**6G** Baire's theorem is one of the foundation stones of twentieth-century analysis; its consequences pervade the subject. I give one as a sample.

**Lemma** Let $(X,\rho)$ be a complete metric space, and $\langle F_n\rangle_{n\in\mathbb{N}}$ a sequence of closed subsets of $X$ covering $X$. Set

$$H_n = \bigcup\{G : G \subseteq X \text{ is open}, G \subseteq F_n\}$$

for each $n$, and

$$H = \bigcup_{n\in\mathbb{N}} H_n.$$

Then $H$ and all the $H_n$ are open and $H$ is dense in $X$.

**proof** Each $H_n$ is a union of open sets, therefore open, so $H$ also must be open. For each $n \in \mathbb{N}$ set

$$G_n = H_n \cup (X \setminus F_n).$$

$H_n$ is a union of open sets so must be open, and $X \setminus F_n$ is open because $F_n$ is closed, so $G_n$ is open. Now $G_n$ is dense. **P** If $x \in X$, $\delta > 0$ then either $U_\delta(x) \subseteq F_n$, in which case

$$x \in U_\delta(x) \subseteq H_n \subseteq G_n,$$

or $U_\delta(x) \cap (X \setminus F_n) \neq \emptyset$, in which case $U_\delta(x) \cap G_n \neq \emptyset$. Thus in both cases $U_\delta(x)$ meets $G_n$. As $\delta$ is arbitrary, $x \in \overline{G}_n$. As $x$ is arbitrary, $G_n$ is dense. **Q**

By Baire's theorem, $E = \bigcap_{n\in\mathbb{N}} G_n$ is dense. Now $E \subseteq H$. **P** Take $x \in E$. Then there is a $n \in \mathbb{N}$ such that $x \in F_n$. Since $x \in G_n$, we must have $x \in H_n$; but of course $H_n \subseteq H$, so $x \in H$. **Q**

So $\overline{H} \supseteq \overline{E} = X$ and $H$ is dense, as claimed.

**6H Theorem** Let $(X,\rho)$ be a complete metric space, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of continuous real-valued functions on $X$ such that $f(x) = \lim_{n\to\infty} f_n(x)$ is defined in $\mathbb{R}$ for every $x \in X$. Then $H = \{x : x \in X, f \text{ is continuous at } x\}$ is dense in $X$.

**proof** For $k$, $m$, $n \in \mathbb{N}$ set

$$F_{kmn} = \{x : x \in X, |f_m(x) - f_n(x)| \le \tfrac{1}{k}\}.$$

Then $F_{kmn}$ is closed because $f_m$, $f_n$ are continuous, so $x \mapsto |f_m(x) - f_n(x)|$ is continuous. For $k$, $m \in \mathbb{N}$ set

$$F_{km} = \bigcap_{n\ge m} F_{kmn};$$

then $F_{km}$ is an intersection of closed sets, so is closed. Note that $\bigcup_{m\in\mathbb{N}} F_{km} = X$ for each $k$. **P** If $x \in X$, there is an $m$ such that $|f_n(x) - f(x)| \le \frac{1}{2k}$ for every $n \ge m$; now if $n \ge m$,

$$|f_m(x) - f_n(x)| \le |f_m(x) - f(x)| + |f_n(x) - f(x)| \le \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k},$$

so $x \in F_{kmn}$. As $n$ is arbitrary, $x \in F_{km}$. As $x$ is arbitrary, $\bigcup_{m\in\mathbb{N}} F_{km} = X$. **Q**

Set

$$H_{km} = \bigcup\{G : G \subseteq X \text{ is open}, G \subseteq F_{km}\}.$$

Of course $H_{km} \subseteq F_{km}$. By 6G,

$$G_k = \bigcup_{m\in\mathbb{N}} H_{km}$$

is open and dense in $X$. By Baire's theorem,

$$E = \bigcap_{k\in\mathbb{N}} G_k$$

is dense.

Now $E \subseteq H$. **P** Take $x_0 \in E$ and $\epsilon > 0$. Then there is a $k \in \mathbb{N}$ such that $\frac{1}{k} \le \frac{\epsilon}{3}$. We know that $x_0 \in G_k$, so there is an $m \in \mathbb{N}$ such that $x_0 \in H_{km}$. Let $\delta_0 > 0$ be such that $N_{\delta_0}(x_0) \subseteq H_{km}$. Note that if $\rho(x, x_0) \le \delta_1$, then $x \in H_{km} \subseteq F_{km}$, so $|f_n(x) - f_m(x)| \le \frac{1}{k}$ for every $n \ge m$, and $|f(x) - f_m(x)| \le \frac{1}{k}$. In particular, $|f(x_0) - f_m(x_0)| \le \frac{1}{k}$.

Let $\delta_1 > 0$ be such that $|f_m(x) - f_m(x_0)| \leq \frac{\epsilon}{3}$ whenever $x \in N_{\delta_1}(x_0)$. Set $\delta = \min(\delta_1, \delta_2) > 0$. If $\rho(x, x_0) \leq \delta$, then, putting the facts above together,

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_m(x_0)| + |f_m(x) - f_m(x_0)| + |f_m(x_0) - f(x_0)| \\ &\leq \frac{1}{k} + \frac{\epsilon}{3} + \frac{1}{k} \\ &\leq \epsilon. \end{aligned}$$

As $\epsilon$ is arbitrary, $f$ is continuous at $x_0$. As $x_0$ is arbitrary, $E \subseteq H$. **Q**

Consequently $\overline{H} \supseteq \overline{E} = X$ and $H$ is dense, as claimed.

Version of 22.3.94

## 7. Normed spaces

**7A Linear spaces** Recall that a **linear space over** $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$ is a set $X$ together with operations $+ : X \times X \to X$, $- : X \to X$, $. : \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}} \times X \to X$ and a distinguished element $0$ of $X$ such that

(a) $(X, +, 0, -)$ is a commutative group, that is,
$(x + y) + z = x + (y + z)$ *associative law*
$x + 0 = 0 + x = x$ *identity*
$x + (-x) = (-x) + x = 0$ *inverse*
$x + y = y + x$ *commutative*
for all $x$, $y$, $z \in X$.

(b) $\alpha(x + y) = \alpha x + \alpha y$ *distributive law*
$(\alpha + \beta)x = \alpha x + \beta x$ *distributive law*
$\alpha(\beta x) = (\alpha\beta)x$
$1x = x$
for all $\alpha$, $\beta \in \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, $x$, $y \in X$.

**7B Useful facts** If $(X, +, \mathbf{0}, -, .)$ is a linear space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, then
(a) $0x = \mathbf{0}$ for all $x \in X$,
(b) $(-