xi_i\| = |\alpha|\|x\|_1,$$

so $\alpha x \in \ell^1(X)$ and $\|\alpha x\|_1 = |\alpha|\|x\|_1$;

$$\sum_{i=1}^{\infty} \|0\| = \sum_{i=1}^{\infty} 0 = 0,$$

so $\mathbf{0} \in \ell^1(X)$. Thus $\ell^1(X)$ is a linear subspace of $X^{\mathbb{N}}$. Moreover, if $\|x\|_1 = 0$ then $\|\xi_i\| = 0$ for every i so $\xi_i = 0$ for every i and $x = \mathbf{0}$; thus $\| \, \|_1$ is a norm.

(b) Now suppose that X is a Banach space. Let $\langle x^{(r)} \rangle_{r \in \mathbb{N}}$ be a Cauchy sequence in $\ell^1(X)$. For each r express $x^{(r)}$ as $(\xi_1^{(r)}, \dots)$.

(i) For each $n \in \mathbb{N}$,

$$\|\xi_n^{(r)} - \xi_n^{(s)}\| \le \|x^{(r)} - x^{(s)}\|_1 \to 0 \text{ as } r,\, s \to \infty,$$

so $\langle \xi_n^{(r)} \rangle_{r \in \mathbb{N}}$ is a Cauchy sequence in X; because X is a Banach space, it has a limit $\xi_n \in X$. Set $x = (\xi_1, \dots)$; then x is a sequence in X.

(ii) Because $\langle x^{(r)} \rangle_{r \in \mathbb{N}}$ is a Cauchy sequence, it must be bounded; say $\|x^{(r)}\|_1 \le M$ for every $r \in \mathbb{N}$. Then for any r, $n \in \mathbb{N}$

$$\begin{aligned}
\sum_{k=1}^{n} \|\xi_k\| &\le \sum_{k=1}^{n} \|\xi_k - \xi_k^{(r)}\| + \sum_{k=1}^{n} \|\xi_k^{(r)}\| \\
&\le \sum_{k=1}^{n} \|\xi_k - \xi_k^{(r)}\| + \|x^{(r)}\|_1 \\
&\le \sum_{k=1}^{n} \|\xi_k - \xi_k^{(r)}\| + M \\
&\to M
\end{aligned}$$

as $r \to \infty$, so $\sum_{k=1}^{n} \|\xi_k\| \le M$. As n is arbitrary, $\sum_{k=1}^{\infty} \|\xi_k\| \le M$ and $x \in \ell^1(X)$.

(iii) Let $\epsilon > 0$. Then there is an r_0 such that $\|x^{(r)} - x^{(s)}\|_1 \le \epsilon$ for every r, $s \ge r_0$. Then for any r, $s \ge r_0$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\sum_{k=1}^{n} \|\xi_k - \xi_k^{(s)}\| &\le \sum_{k=1}^{n} \|\xi_k - \xi_k^{(r)}\| + \sum_{k=1}^{n} \|\xi_k^{(r)} - \xi_k^{(s)}\| \\
&\le \sum_{k=1}^{n} \|\xi_k - \xi_k^{(r)}\| + \|x^{(r)} - \xi^{(s)}\|_1 \\
&\le \sum_{k=1}^{n} \|\xi_k - \xi_k^{(r)}\| + \epsilon \\
&\to \epsilon
\end{aligned}$$

as $r \to \infty$, so $\sum_{k=1}^{n} \|\xi_k - \xi_k^{(s)}\| \le \epsilon$. As n is arbitrary, $\|x - x^{(s)}\| \le \epsilon$, for every $s \ge r_0$. As ϵ is arbitrary, $\langle x^{(r)} \rangle_{r \in \mathbb{N}} \to x$. As $\langle x^{(r)} \rangle_{r \in \mathbb{N}}$ is arbitrary, $\ell^1(X)$ is complete.

40. **(a)** $C_b(X)$ is a subset of ℓ_X^∞ by definition – $C_b(X)$ is precisely the set of continous functions belonging to ℓ_X^∞. If x, $y \in C_b(X)$ and $\alpha \in \mathbb{R}$ then $x+y$, αx are continuous functions, so belong to $C_b(X)$; also the constant zero function is continuous. Thus $C_b(X)$ is a linear subspace of ℓ_X^∞.

(b) Let $x \in \ell_X^\infty$ be any point in the closure of $C_b(X)$. Let $t_0 \in X$, $\epsilon > 0$. Then there is a $y \in C_b(X)$ such that $\|x-y\|_\infty \le \frac{1}{3}\epsilon$. Because y is continuous, there is a $\delta > 0$ such that $|y(t)-y(t_0)| \le \frac{1}{3}\epsilon$ whenever $\rho(t,t_0) \le \delta$. Now if $\rho(t,t_0) \le \delta$

$$\begin{aligned}|x(t)-x(t_0)| &\le |x(t)-y(t)| + |y(t)-y(t_0)| + |y(t_0)-x(t_0)| \\ &\le \|x-y\|_\infty + \frac{1}{3}\epsilon + \|x-y\|_\infty \\ &\le \frac{1}{3}\epsilon + \frac{1}{3}\epsilon + \frac{1}{3}\epsilon \\ &= \epsilon.\end{aligned}$$

As t_0, ϵ are arbitrary, x is continuous. Thus $C_b(X) = \overline{C_b(X)}$ is a closed subset of ℓ_X^∞.

41. **(a)** Suppose that f, $g \in C_{bu}(X)$. If $\epsilon > 0$ then (because f and g are uniformly continuous) there are δ_1, $\delta_2 > 0$ such that $|f(x)-f(y)| \le \frac{1}{2}\epsilon$ whenever $\rho(x,y) \le \delta_1$, $|g(x)-g(y)| \le \frac{1}{2}\epsilon$ whenever $\rho(x,y) \le \delta_2$. Set $\delta = \min(\delta_1,\delta_2) > 0$; then if $\rho(x,y) \le \delta$,

$$\begin{aligned}|(f+g)(x)-(f+g)(y)| &= |f(x)+g(x)-f(y)-g(y)| \\ &\le |f(x)-f(y)| + |g(x)-g(y)| \\ &\le \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.\end{aligned}$$

As ϵ is arbitrary, $f+g$ is uniformly continuous and belongs to $C_{bu}(X)$.

(b) Suppose that $f \in C_{bu}(X)$ and $\alpha \in \mathbb{R}$. If $\epsilon > 0$ then (because f is uniformly continuous) there is a $\delta > 0$ such that $|f(x)-f(y)| \le \frac{\epsilon}{1+|\alpha|}$ whenever $\rho(x,y) \le \delta$. So if $\rho(x,y) \le \delta$,

$$\begin{aligned}|(\alpha f)(x)-(\alpha f)(y)| &= |\alpha f(x)-\alpha f(y)| = |\alpha||f(x)-f(y)| \\ &\le |\alpha|\tfrac{\epsilon}{1+|\alpha|} \le \epsilon.\end{aligned}$$

As ϵ is arbitrary, αf is uniformly continuous and belongs to $C_{bu}(X)$.

(c) Writing $\mathbf{0}$ for the zero function in $C_b(X)$, then for any $\epsilon > 0$ we have $|\mathbf{0}(x)-\mathbf{0}(y)| = 0 \le \epsilon$ whenever $\rho(x,y) \le 1$, so $\mathbf{0}$ is uniformly continuous and belongs to $C_{bu}(X)$.
Thus $C_{bu}(X)$ is a linear subspace of $C_b(X)$.

(d) Now suppose $f \in C_b(X)$ belongs to the closure of $C_{bu}(X)$ in $C_b(X)$. Then there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in $C_{bu}(X)$ converging to f for the norm $\| \, \|_\infty$ of $C_b(X)$. Let $\epsilon > 0$. Then there is an $n \in \mathbb{N}$ such that $\|f-f_n\|_\infty \le \frac{1}{3}\epsilon$. Next, there is a $\delta > 0$ such that $|f_n(x)-f_n(y)| \le \frac{1}{3}\epsilon$ whenever $\rho(x,y) \le \delta$. If now $\rho(x,y) \le \delta$,

$$\begin{aligned}|f(x)-f(y)| &\le |f(x)-f_n(x)| + |f_n(x)-f_n(y)| + |f_n(y)-f(y)| \\ &\le \|f-f_n\|_\infty + \frac{1}{3}\epsilon + \|f_n-f\|_\infty \\ &\le \frac{1}{3}\epsilon + \frac{1}{3}\epsilon + \frac{1}{3}\epsilon = \epsilon.\end{aligned}$$

As ϵ is arbitrary, $f \in C_{bu}(X)$. As f is arbitrary, $C_{bu}(X)$ is closed in $C_b(X)$.

42. (Not available.)

43. (a) Given $y \in \mathbb{R}^2$, $\epsilon > 0$ set $\delta = \epsilon/2 > 0$. Then if $x \in \mathbb{R}^2$ and $\rho_2(y,x) \leq \delta$, we have

$$|\xi_1 - \eta_1| \leq \rho_2(x,y) \leq \delta, \quad |\xi_2 - \eta_2| \leq \rho_2(x,y) \leq \delta,$$

so

$$|(\xi_1 + \xi_2) - (\eta_1 + \eta_2)| \leq |\xi_1 - \eta_1| + |\xi_2 - \eta_2| \leq 2\delta = \epsilon.$$

As ϵ is arbitrary, the function $x \mapsto \xi_1 + \xi_2$ is continuous at y; as y is arbitrary, it is continuous.

(b) The point is that, for any $x = (\xi_1, \xi_2)$, $Y = (\eta_1, \eta_2) \in \mathbb{R}^2$,

$$\xi_1 \leq \eta_1 + |\xi_1 - \eta_1| \leq \eta_1 + \rho_2(x,y) \leq \max(\eta_1, \eta_2) + \rho_2(x,y),$$

$$\xi_2 \leq \eta_2 + |\xi_2 - \eta_2| \leq \eta_2 + \rho_2(x,y) \leq \max(\eta_1, \eta_2) + \rho_2(x,y),$$

so

$$\max(\xi_1, \xi_2) \leq \max(\eta_1, \eta_2) + \rho_2(x,y),$$

and similarly

$$\max(\eta_1, \eta_2) \leq \max(\xi_1, \xi_2) + \rho_2(x,y),$$

so

$$|\max(\xi_1, \xi_2) - \max(\eta_1, \eta_2)| \leq \rho_2(x,y).$$

Now, given $y \in \mathbb{R}^2$, $\epsilon > 0$ set $\delta = \epsilon > 0$. Then if $x \in \mathbb{R}^2$ and $\rho_2(y,x) \leq \delta$, we have

$$|\max(\xi_1, \xi_2) - \max(\eta_1, \eta_2)| \leq \rho_2(x,y) \leq \delta = \epsilon.$$

As ϵ is arbitrary, the function $x \mapsto \max(\xi_1, \xi_2)$ is continuous at y; as y is arbitrary, it is continuous.

(c) Given $y \in \mathbb{R}$, $\epsilon > 0$ set $\delta = \min(1, \epsilon/\sqrt{1 + (2y+1)^2}) > 0$. Then if $|x - y| \leq \delta$

$$\begin{aligned}
\rho_2((x,x^2),(y,y^2)) &= \sqrt{(x-y)^2 + (x^2 - y^2)^2} \\
&= |x-y|\sqrt{1 + (x+y)^2} \\
&\leq \delta\sqrt{1 + (2y+\delta)^2} \\
&\leq \delta\sqrt{1 + (2y+1)^2} \\
&\leq \epsilon.
\end{aligned}$$

As ϵ is arbitrary, the function $x \mapsto (x, x^2)$ is continuous at y; as y is arbitrary, it is continuous.

44. **(a) Picard's theorem** Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function which is Lipschitz in the second variable, that is, there is a $K \ge 0$ such that $|f(t, \xi) - f(t, \eta)| \le K|\xi - \eta|$ whenever $t \in [a, b]$ and ξ, $\eta \in [c, d]$. If $t_0 \in [a, b]$ and $x_0 \in]c, d[$, then there is a $\delta > 0$ such that the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(u, x(u))du$$

has a unique solution as a continuous function from $[a, b] \cap [t_0 - \delta, t_0 + \delta]$ to $[c, d]$.

proof (i) Because f is continuous and $[a, b] \times [c, d]$ is compact, f is bounded; say $|f(t, \xi)| \le M$ for every $t \in [a, b]$, $\xi \in [c, d]$. Take $\delta > 0$ such that $\delta K < 1$ and $c \le x_0 - \delta M \le x_0 + \delta M \le d$.

(ii) Set $a' = \max(a, t_0 - \delta)$, $b' = \min(b, t_0 + \delta)$. Let X be the set of continuous functions from $[a', b']$ to $[c, d]$. Then X is a closed subset of $C([a', b'])$. **P** If $\langle x_n \rangle_{n \in \mathbb{N}}$ is a sequence in X converging to $x \in C([a', b'])$, then for any $t \in [a', b']$

$$|x(t) - x_n(t)| \le \|x - x_n\|_\infty \to 0,$$

so

$$x(t) = \lim_{n \to \infty} x_n(t) \in [c, d]. \textbf{ Q}$$

Consequently X is a complete metric space.

(iii) Now we have a function $T : X \to X$ defined by writing

$$(Tx)(t) = x_0 + \int_{t_0}^{t} f(u, x(u))du$$

for every $t \in [a', b']$. **P** Note first that the function $u \mapsto (u, x(u)) : [a', b'] \to \mathbb{R}^2$ is continuous; because f is continuous, $u \mapsto f(u, x(u)) : [a', b'] \to \mathbb{R}$ is continuous, therefore integrable. So $(Tx)(t)$ is defined as a real number. Next, if t, $t' \in [a', b']$ then

$$\begin{aligned}
|(Tx)(t) - (Tx)(t')| &= |\int_{\min(t,t')}^{\max(t,t')} f(u, x(u))du| \\
&\le \int_{\min(t,t')}^{\max(t,t')} |f(u, x(u))|du \\
&\le \int_{\min(t,t')}^{\max(t,t')} M\,du \\
&= M|t - t'|,
\end{aligned}$$

so Tx is continuous. Moreover,

$$|(Tx)(t) - x_0| = |(Tx)(t) - (Tx)(t_0)| \le M|t - t_0| \le \delta M$$

so

$$c \le x_0 - \delta M \le (Tx)(t) \le x_0 + \delta M \le d$$

for every $t \in [a', b']$, so $Tx \in X$. **Q**

(iv) Now $T : X \to X$ is a contraction. **P** If x, $y \in X$ and $t \in [a', b']$, then

$$\begin{aligned}
|(Tx)(t) - (Ty)(t)| &= |\int_{t_0}^{t} f(u, x(u)) - f(u, y(u))du| \\
&\le \int_{\min(t,t_0)}^{\max(t,t_0)} |f(u, x(u)) - f(u, y(u))|du \\
&\le \int_{\min(t,t_0)}^{\max(t,t_0)} K|x(u) - y(u)|du \\
&\le \int_{\min(t,t_0)}^{\max(t,t_0)} K\|x - y\|_\infty du \\
&\le K|t - t_0|\|x - y\|_\infty \\
&\le K\delta\|x - y\|_\infty.
\end{aligned}$$

As t is arbitrary, $\|Tx - Ty\|_\infty \le K\delta\|x - y\|_\infty$. But $\delta K < 1$, by the choice of δ, so T is a contraction. **Q**

(v) By the contraction mapping theorem, there is a unique $x \in X$ such that $Tx = x$, that is,

$$x(t) = x_0 + \int_{t_0}^{t} f(u, x(u))du$$

for every $t \in [a', b'] = [a, b] \cap [t_0 - \delta, t_0 + \delta]$.

(b) The integral equation corresponding to the given differential equation is

$$x(t) = 1 + \int_0^t x(u)^2 du,$$

corresponding to the operator T given by

$$(Tx)(t) = 1 + \int_0^t x(u)^2 du.$$

Starting with $x_0(t) = 1$, we get

$$x_1(t) = (Tx_0)(t) = 1 + \int_0^t 1\,du = 1 + t,$$

$$x_2(t) = (Tx_1)(t) = 1 + \int_0^t (1 + u)^2 du = 1 + \int_0^t 1 + 2u + u^2 du = 1 + t + t^2 + \tfrac{1}{3}t^3,$$

$$\begin{aligned}
x_3(t) &= (Tx_2)(t) \\
&= 1 + \int_0^t (1 + u + u^2 + \tfrac{1}{3}u^3)^2 du \\
&= 1 + \int_0^t 1 + 2u + 3u^2 + \tfrac{8}{3}u^3 + \tfrac{5}{3}u^4 + \tfrac{2}{3}u^5 + \tfrac{1}{9}u^6 du \\
&= 1 + t + t^2 + t^3 + \tfrac{2}{3}t^4 + \tfrac{1}{3}t^5 + \tfrac{1}{9}t^6 + \tfrac{1}{63}t^7.
\end{aligned}$$

As for the exact solution, it is given by separation of variables:

$$\frac{dx}{dt} = x^2$$

corresponds to

$$\frac{dx}{x^2} = dt, \quad -\frac{1}{x} = t + c;$$

setting $t = 0$, $x = 1$ we get $c = -1$,

$$x = \frac{1}{1-t} = 1 + t + t^2 + t^3 + t^4 + \dots \text{ for } |t| < 1.$$

Thus for t close to 0, the polynomials $x_n(t)$ are visibly approximating the series solution for the exact solution $1/(1 - t)$.

(Diagram unavailable.)

45. (Not available.)

46. (Not available.)

47. (a)

$$\begin{aligned} H \text{ is closed in } (F,\rho_F) & \\ &\iff F \setminus H \text{ is open in } (F,\rho_F) \\ &\iff \text{there is a } G \subseteq X, \text{ open in } (X,\rho), \\ &\qquad \text{such that } F \setminus H = F \cap G \\ &\iff \text{there is a } G \subseteq X, \text{ open in } (X,\rho), \\ &\qquad \text{such that } H = F \cap (X \setminus G) \\ &\iff \text{there is an } E \subseteq X, \text{ closed in } (X,\rho), \\ &\qquad \text{such that } H = F \cap E \\ &\iff H \text{ is closed in } (X,\rho). \end{aligned}$$

(b)(i) If f is continuous and $H \subseteq Y$ is closed, then $Y \setminus H$ and $f^{-1}[Y \setminus H]$ are open, so

$$f^{-1}[H] = X \setminus f^{-1}[Y \setminus H]$$

is closed. (ii) If the inverse image of any closed set is closed, then for any open set $G \subseteq Y$

$$f^{-1}[G] = X \setminus f^{-1}[Y \setminus G]$$

is open; as G is arbitrary, f is continuous.

(c) Let $H \subseteq Y$ be closed. Then

$$f^{-1}[H] = (E \cap f^{-1}[H]) \cup (F \cap f^{-1}[H]) = f_E^{-1}[H] \cup f_F^{-1}[H],$$

writing f_E, f_F for the restrictions of f to E, F respectively. But f_E, regarded as a function from (E,ρ_E) to (Y,σ), is continuous, so $f_E^{-1}[H]$ is ρ_E-closed (by part (b)) and therefore ρ-closed (by (a)). Similarly, $f_F^{-1}[H]$ is ρ-closed, and now $f^{-1}[H]$, being the union of two ρ-closed sets, is ρ-closed. As H is arbitrary, f is continuous, by (b) in the other direction.

48. **(a)** Let (X,ρ) and (Y,σ) be metric spaces, and $f : X \to Y$ a function.

(i) For $x_0 \in X$, f is continuous at x_0 if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\sigma(f(x), f(x_0)) \le \epsilon$ whenever $\rho(x, x_0) \le \delta$. f is continuous if it is continuous at every point of X.

(ii) f is continuous if $f^{-1}[G]$ is open in X for every open set $G \subseteq Y$.

(iii) f is continuous if $\lim_{n\to\infty} f(x_n) = f(x)$ in Y whenever $\langle x_n \rangle_{n\in\mathbb{N}}$ is a sequence in X converging to $x \in X$.

(b)(i) If X is a normed space, a set $A \subseteq X$ is **bounded** if there is an $M \ge 0$ such that $\|x\| \le M$ for every $x \in A$.

(ii) A linear operator T from a normed space X to a normed space Y is continuous iff $\{Tx : x \in X, \|x\| \le 1\}$ is a bounded subset of Y.

(c)(i) If $x = (\xi_1, \dots, y = (\eta_1, \dots) \in \boldsymbol{c}$ and $\alpha \in \mathbb{R}$, then

$$f(x + y) = \lim_{i\to\infty} \xi_i + \eta_i = \lim_{i\to\infty} \xi_i + \lim_{i\to\infty} \eta_i = f(x) + f(y),$$

$$f(\alpha x) = \lim_{i\to\infty} \alpha\xi_i = \alpha \lim_{i\to\infty} \xi_i = \alpha f(x),$$

so f is linear. If $x = (\xi_1, \dots) \in \boldsymbol{c}$ and $\|x\|_\infty \le 1$, then $-1 \le \xi_i \le 1$ for every i, so

$$|f(x)| = |\lim_{i\to\infty} \xi_i| \le 1.$$

Thus $\{f(x) : \|x\|_\infty \le 1\} \subseteq [-1, 1]$ is bounded and f is continuous.

49. [answer to former version]

(a) For $S \in B(X;Y)$ set $\|S\| = \sup\{\|Sx\| : x \in X, \|x\| \le 1\}$. For $S, T \in B(X;Y)$ and $\alpha \in \mathbb{R}$, define operators $S+T : X \to Y$ and $\alpha S : X \to Y$ by setting

$$(S+T)(x) = Sx + Tx, \; (\alpha S)(x) = \alpha S(x)$$

for all $x \in X$; also define a zero operator $\mathbf{0} : X \to Y$ by setting $\mathbf{0}(x) = 0$ for every x, and $-S : X \to Y$ by setting $(-S)(x) = -Sx$ for every $x \in X$.

(i) $S+T$ is linear. **P** If $x, y \in X$ and $\gamma \in \mathbb{R}$, then

$$\begin{aligned}(S+T)(x+y) &= S(x+y) + T(x+y)\\ &= Sx + Sy + Tx + Ty\\ &= Sx + Tx + Sy + Ty\\ &= (S+T)(x) + (S+T)(y),\end{aligned}$$

$$(S+T)(\gamma x) = S(\gamma x) + T(\gamma x) = \gamma Sx + \gamma Tx = \gamma(Sx+Tx) = \gamma(S+T)x. \text{ **Q**}$$

(ii) $S+T$ is bounded, and $\|S+T\| \le \|S\| + \|T\|$. **P** If $\|x\| \le 1$, then

$$\|(S+T)(x) = \|Sx + Tx\| \le \|Sx\| + \|Tx\| \le \|S\| + \|T\|. \text{ **Q**}$$

(iii) αS is linear. **P** If $x, y \in X$ and $\gamma \in \mathbb{R}$, then

$$(\alpha S)(x+y) = \alpha S(x+y) = \alpha(Sx + Sy) = \alpha Sx + \alpha Sy = (\alpha S)(x) + (\alpha S)(y),$$

$$(\alpha S)(\gamma x) = \alpha S(\gamma x) = \alpha\gamma Sx = \gamma\alpha Sx = \gamma(\alpha S)x. \text{ **Q**}$$

(iv) αS is bounded, and $\|\alpha S\| \le |\alpha|\|S\|$. **P** If $\|x\| \le 1$, then

$$\|(\alpha S)(x)\| = \|\alpha Sx\| = |\alpha|\|Sx\| \le |\alpha|\|S\|. \text{ **Q**}$$

(v) $\mathbf{0}$ is linear, because

$$\mathbf{0}(x+y) = 0 = 0 + 0 = \mathbf{0}(x) + \mathbf{0}(y), \quad \mathbf{0}(\gamma x) = 0 = \gamma 0 = \gamma\mathbf{0}(x)$$

for all $x, y \in X$ and $\gamma \in \mathbb{R}$, and bounded because

$$\|\mathbf{0}x\| = \|0\| = 0$$

whenever $\|x\| \le 1$.

(vi) Finally, $-S = (-1)S$ is linear and bounded by (iii)-(iv) above.

(vii) A fact which will be useful is the following: if $T \in B(X;Y)$ and $x \in X$, then $\|Tx\| \le \|T\|\|x\|$. **P** Set $\gamma = \|x\|$. If $\gamma = 0$ then $x = 0$ so $Tx = 0$ and $\|Tx\| = 0 = \|T\|\|x\|$. If $\gamma > 0$ then

$$\|Tx\| = \|T(\gamma\gamma^{-1}x)\| = \|\gamma T(\gamma^{-1}x)\| = \gamma\|T(\gamma^{-1}x)\| \le \gamma\|T\| = \|T\|\|x\|$$

because $\|\gamma^{-1}x\| = \gamma^{-1}\|x\| = 1$.

(b) This defines addition, scalar multiplication, zero and negatives in $B(X;Y)$. To see that $B(X;Y)$ is a linear space we have to check that, for $S, T, U \in B(X;Y)$, $\alpha, \beta \in \mathbb{R}$ we have

$$\begin{aligned}(S+(T+U))(x) &= Sx + (T+U)(x)\\ &= Sx + (Tx + Ux)\\ &= (Sx + Tx) + Ux\\ &= ((S+T)+U)x\end{aligned}$$

for every $x \in X$, so $S + (T+U) = (S+T) + U$;

$$(S+T)(x) = Sx + Tx = Tx + Sx = (T+S)(x)$$

for every $x \in X$, so $S + T = T + S$;

$$(S+\mathbf{0})(x) = Sx + \mathbf{0}x = Sx + 0 = Sx$$

for every $x \in X$, so $S + \mathbf{0} = S$;

$$(S+(-S))(x) = Sx + (-S)x = Sx + (-Sx) = 0 = \mathbf{0}x$$

for every $x \in X$, so $S + (-S) = \mathbf{0}$;

$$\begin{aligned}(\alpha(S+T))(x) &= \alpha(S+T)(x)\\ &= \alpha(Sx+Tx)\\ &= \alpha Sx + \alpha Tx\\ &= (\alpha S)(x) + (\alpha T)(x)\\ &= (\alpha S + \alpha T)(x)\end{aligned}$$

for every $x \in X$, so $\alpha(S+T) = \alpha S + \alpha T$;

$$\begin{aligned}((\alpha+\beta)S)(x) &= (\alpha+\beta)(Sx)\\ &= \alpha Sx + \beta Sx\\ &= (\alpha S)(x)x + (\beta S)(x)\\ &= (\alpha S + \beta S)(x)\end{aligned}$$

for every $x \in X$, so $(\alpha+\beta)S = \alpha S + \beta S$;

$$(\alpha(\beta S))(x) = \alpha(\beta S)(x) = \alpha\beta Sx = (\alpha\beta S)(x)$$

for every $x \in X$, so $(\alpha\beta)S = (\alpha\beta)S$;

$$(1S)(x) = 1.Sx = Sx$$

for every $x \in X$, so $1S = S$.

(c) Thus $B(X;Y)$ is a linear space. To see that it is a normed space, we have done most of the work in (a); we already know that $\|S+T\| \le \|S\| + \|T\|$, $\|\alpha S\| = |\alpha|\|S\|$ for $S, T \in B(X;Y)$ and $\alpha \in \mathbb{R}$. The remaining fact to check is that

$$\|S\| = 0 \Rightarrow S = \mathbf{0}.$$

But if $\|S\| = 0$, then $\|Sx\| \le \|S\|\|x\| = 0$ for every $x \in X$, so $Sx = 0$ for every x and $S = \mathbf{0}$.

(d) Now suppose that Y is a Banach space. Let $\langle T_n\rangle_{n\in\mathbb{N}}$ be a Cauchy sequence in $B(X;Y)$.

(i) For any $x \in X$, we have

$$\|T_m x - T_n x\| \le \|T_m - T_n\|\|x\| \to 0$$

as $m, n \to \infty$, so $\langle T_n x\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in Y; because Y is complete, $Tx = \lim_{n\to\infty} T_n x$ exists in Y. This defines a function $T : X \to Y$.

(ii)(α) Suppose that $x, y \in X$ and $\alpha \in \mathbb{R}$. Then, for any $n \in \mathbb{N}$,

$$\begin{aligned}\|T(x+y) - Tx - Ty\| &= \|T(x+y) - T_n(x+y) - Tx + T_n x - Ty + T_n y\|\\ &\le \|T(x+y) - T_n(x+y)\| + \|Tx - T_n x\| + \|Ty - T_n y\|\\ &\to 0\end{aligned}$$

as $n \to \infty$, so $\|T(x+y) - Tx - Ty\| = 0$ and $T(x+y) = Tx + Ty$. In the same way,

$$\begin{aligned}\|T(\alpha x) - \alpha Tx\| &= \|T(\alpha x) - T_n(\alpha x) - \alpha Tx + \alpha T_n x\|\\ &\le \|T(\alpha x) - T_n(\alpha x)\| + \|\alpha Tx - \alpha T_n x\|\\ &= \|T(\alpha x) - T_n(\alpha x)\| + |\alpha|\|Tx - T_n x\|\\ &\to 0\end{aligned}$$

as $n \to \infty$, so $\|T(\alpha x) - \alpha Tx\| = 0$ and $T(\alpha x) = \alpha Tx$. As x, y and α are arbitrary, T is linear.

(β) Because Cauchy sequences, in any normed space, are bounded, there is an $M \ge 0$ such that $\|T_n\| \le M$ for every $n \in \mathbb{N}$. Now, if $\|x\| \le 1$ and $n \in \mathbb{N}$,

$$\|Tx\| \le \|Tx - T_n x\| + \|T_n x\| \le \|Tx - T_n x\| + \|T_n\| \le \|Tx - T_n x\| + M \to M$$

as $n \to \infty$. As x is arbitrary, T is bounded and belongs to $B(X;Y)$.

(iii) Now let $\epsilon > 0$. Then there is an n_0 such that $\|T_m - T_n\| \le \epsilon$ whenever $m, n \ge n_0$. If $m, n \ge n_0$ and $\|x\| \le 1$, then

$$\|Tx - T_m x\| \le \|Tx - T_n x\| + \|T_n x - T_m x\| \le \|Tx - T_n x\| + \|T_n - T_m\| \le \|Tx - T_n x\| + \epsilon.$$

Letting $n \to \infty$,

$$\|Tx - T_m x\| \le \epsilon \text{ for all } m \ge n_0, \; \|x\| \le 1.$$

As x is arbitrary, $\|T - T_m\| \le \epsilon$ for all $m \ge n_0$. As ϵ is arbitrary, $T = \lim_{m\to\infty} T_m$ in $B(X;Y)$. As $\langle T_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $B(X;Y)$ is complete, and is a Banach space.

50. If X is a normed space, a set $A \subseteq X$ is *bounded* if there is some $M \geq 0$ such that $\|x\| \leq M$ for every $x \in A$.

If X and Y are normed spaces, a linear operator $T : X \to Y$ is *bounded* if $\{Tx : \|x\| \leq 1\}$ is a bounded subset of Y.

If X and Y are normed spaces, we have a linear structure on $B(X;Y)$ defined by writing

$$(S+T)(x) = Sx + Tx, \quad (\alpha T)(x) = \alpha Tx$$

whenever S, $T \in B(X;Y)$ and $x \in X$; the zero $\mathbf{0}$ of $B(X;Y)$ is defined by the formula $\mathbf{0}(x) = 0$ for every $x \in X$, and $-T$ is defined by the formula $(-T)(x) = -Tx$ for every $x \in X$. The norm of $B(X;Y)$ is defined by the formula $\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$.

Consider the linear space ℓ^1 of absolutely summable sequences, with the norm $\| \|_\infty$. For $x = (\xi_1, \xi_2, \dots) \in \ell^1$, set $Tx = \sum_{i=1}^\infty \xi_i \in \mathbb{R}$. Then T is a linear operator from ℓ^1 to $\mathbb{R}$. But if we set $x_n = (1, 1, \dots, 1, 0, 0, \dots)$, with n 1's and then zeroes, $Tx_n = n$ while $\|x_n\|_\infty = 1$; so that T is not bounded.

51. We have $\mathbf{A}x = (\xi_1 + \xi_2, \xi_2)$. Write $\|\mathbf{A}\|_{jk}$ for the norm of $\mathbf{A}$ when the domain is given the norm $\|\,\|_j$ and the range is given the norm $\|\,\|_k$.

(a) Suppose that the domain of $\mathbf{A}$ is given the norm $\|\,\|_1$, so that its unit ball is the set $\{(\xi_1, \xi_2) : |\xi_1| + |\xi_2| \leq 1\}$.

(i)

$$\|\mathbf{A}x\|_1 = |\xi_1 + \xi_2| + |\xi_2| \leq 2(|\xi_1| + |\xi_2|),$$

so $\|\mathbf{A}\|_{11} \leq 2$; examining $x = (0,1)$, $\mathbf{A}x = (1,1)$ we see that $\|\mathbf{A}\|_{11} = 2$ exactly.

(ii)

$$\|\mathbf{A}x\|_2 = \sqrt{|\xi_1 + \xi_2|^2 + |\xi_2|^2} \leq \sqrt{1 + |\xi_2|^2} \leq \sqrt{2}$$

if $\|x\|_1 \leq 1$, so $\|\mathbf{A}\|_{12} \leq \sqrt{2}$; examining $x = (0,1)$, $\mathbf{A}x = (1,1)$ we see that $\|\mathbf{A}\|_{12} = \sqrt{2}$ exactly.

(iii) $\|\mathbf{A}x\|_\infty = \max(|\xi_1 + \xi_2|, |\xi_2|) \leq 1$ if $\|x\|_1 \leq 1$, so $\|\mathbf{A}\|_{1\infty} \leq 1$; examining $x = (0,1)$, $\mathbf{A}x = (1,1)$ we see that $\|\mathbf{A}\|_{1\infty} = 1$ exactly.

(b) Suppose that the domain of $\mathbf{A}$ is given the norm $\|\,\|_2$, so that its unit ball is the set $\{(r\cos\theta, r\sin\theta) : 0 \leq r \leq 1, \theta \in \mathbb{R}\}$.

(i)

$$\|\mathbf{A}x\|_1 = r(|\cos\theta + \sin\theta| + |\cos\theta|)$$

takes its greatest value when $r = 1$ and $\cos\theta$ and $\sin\theta$ have the same sign (because for any value of $\cos\theta$ the corresponding possible values of $\sin\theta$ are $\pm\sqrt{1 - \cos^2\theta}$), so

$$\|\mathbf{A}\|_{21} = \sup_{\theta\in\mathbb{R}} 2\cos\theta + \sin\theta = \sup_{\theta\in\mathbb{R}} s\cos(\theta - \phi) = s,$$

where $s\cos\phi = 2$, $s\sin\phi = 1$, so $s = \sqrt{5}$.

(ii)

$$\|\mathbf{A}x\|_2 = r\sqrt{|\cos\theta + \sin\theta|^2 + |\sin\theta|^2} = r\sqrt{|1 + 2\cos\theta\sin\theta| + \sin^2\theta}$$

takes its greatest value when $r = 1$, $\cos\theta\sin\theta \geq 0$, so

$$\|\mathbf{A}\|_{22} = \sup_{\theta\in\mathbb{R}} \sqrt{1 + \sin^2\theta + 2\sin\theta\cos\theta}.$$

To find this, we need to maximise

$$1 + \sin^2\theta + 2\sin\theta\cos\theta = 1 + s^2 \pm 2s\sqrt{1 - s^2} = 1 + t \pm 2\sqrt{t - t^2},$$

where $t = \sin^2\theta \in [0,1]$. This has greatest value $\frac{1}{2}(3 + \sqrt{5})$ when $1 + \frac{1-2t}{\sqrt{t-t^2}} = 0$, $t = \frac{1}{10}(5 + \sqrt{5})$, so $\|\mathbf{A}\|_{22} = \sqrt{\frac{1}{2}(3 + \sqrt{5})} \simeq 1.62$.

(iii) $\|\mathbf{A}x\|_\infty = r\max(|\cos\theta + \sin\theta|, |\cos\theta|)$ has greates value when $\cos\theta$, $\sin\theta$ have the same sign and $r = 1$; so we have to maximise $\cos\theta + \sin\theta = \sqrt{2}\cos(\theta - \frac{\pi}{4})$. Thus $\|\mathbf{A}\|_{2\infty} = \sqrt{2}$.

(c) Suppose that the domain of $\mathbf{A}$ is given the norm $\|\,\|_\infty$, so that its unit ball is the set $\{(\xi_1, \xi_2) : |\xi_1| \leq 1, |\xi_2| \leq 1\}$.

(i)

$$\|\mathbf{A}x\|_1 = |\xi_1 + \xi_2| + |\xi_2| \leq |\xi_1| + 2|\xi_2|,$$

so $\|\mathbf{A}\|_{\infty 1} \leq 3$; examining $x = (1,1)$, $\mathbf{A}x = (2,1)$ we see that $\|\mathbf{A}\|_{\infty 1} = 3$ exactly.

(ii)

$$\|\mathbf{A}x\|_2 = \sqrt{|\xi_1 + \xi_2|^2 + |\xi_2|^2} \leq \sqrt{5}$$

if $\|x\|_\infty \leq 1$, so $\|\mathbf{A}\|_{\infty 2} \leq \sqrt{5}$; examining $x = (1,1)$, $\mathbf{A}x = (2,1)$ we see that $\|\mathbf{A}\|_{\infty 2} = \sqrt{5}$ exactly.

(iii) $\|\mathbf{A}x\|_\infty = \max(|\xi_1 + \xi_2|, |\xi_2|) \leq 2$ if $\|x\|_\infty \leq 1$, so $\|\mathbf{A}\|_{\infty\infty} \leq 2$; examining $x = (1,1)$, $\mathbf{A}x = (2,1)$ we see that $\|\mathbf{A}\|_{\infty\infty} = 2$ exactly.

52. (i) Suppose that X is compact and that $\mathcal{E}$ is a non-empty collection of closed sets with the finite intersection property. Consider $\mathcal{G} = \{X \setminus E : E \in \mathcal{E}\}$. Then $\mathcal{G}$ is a family of open subsets of X. **?** If $\bigcap \mathcal{E} = \emptyset$, then $\bigcup \mathcal{G} = X \setminus \bigcap \mathcal{E} = X$, so $\mathcal{G}$ is an open cover of X. As X is compact, it has a finite subcover $\{G_0, \dots, G_n\}$ say. Now $X \setminus G_0, \dots, X \setminus G_n$ all belong to $\mathcal{E}$, and their intersection is

$$\bigcap_{i \le n} X \setminus G_i = X \setminus \bigcup_{i \le n} G_i = X \setminus X = \emptyset,$$

which is impossible, because $\mathcal{E}$ is supposed to have the finite intersection property. **X** Thus $\bigcap \mathcal{E}$ is non-empty, as required.

(ii) Suppose that X has the given property. Let $\mathcal{G}$ be an open cover of X. If $\mathcal{G} = \emptyset$ then it is already finite so surely has a finite subcover. Otherwise, set $\mathcal{E} = \{X \setminus G : G \in \mathcal{G}\}$. This is a non-empty family of closed subsets of X. Also $\bigcap \mathcal{E} = X \setminus \bigcup \mathcal{G} = \emptyset$. So $\mathcal{E}$ cannot have the finite intersection property, and there are $E_0, \dots, E_n \in \mathcal{E}$ such that $\bigcap_{i \le n} E_i = \emptyset$. Now $X \setminus E_0, \dots, X \setminus E_n \in \mathcal{G}$ and

$$\bigcup_{i \le n} X \setminus E_i = X \setminus \bigcap_{i \le n} E_i = X,$$

so $\{X \setminus E_0, \dots, X \setminus E_n\}$ is a finite subcover. As $\mathcal{G}$ is arbitrary, X is compact.

53. (Not available.)

54. **(a)** If X is a normed space its **bidual** X^{**} is the Banach space $(X^*)^* = B(X^*; \mathbb{R})$, where $X^* = B(X; \mathbb{R})$ is the dual of X.

If $x \in X$, we can write $\hat{x}(f) = f(x)$ for every $f \in X^*$; this defines $\hat{x}$ as a function from X^* to $\mathbb{R}$. Now we find (i) that $\hat{x} \in X^{**}$ for every $x \in X$ (ii) that the map $x \mapsto \hat{x} : X^{**} \to X$ is a norm-preserving linear operator, so that $\{\hat{x} : x \in X\}$ is a linear subspace of X^{**} isomorphic, as normed space, to X.

(b) If X and Y are normed spaces and $T : X \to Y$ is a bounded linear operator, then for every $g \in Y^*$ we write $T^\times(g) = gT$; this is a bounded linear operator from X to $\mathbb{R}$. Thus we have a function $T^\times : Y^* \to X^*$.

It turns out that $T^\times$ is a bounded linear operator from Y^* to X^*, so has itself an adjoint $T^{\times\times} : X^{**} \to Y^{**}$. Now suppose that $x \in X$. Then $\hat{x} \in X^{**}$ so we can speak of $T^{\times\times}\hat{x} \in Y^{**}$. To calculate it, take any $g \in Y^*$; then

$$(T^{\times\times}\hat{x})(g) = \hat{x}(T^\times g) = (T^\times g)(x) = g(Tx) = \widehat{Tx}(g).$$

As g is arbitrary, $T^{\times\times}\hat{x} = \widehat{Tx}$.

55. **(a) Hahn-Banach theorem**: if X is a normed space, $Y \subseteq X$ is a linear subspace and $f \in Y^*$ a bounded linear functional, then there is a $g \in X^*$ such that $g(y) = f(y)$ for every $y \in Y$ and $\|g\| = \|f\|$.

(b) In particular, if $X = \ell^\infty$ and $Y = \boldsymbol{c}$, the space of convergent real sequences, we have a bounded linear functional $f : Y \to \mathbb{R}$ given by setting $f(y) = \lim_{n\to\infty} \eta_n$ if $y = (\eta_1, \eta_2, \dots) \in \boldsymbol{c}$, and $\|f\| = 1$. So the Hahn-Banach theorem tells us that there is a $g \in (\ell^\infty)^*$ extending f.

Now consider ℓ^1. We know that we can identify $(\ell^1)^*$ with ℓ^∞, if for $y = (\eta, \dots) \in \ell^\infty$, $x = (\xi_1, \dots) \in \ell^1$ we write $f_y(x) = \sum_{i=1}^\infty \xi_i \eta_i$. Accordingly we have a $\phi \in (\ell^1)^{**}$ corresponding to g above, so that $\phi(f_y) = \lim_{n\to\infty} \eta_n$ whenever $y = (\eta_1, \dots) \in \ell^\infty$.

? If $\phi = \hat{x}$, where $x = (\xi_1, \dots) \in \ell^1$, then for each $n \in \mathbb{N}$ set $d_n = (\delta_{n1}, \delta_{n2}, \dots)$, where $\delta_{ni} = 1$ if $i = n$ and 0 otherwise. Then $d_n \in \ell^\infty$ and

$$\begin{aligned} 0 &= \lim_{i\to\infty} \delta_{ni} = g(d_n) = \phi(f_{d_n}) \\ &= \hat{x}(f_{d_n}) = f_{d_n}(x) = \sum_{i=1}^\infty \delta_{ni}\xi_i = \xi_n \end{aligned}$$

for every n. But also, setting $e = (1, 1, 1, \dots)$, we have

$$\begin{aligned} 1 &= \lim_{i\to\infty} 1 = g(e) = \phi(f_e) \\ &= \hat{x}(f_e) = f_e(x) = \sum_{i=1}^\infty \xi_i, \end{aligned}$$

so some ξ_i is non-zero. **X**

Thus $\phi \neq \hat{x}$ for any $x \in \ell^1$ and the map $x \mapsto \hat{x} : \ell^1 \to (\ell^1)^{**}$ is not surjective, that is, ℓ^1 is not reflexive.

56. (Not available.)

57. **(a)(i)** ℓ^2 is the space of square-summable real sequences, that is, the space of sequences $y = (\eta_1, \eta_2, \dots) \in \mathbb{R}^{\mathbb{N}}$ such that $\sum_{i=1}^{\infty} \eta_i^2 < \infty$.

(ii) If $x = (\xi_1, \dots)$ and $y = (\eta_1, \dots)$ belong to ℓ^2, and $\alpha \in \mathbb{R}$, then

$$(\xi_n + \eta_n)^2 \le (|\xi_n| + |\eta_n|)^2 \le (2\max(|\xi_n|, |\eta_n|)^2 = 4\max(|\xi_n|^2, |\eta_n|^2) \le 4(\xi_n^2 + \eta_n^2)$$

for every n, so

$$\textstyle\sum_{n=1}^{\infty}(\xi_n + \eta_n)^2 \le \sum_{n=1}^{\infty} 4(\xi_n^2 + \eta_n^2) = 4\sum_{n=1}^{\infty} \xi_n^2 + 4\sum_{n=1}^{\infty} \eta_n^2 < \infty,$$

while also

$$\textstyle\sum_{n=1}^{\infty}(\alpha\xi_n)^2 = \alpha^2 \sum_{n=1}^{\infty} \xi_n^2 < \infty,$$

so $x + y$ and αx belong to ℓ^2. Also, of course, the constant sequence 0 belongs to ℓ^2. Thus ℓ^2 is a linear subspace of $\mathbb{R}^{\mathbb{N}}$.

(b) For $x = (\xi_1, \dots) \in \ell^2$, set $\|x\|_2 = \sqrt{\sum_{i=1}^{\infty} \xi_i^2}$. Then $\| \|_2$ is a norm on ℓ^2. **P** **(i)** By the definition of ℓ^2, the sum $\sum_{i=1}^{\infty} \xi_i^2$ is finite, and of course it is non-negative; so $\|x\|_2$ is defined and non-negative. **(ii)** It is worth noting at this point that if $\|x\|_2 = 0$, where $x = (\xi_1, \dots)$, then $\sum_{i=1}^{\infty} \xi_i^2 = 0$ so every ξ_i is zero and $x = 0$. **(iii)** If $x = (\xi_1, \dots)$ and $y = (\eta_1, \dots)$ belong to ℓ^2, set $(x|y) = \sum_{i=1}^{\infty} \xi_i\eta_i$; this is well-defined because

$$\textstyle\sum_{i=1}^{\infty} |\xi_i\eta_i| \le \sum_{i=1}^{\infty} \xi_i^2 + \eta_i^2 = \sum_{i=1}^{\infty} \xi_i^2 + \sum_{i=1}^{\infty} \eta_i^2 < \infty,$$

so we have an absolutely summable series which must be summable. **(iv)** If $x = (\xi_1, \dots)$ and $y = (\eta_1, \dots)$ belong to ℓ^2, and α, $\beta \in \mathbb{R}$, then

$$\begin{aligned} 0 &\le \sum_{i=1}^{\infty}(\beta\xi_i - \alpha_i\eta_i)^2 \\ &= \sum_{i=1}^{\infty} \beta^2\xi_i^2 - 2\alpha\beta\xi_i\eta_i + \alpha_i^2\eta_i^2 \\ &= \beta^2 \sum_{i=1}^{\infty} \xi_i^2 - 2\alpha_i\beta_i \sum_{i=1}^{\infty} \xi_i\eta_i + \alpha_i^2 \sum_{i=1}^{\infty} \eta_i^2 \\ &= \beta^2\|x\|_2^2 - 2\alpha\beta(x|y) + \alpha^2\|y\|_2^2, \end{aligned}$$

so that

$$2\alpha\beta(x|y) \le \beta^2\|x\|_2^2 + \alpha^2\|y\|_2^2.$$

(v) Setting $\alpha = \|x\|_2$, $\beta = \|y\|_2$ then we get

$$2\alpha\beta(x|y) \le \beta^2\alpha^2 + \alpha^2\beta^2 = 2\alpha^2\beta^2;$$

so if α, β are both non-zero then

$$(x|y) \le \alpha\beta = \|x\|_2\|y\|_2.$$

On the other hand, if $\alpha = 0$ then $\xi_i = 0$ for every i and $(x|y) = 0$, while if $\beta = 0$ then $\eta_i = 0$ for every i and again $(x|y) = 0$. Thus we have

$$(x|y) \le \|x\|_2\|y\|_2$$

for all x, $y \in \ell^2$. **(vi)** If $x = (\xi_1, \dots)$ and $y = (\eta_1, \dots)$ belong to ℓ^2, then

$$\begin{aligned} \|x + y\|_2^2 &= \sum_{i=1}^{\infty}(\xi_i + \eta_i)^2 = \sum_{i=1}^{\infty} \xi_i^2 + 2\xi_i\eta_i + \eta_i^2 \\ &= \sum_{i=1}^{\infty} \xi_i^2 + 2\sum_{i=1}^{\infty} \xi_i\eta_i + \sum_{i=1}^{\infty} \eta_i^2 = \|x\|_2^2 + 2(x|y) + \|y\|_2^2 \\ &\le \|x\|_2^2 + 2\|x\|_2\|y\|_2 + \|y\|_2^2 = (\|x\|_2 + \|y\|_2)^2; \end{aligned}$$

taking square roots,

$$\|x + y\|_2 \le \|x\|_2 + \|y\|_2.$$

(vii) Finally, if $x = (\xi_1, \dots) \in \ell^2$ and $\alpha \in \mathbb{R}$, then

$$\|\alpha x\|_2 = \sqrt{\textstyle\sum_{i=1}^{\infty}(\alpha\xi_i^2)} = \sqrt{\alpha^2 \textstyle\sum_{i=1}^{\infty} \xi_i^2} = |\alpha|\sqrt{\textstyle\sum_{i=1}^{\infty} \xi_i^2} = |\alpha|\|x\|_2.$$

Thus $\| \|_2$ is a norm on ℓ^2. **Q**

(c) Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be a Cauchy sequence in ℓ^2. Express each x_n as $(\xi_1^{(n)}, \dots)$.

(i) For each $i \in \mathbb{N}$,

$$|\xi_i^{(m)} - \xi_i^{(n)}| \le \sqrt{\textstyle\sum_{j=1}^{\infty} |\xi_j^{(m)} - \xi_j^{(n)}|^2} = \|x_m - x_n\|_2 \to 0$$

as m, $n \to \infty$, that is, $\langle\xi_i^{(n)}\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Because $\mathbb{R}$ is complete, it has a limit $\xi_i \in \mathbb{R}$. Set

$$x = (\xi_1, \xi_2, \dots) \in \mathbb{R}^{\mathbb{N}}.$$

(ii) Cauchy sequences in normed spaces are bounded, so there is an $M \ge 0$ such that $\|x_n\|_2 \le M$ for every $n \in \mathbb{N}$. Now

$$\textstyle\sum_{i=1}^{k}(\xi_i^{(n)})^2 \le \sum_{i=1}^{\infty}(\xi_i^{(n)})^2 = \|x_n\|_2^2 \le M^2 \text{ for all } k,\ n \in \mathbb{N};$$

letting $n \to \infty$,

$$\textstyle\sum_{i=1}^{k} \xi_i^2 = \lim_{n\to\infty} \sum_{i=1}^{k}(\xi_i^{(n)})^2 \le M^2 \text{ for every } k \in \mathbb{N};$$

letting $k \to \infty$,

$$\textstyle\sum_{i=1}^{\infty} \xi_i^2 = \lim_{n\to\infty} \sum_{i=1}^{k} \xi_i^2 \le M^2 < \infty.$$

So $x \in \ell^2$.

(iii) Let $\epsilon > 0$. Then there is an $n_0 \in \mathbb{N}$ such that $\|x_m - x_n\|_2 \le \epsilon$ for all m, $n \ge n_0$. Accordingly

$$\textstyle\sum_{i=1}^{k}(\xi_i^{(m)} - \xi_i^{(n)})^2 \le \sum_{i=1}^{\infty}(\xi_i^{(m)} - \xi_i^{(n)})^2 = \|x_m - x_n\|_2^2 \le \epsilon^2 \text{ for all } k \in \mathbb{N},\ m,\ n \ge n_0;$$

letting $n \to \infty$,

$$\textstyle\sum_{i=1}^{k}(\xi_i^{(m)} - \xi_i)^2 = \lim_{n\to\infty} \sum_{i=1}^{k}(\xi_i^{(m)} - \xi_i^{(n)})^2 \le \epsilon^2 \text{ for every } k \in \mathbb{N},\ m \ge n_0;$$

letting $k \to \infty$,

$$\textstyle\|x_m - x\|_2^2 = \sum_{i=1}^{\infty}(\xi_i^{(m)} - \xi_i)^2 = \lim_{k\to\infty} \sum_{i=1}^{k}(\xi_i^{(m)} - \xi_i)^2 \le \epsilon^2 \text{ for every } m \ge n_0,$$

and

$$\|x_m - x\|_2 \le \epsilon \text{ for every } m \ge n_0.$$

As ϵ is arbitrary, $\langle x_n\rangle_{n\in\mathbb{N}}$ converges to x. As $\langle x_n\rangle_{n\in\mathbb{N}}$ is arbitrary, ℓ^2 is complete.

(d) For $x = (\xi_1, \xi_2, \dots) \in \ell^2$, $y = (\eta_1, \eta_2, \dots) \in \ell^2$ set $f_y(x) = \sum_{i=1}^{\infty} \xi_i\eta_i$. Then $y \mapsto f_y$ is a normed space isomorphism between ℓ^2 and $(\ell^2)^*$.

(i) The first thing to check is that $f_y(x)$ is always defined, that is, that $\langle\xi_i\eta_i\rangle_{i\in\mathbb{N}}$ is always summable. But this is done in (b-iii) above, because $f_y(x)$ is just what was there called $(x|y)$. Note that we saw also, in (b-v), that $|f_y(x)| = |(x|y)| \le \|x\|_2\|y\|_2$ for all x, $y \in \ell^2$.

(ii) Thus $f_y : \ell^2 \to \mathbb{R}$ is defined for each $y \in \ell^2$. Now each f_y is linear. **P** If x, $z \in \ell^2$ and $\alpha \in \mathbb{R}$, then

$$\begin{aligned} f_y(x + z) &= \sum_{i=1}^{\infty} \eta_i(\xi_i + \zeta_i) = \sum_{i=1}^{\infty} \eta_i\xi_i + \eta_i\zeta_i \\ &= \sum_{i=1}^{\infty} \eta_i\xi_i + \sum_{i=1}^{\infty} \eta_i\zeta_i = f_y(x) + f_y(z), \end{aligned}$$

$$\begin{aligned} f_y(\alpha x) &= \sum_{i=1}^{\infty} \eta_i(\alpha\xi_i) \\ &= \alpha\sum_{i=1}^{\infty} \eta_i\xi_i = \alpha f_y(x). \textbf{ Q} \end{aligned}$$

Also, because $|f_y(x)| \le \|y\|_2\|x\|_2$ for every x, f_y is a bounded operator, and $f_y \in (\ell^2)^*$, with $\|f_y\| \le \|y\|_2$.

(iii) The map $y \mapsto f_y : \ell^2 \to (\ell^2)^*$ is linear. **P** Let y, $z \in \ell^2$, $\alpha \in \mathbb{R}$. Then, for any $x \in \ell^2$,

$$\begin{aligned} f_{y+z}(x) &= \sum_{i=0}^{\infty}(\eta_i + \zeta_i)\xi_i = \sum_{i=0}^{\infty} \eta_i\xi_i + \sum_{i=0}^{\infty} \zeta_i\xi_i \\ &= f_y(x) + f_z(x) = (f_y + f_z)(x), \end{aligned}$$

$$\begin{aligned} f_{\alpha y}(x) &= \sum_{i=0}^{\infty}(\alpha\eta_i)\xi_i = \alpha\sum_{i=0}^{\infty} \eta_i\xi_i \\ &= \alpha f_y(x) = (\alpha f_y)(x). \end{aligned}$$

As x is arbitrary, $f_{y+z} = f_y + f_z$ and $f_{\alpha y} = \alpha f_y$; as y, z and α are arbitrary, $y \mapsto f_y$ is linear. **Q**

(iv) Now $\|f_y\| = \|y\|_2$ for every $y \in \ell^2$. **P** We already know that $\|f_y\| \le \|y\|_2$. If $y = 0$, then $f_y = 0$ so $\|f_y\| = 0 = \|y\|_2$. Otherwise, look at

$$\textstyle\|y\|_2^2 = \sum_{i=i}^{\infty} \eta_i^2 = f_y(y) \le \|f_y\|\|y\|_2.$$

Dividing both sides by $\|y\|_2$, $\|f_y\| \ge \|y\|_2$ and the two norms are equal. **Q**

Accordingly the map $y \mapsto f_y : \ell^2 \to (\ell^2)^*$ is a norm-preserving linear operator.

(v) Every $f \in (\ell^2)^*$ is expressible as f_y for some $y \in \ell^2$. **P** Set $d_n = (\delta_{n1}, \delta_{n2}, \dots)$ where $\delta_{ni} = 1$ if $i = n$, 0 otherwise. Then $d_i \in \ell^2$ for each i. Set $\eta_i = f(d_i)$ for each $i \in \mathbb{N}$. If $n \in \mathbb{N}$, set $\epsilon_j^{(n)} = \eta_j$ if $j \le n$, 0 otherwise; set $e^{(n)} = \langle\epsilon_j^{(n)}\rangle_{j\in\mathbb{N}}$; then

$$\textstyle\sum_{j=1}^{n} \eta_i^2 = f(e^{(n)}) \le \|f\|\|e^{(n)}\|_2 = \|f\|\sqrt{\sum_{j=1}^{n} \eta_i^2}.$$

But this means that $\sqrt{\sum_{j=1}^{n} \eta_j^2} \le \|f\|$. Taking the limit as $n \to \infty$,

$$\textstyle\|y\|_2^2 = \lim_{n\to\infty} \sum_{j=1}^{n} \eta_j^2 \le \|f\|^2 < \infty$$

and $y \in \ell^2$.

Consider $f_y \in (\ell^2)^*$. We have $f_y(d_i) = \eta_i = f(d_i)$ for every i. Now let $x = (\xi_1, \xi_2, \dots)$ be any member of ℓ^2. For each $n \in \mathbb{N}$, define $x^{(n)}$ by setting $x^{(n)} = \sum_{i=1}^{n} \xi_i d_i$, that is, $x^{(n)}$ is the sequence $(\xi_1, \dots, \xi_n, 0, 0, \dots)$. Then, because f and f_y are both linear,

$$\textstyle f(x^{(n)}) = \sum_{i=0}^{n} \xi_i f(d_i) = \sum_{i=0}^{n} \xi_i\eta_i = f_y(x^{(n)}).$$

And this is true for every n. On the other hand, for each n,

$$\textstyle\|x - x^{(n)}\|_2^2 = \sum_{i=n+1}^{\infty} \xi_i^2 \to 0$$

as $n \to \infty$. So $x = \lim_{n\to\infty} x^{(n)}$ in ℓ^2. Since f and f_y are both bounded linear operators, therefore continuous,

$$f(x) = \lim_{n\to\infty} f(x^{(n)}) = \lim_{n\to\infty} f_y(x^{(n)}) = f_y(x).$$

As x is arbitrary, $f = f_y$. **Q**

So the map $y \mapsto f_y : \ell^2 \to (\ell^2)^*$ is surjective and is a normed space isomorphism.

58. Set
$$\mathcal{G} = \{\left]-\infty, \sqrt{2}\right[\} \cup \{\left]\frac{1}{n} + \sqrt{2}, \infty\right[: n \in \mathbb{N}\}.$$

59. **(a) Uniform Boundedness Theorem** Let X be a Banach space, Y a normed space and $A \subseteq B(X;Y)$ a set such that $\{Tx : T \in A\}$ is bounded for every $x \in X$. Then A is bounded in $B(X;Y)$.

proof (i) For each $n \in \mathbb{N}$, set

$$F_n = \{x : x \in X,\ \|Tx\| \le n \text{ for every } T \in A\}.$$

Then F_n is a closed subset of X. **P** For each $T \in A$, the functions $x \mapsto Tx : X \to Y$ and $y \mapsto \|y\| : Y \to \mathbb{R}$ are continuous (see 11F, 8B), so their composition $x \mapsto \|Tx\| : X \to \mathbb{R}$ is continuous. Now $[-n, n]$ is a closed subset of $\mathbb{R}$, so its inverse image

$$F_{nT} = \{x : x \in X,\ \|Tx\| \in [-n, n]\}$$

must be closed in X. Consequently

$$F_n = \{x : x \in X,\ x \in F_{nT} \text{ for every } T \in A\}$$

is an intersection of closed sets, and therefore closed. **Q**

(ii) Now the hypothesis '$\{Tx : T \in A\}$ is bounded for every $x \in X$' means just that

for every $x \in X$ there is an $n \in \mathbb{N}$ such that $\|Tx\| \le n$ for every $T \in A$,

that is, that $\bigcup_{n\in\mathbb{N}} F_n = X$.

By Baire's theorem there is an $n \in \mathbb{N}$ such that F_n includes a non-trivial ball; that is, there are an $x_0 \in X$ and a $\delta > 0$ such that $N_\delta(x_0) \subseteq F_n$.

(iii) Now take any $T \in A$, $x \in X$ such that $\|x\| \le 1$. Then x_0 and $x_0 + \delta x$ both belong to $N_\delta(x_0) \subseteq F_n$. So

$$\begin{aligned}\|Tx\| &= \tfrac{1}{\delta}\|T(\delta x)\| = \tfrac{1}{\delta}\|T(x_0+\delta x) - Tx_0\| \\ &\le \tfrac{1}{\delta}(\|T(x_0+\delta x)\| + \|Tx_0\|) \le \tfrac{1}{\delta}(n+n) = \tfrac{2n}{\delta}.\end{aligned}$$

As x is arbitrary, $\|T\| \le 2n/\delta$. As T is arbitrary, A is bounded.

(b)(i) If X is a normed space and $A \subseteq X$ is such that $\{f(x) : x \in A\}$ is bounded for every $f \in X^*$, then A is bounded. **P** Look at $\{\hat{x} : x \in A\} \subseteq X^{**} = B(X^*;\mathbb{R})$. For any $f \in X^*$, $\{\hat{x}(f) : x \in A\} = \{f(x) : x \in A\}$ is bounded, and X^* is a Banach space, so $\{\hat{x} : x \in A\}$ is bounded. But $\|\hat{x}\| = \|x\|$ for every x, so A must be bounded. **Q**

(ii) If X is a Banach space and $A \subseteq X^*$ is such that $\{f(x) : f \in A\}$ is bounded for every $x \in X$, then A is bounded. (This is just the case $Y = \mathbb{R}$ of the Uniform Boundedness Theorem.)

(c) Set $X = C([0,1])$ with $\|\,\|_1$. For each $n \in \mathbb{N}$, set $f_n(x) = n(\int_0^{1/2n} x(t)dt - \int_{1/2n}^{1/n} x(t)dt)$ for each $x \in C([0,1])$. Then f_n is a linear operator (because integration is a linear functional) and is bounded (because $|f_n(x)| \le n\|x\|_1$ for every x). In fact $\|f_n\| = n$. **P** Set $x_n(t) = \sin 2n\pi t$ if $0 \le \frac{1}{2n}$, 0 otherwise; then $f_n(x_n) = n\int_0^{1/2n} x_n(t)dt = n\|x_n\|_1$. **Q**

However, $\lim_{n\to\infty} f_n(x) = 0$ for every $x \in C([0,1])$. **P** Let $\epsilon > 0$. Set $\alpha = x(0)$. Then there is an $n_0 \in \mathbb{N}$ such that $|x(t) - \alpha| \le \epsilon$ whenever $0 \le t \le \frac{1}{n_0}$. If $n \ge n_0$, then

$$\begin{aligned}|f_n(x)| &= n\Big|\int_0^{1/2n}(x(t)-\alpha)dt - \int_{1/2n}^{1/n}(x(t)-\alpha)dt\Big| \\ &= n\Big|\int_0^{1/2n}(x(t)-\alpha)dt + \int_{1/2n}^{1/n}(\alpha - x(t))dt\Big| \\ &\le n\int_0^{1/n}|x(t)-\alpha|dt \le n\int_0^{1/n}\epsilon dt = \epsilon.\end{aligned}$$

As ϵ is arbitrary, $\lim_{n\to 0} f_n(x) = 0$. **Q**

remark This shows that $C([0,1])$, under $\|\,\|_1$, cannot be a Banach space.

60. (a) We define the linear structure on $X \times Y$ by setting

$$(x, y) + (x', y') = (x + x', y + y'), \quad \alpha(x, y) = (\alpha x, \alpha y);$$

the zero of $X \times Y$ is $(0, 0)$ and the additive inverse of (x, y) is $(-x, -y)$.

A standard norm on $X \times Y$ is given by

$$\|(x, y)\| = \max(\|x\|, \|y\|).$$

(b) If $T : X \to Y$ is a linear operator, then for any two points (x, Tx) and (x', Tx') of its graph Γ, and any $\alpha \in \mathbb{R}$, we have

$$(x, Tx) + (x', Tx') = (x + x', Tx + Tx') = (x + x', T(x + x')) \in \Gamma,$$

$$\alpha(x, Tx) = (\alpha x, \alpha Tx) = (\alpha x, T(\alpha x)) \in \Gamma.$$

So Γ is a linear subspace.

(c) If T is continuous, it is a bounded operator; say $\|Tx\| \le M\|x\|$ for any $x \in X$. Suppose that $(x, y) \in \overline{\Gamma}$. Then for any $\delta > 0$ there is an point (x', Tx') of Γ such that $\|(x, y) - (x', Tx')\| \le \delta$, that is, $\|x - x'\| \le \delta$ and $\|y - Tx'\| \le \delta$. But now

$$\|y - Tx\| \le \|y - Tx'\| + \|Tx' - Tx\| \le \delta + M\|x' - x\| \le \delta(1 + M).$$

As δ is arbitrary, $\|y - Tx\| = 0$ and $(x, y) = (x, Tx) \in \Gamma$. As (x, y) is arbitrary, Γ is closed.

(d) Try $X = C([0, 1])$, with $\| \|_1$, and $Y = \mathbb{R}$; set $T = f(1)$. If $f_n(t) = t^n$ for $t \in [0, 1]$, $n \in \mathbb{N}$, then $\langle f_n \rangle_{n \in \mathbb{N}} \to \mathbf{0}$, the zero of X, for $\| \|_1$, while $(f_n, 1) \in \Gamma$, the graph of T, for every $n \in \mathbb{N}$. Accordingly $(\mathbf{0}, 1)$ belongs to the closure of Γ; but it certainly does not belong to Γ, so Γ is not closed.

61. (Not available.)

62. (Not available.)

63. Let $K \subseteq \mathbb{R}^n$ be a bounded set. For each $k \in \mathbb{N}$ set $G_k = U_k(0) = \{x : \rho_2(x, 0) < k\}$; then every G_k is open, and $K \subseteq \mathbb{R}^n = \bigcup_{k \in \mathbb{N}} G_k$. So there is a finite subcover $\{G_{k_0}, \ldots, G_{k_m}\}$ of K. Set $k = \max(k_0, \ldots, k_m)$; then $K \subseteq G_k$ and $|\xi_i| \le \rho_2(x, 0) \le k$ whenever $x = (\xi_1, \ldots, \xi_n) \in K$ and $i \le n$. Thus K is bounded.

64. (a) $\rho(x, A) = 0$ iff for every $\delta > 0$ there is a $y \in A$ such that $\rho(x, y) \leq \delta$, that is, iff $A \cap N_\delta(x) \neq \emptyset$ for every $\delta > 0$, where $N_\delta(x)$ is the closed ball with centre x and radius δ; but this is the standard criterion for x to belong to $\overline{A}$.

(b) For any $\epsilon > 0$, there is a $z \in A$ such that $\rho(y, z) \leq \rho(y, A) + \epsilon$. Now

$$\rho(x, A) \leq \rho(x, z) \leq \rho(x, y) + \rho(y, z) \leq \rho(x, y) + \rho(y, A) + \epsilon.$$

As ϵ is arbitrary,

$$\rho(x, A) \leq \rho(x, y) + \rho(y, A), \text{ i.e., } \rho(x, A) - \rho(y, A) \leq \rho(x, y).$$

Similarly,

$$\rho(y, A) - \rho(x, A) \leq \rho(y, x) = \rho(x, y),$$

so $|\rho(x, A) - \rho(y, A)| \leq \epsilon$.

(c) Set $f(x) = \rho(x, F)$ for $x \in X$. Then, by (b),

$$|f(x) - f(y)| \leq \rho(x, y) \text{ for all } x, y \in X,$$

so f is continuous. If $x \in K$, then $x \notin F = \overline{F}$ (because F is closed), so $f(x) > 0$ (by (a)). Now, because K is compact and f is continuous, there is an $x_0 \in K$ such that $f(x) \geq f(x_0)$ for every $x \in K$. So if $x \in K$, $y \in F$,

$$\rho(x, y) \geq \rho(x, F) = f(x) \geq f(x_0).$$

As x and y are arbitrary, $\rho(, F) \geq f(x_0) > 0$.

(d) Try

$$E = \{(x, y) : xy = 0\}, \; F = \{(x, y) : xy = 1\}.$$

65. **(i)⇒(ii)** Suppose that f is continuous at x^* and that $\langle x_n \rangle_{n \in \mathbb{N}} \to x^*$. Let $\epsilon > 0$. Then there is a $\delta > 0$ such that $\sigma(f(x), f(x^*)) \leq \epsilon$ whenever $\rho(x, x^*) \leq \delta$. Now there is an $n_0 \in \mathbb{N}$ such that $\rho(x_n, x^*) \leq \delta$ whenever $n \geq n_0$. Now if $n \geq n_0$, $\rho(x_n, x^*) \leq \delta$ so $\sigma(f(x_n), f(x^*)) \leq \epsilon$. As ϵ is arbitrary, $\langle f(x_n) \rangle_{n \in \mathbb{N}} \to f(x^*)$.

(ii)⇒(iii) Suppose that (ii) is true, and that $A \subseteq X$, $x^* \in \overline{A}$. Then there is a sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in A converging to x^*. Now $\langle f(x_n) \rangle_{n \in \mathbb{N}}$ is a sequence in $f[A]$ converging to $f(x^*)$, so $f(x^*) \in \overline{f[A]}$.

(iii)⇒(i) Assume (iii), and let $\epsilon > 0$. Consider $A = X \setminus f^{-1}[N_\epsilon(f(x^*))]$, where $N_\epsilon(f(x^*))$ is the closed ball of radius ϵ and centre $f(x^*)$. Then $f[A] \cap N_\epsilon(f(x^*)) = \emptyset$, so $f(x^*) \notin \overline{f[A]}$. From (iii) it follows that $x^* \notin \overline{A}$. But this means that there is a $\delta > 0$ such that $N_\delta(x) \cap A = \emptyset$. Now suppose that $\rho(x, x^*) \leq \delta$. Then $x \notin A$ so $x \in f^{-1}[N_\epsilon(f(x^*))]$, that is, $f(x) \in N_\epsilon(f(x^*))$, that is, $\sigma(f(x), f(x^*)) \leq \epsilon$. As ϵ is arbitrary, f is continuous at x_0.

66. (a) Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be a convergent sequence in X with limit x. Then for any $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $\rho(x_n, x) \le \frac{1}{2}\epsilon$ for every $n \ge n_0$. If now $m, n \ge n_0$,

$$\rho(x_m, x_n) \le \rho(x_n, x) + \rho(x, x_m) \le \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.$$

As ϵ is arbitrary, $\langle x_n\rangle_{n\in\mathbb{N}}$ is Cauchy.

(b) Take any $x \in \overline{A}$. Then there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in A converging to x for ρ. By (a), $\langle x_n\rangle_{n\in\mathbb{N}}$ is ρ-Cauchy, that is, $\lim_{m,n\to\infty} \rho(x_m, x_n) = 0$. But as every x_n belongs to A, $\lim_{m,n\to\infty} \rho_A(x_m, x_n) = 0$, and $\langle x_n\rangle_{n\in\mathbb{N}}$ is ρ_A-Cauchy. There is therefore some $x^* \in A$ such that x^* is the ρ_A-limit of $\langle x_n\rangle_{n\in\mathbb{N}}$. But now $\lim_{n\to\infty} \rho(x^*, x_n) = \lim_{n\to\infty} \rho_A(x^*, x_n) = 0$ and x^* is the ρ-limit of $\langle x_n\rangle_{n\in\mathbb{N}}$, that is, $x^* = x$. So $x = x^* \in A$. As x is arbitrary, $\overline{A} = A$ and A is closed.

(c) Take any ρ_F-Cauchy sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in F. Then

$$\lim_{m,n\to\infty} \rho(x_m, x_n) = \lim_{m,n\to\infty} \rho_F(x_m, x_n) = 0$$

so $\langle x_n\rangle_{n\in\mathbb{N}}$ is ρ-Cauchy and has a ρ-limit $x \in X$. Because x is the limit of a sequence in F, it belongs to $\overline{F} = F$. Now $\lim_{n\to\infty} \rho_F(x_n, x) = \lim_{n\to\infty} \rho(x, x_n) = 0$ and x is the ρ_F-limit of $\langle x_n\rangle_{n\in\mathbb{N}}$. As $\langle x_n\rangle_{n\in\mathbb{N}}$ is arbitrary, F is ρ_F-complete.

67. (a)(i) Of course ρ_2 takes values in $[0,\infty[$. (ii) $\rho_2(m,m) = |\frac{1}{m} - \frac{1}{m}| = 0$. (iii) $\rho_2(n,m) = |\frac{1}{n} - \frac{1}{m}| = |\frac{1}{m} - \frac{1}{n}| = \rho_2(m,n)$. (iv) $\rho_2(m,p) = |\frac{1}{m} - \frac{1}{p}| \le |\frac{1}{m} - \frac{1}{n}| + |\frac{1}{n} - \frac{1}{p}| = \rho_2(m,n) + \rho_2(n,p)$. (v) If $\rho_2(m,n) = 0$ then $|\frac{1}{m} - \frac{1}{n}| = 0$ so $\frac{1}{m} = \frac{1}{n}$ and $m = n$.

Let $n_0 \in \mathbb{N}$, $\epsilon > 0$. Set

$$\delta = \frac{1}{2n_0(n_0+1)} > 0.$$

If $\rho_1(n, n_0) \le \delta$, then (because $\delta < 1$) $n = n_0$ and $\rho_2(n, n_0) = 0 \le \epsilon$. If $\rho_2(n, n_0) \le \delta$, then

$$|\frac{1}{n} - \frac{1}{n_0}| \le \delta < |\frac{1}{n_0} - \frac{1}{n_0+1}| < |\frac{1}{n_0} - \frac{1}{n_0+1}|,$$

so again $n = n_0$ and $\rho_1(n, n_0) = 0 \le \epsilon$. As n_0, ϵ are arbitrary, ρ_1 and ρ_2 are equivalent.

(b) A sequence in $\mathbb{N}$ is Cauchy for ρ_1 iff it is eventually constant. It is Cauchy for ρ_2 iff *either* it is eventually constant *or* is diverges to ∞.

(c) Under ρ_1, $\mathbb{N}$ is complete, not totally bounded, not compact. Under ρ_2 it is totally bounded, not complete, not compact.

68. (i)⇒(ii) Let $\langle x_{r_k}\rangle_{k\in\mathbb{N}}$ be a subsequence converging to x. Then for any $n \in \mathbb{N}$, $\langle x_{r_{n+k}}\rangle_{k\in\mathbb{N}}$ is a sequence in $A_n = \{x_i : i \ge n\}$ converging to x, so $x \in \overline{A_n}$.

(ii)⇒(i) Suppose that $x \in \overline{A_n}$ for every n. Choose $\langle r_k\rangle_{k\in\mathbb{N}}$ inductively, as follows. Because $x \in \overline{A_1}$, there is an $r_1 \in \mathbb{N}$ such that $\rho(x, x_{r_1}) \le 1$. Given r_k, then, because $x \in \overline{A_{r_k+1}}$, there is an $r_{k+1} \ge r_k + 1$ such that $\rho(x, x_{r_{k+1}}) \le \frac{1}{k+1}$. In this way we choose a strictly increasing sequence $\langle r_k\rangle_{k\in\mathbb{N}}$ such that $\rho(x, x_{r_k}) \le \frac{1}{k}$ for every k, so that $\langle x_{r_k}\rangle_{k\in\mathbb{N}} \to x$.

69. Set $A_n = \{x_i : i \ge n\}$ for each n. By the last question, x is the limit of a subsequence of $\langle x_n\rangle_{n\in\mathbb{N}}$ iff it belongs to $F = \bigcap_{n\in\mathbb{N}} \overline{A_n}$. But F is the intersection of a family of closed sets, so is closed.

70. Let $\epsilon > 0$. Then there is an $n_0 \in \mathbb{N}$ such that $\rho(x_m, x_n) \le \frac{1}{2}\epsilon$ whenever $m, n \ge n_0$. Now x belongs to the closure of $\{x_i : i \ge n_0\}$, so there is an $n \ge n_0$ such that $\rho(x, x_n) \le \frac{1}{2}\epsilon$. If $m \ge n_0$, then

$$\rho(x, x_m) \le \rho(x, x_n) + \rho(x_n, x_m) \le \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.$$

As ϵ is arbitrary, $x = \lim_{n\to\infty} x_n$.

71. Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a Cauchy sequence in $C([0,1])$. For any $t\in[0,1]$, $|f_n(t)-f_m(t)|\le\rho_\infty(f_n,f_m)$ for every m, $n\in\mathbb{N}$, so $\langle f_n(t)\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. By the General Principle of Convergence, it has a limit $f(t)\in\mathbb{R}$. This defines a function $f:[0,1]\to\mathbb{R}$.

Now $\langle f_n\rangle_{n\in\mathbb{N}}$ converges uniformly to f. **P** Let $\epsilon>0$. Then there is an $n_0\in\mathbb{N}$ such that $\rho_\infty(f_m,f_n)\le\epsilon$ for all m, $n\ge n_0$, that is, $|f_m(t)-f_n(t)|\le\epsilon$ for all m, $n\ge n_0$ and every $t\in[0,1]$. Letting $m\to\infty$, we see that $|f(t)-f_n(t)|\le\epsilon$ for every $n\ge n_0$, $t\in[0,1]$. As ϵ is arbitrary, $\langle f_n\rangle_{n\in\mathbb{N}}$ converges uniformly to f. **Q**

Consequently f is a uniform limit of continuous functions and is continuous.

Finally, f is the limit of $\langle f_n\rangle_{n\in\mathbb{N}}$ for ρ_∞. **P** Let $\epsilon>0$. Then there is an $n_0\in\mathbb{N}$ such that $\rho_\infty(f_m,f_n)\le\epsilon$ for all m, $n\ge n_0$, that is, $|f_m(t)-f_n(t)|\le\epsilon$ for all m, $n\ge n_0$ and every $t\in[0,1]$. Letting $m\to\infty$, we see that $|f(t)-f_n(t)|\le\epsilon$ for every $n\ge n_0$, $t\in[0,1]$, that is, $\rho_\infty(f,f_n)\le\epsilon$ for every $n\ge n_0$. As ϵ is arbitrary, $\langle f_n\rangle_{n\in\mathbb{N}}$ converges to f for ρ_∞. **Q**

As $\langle f_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $C([0,1])$ is complete.

72. (a) Take any $x\in\overline{K+F}$. Then there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in $K+F$ converging to x. Express $\langle x_n\rangle_{n\in\mathbb{N}}$ as $\langle y_n+z_n\rangle_{n\in\mathbb{N}}$, where $y_n\in K$ and $z_n\in F$ for every n. Because K is compact, $\langle y_n\rangle_{n\in\mathbb{N}}$ has a convergent subsequence $\langle y_{n_k}\rangle_{k\in\mathbb{N}}$ say with limit $y\in K$. Now

$$z_{n_k}=x_{n_k}-y_{n_k}\to x-y.$$

Because F is closed, $x-y\in F$ and $x=y+(x-y)\in K+F$. As x is arbitrary, $K+F$ is closed.

(b) By (a), $K+L$ is closed. Also K and L are both bounded, so $K+L$ is bounded, therefore compact.

(c) Set $E=\mathbb{N}$, $F=\{2^{-n}-n:n\in\mathbb{N}\}$. Then $0=\lim_{n\to\infty}n+(2^{-n}-n)\in\overline{E+F}$, but $0\notin E+F$ because no member of F is an integer.

73. (Not available.)

74. (Not available.)

75. (Not available.)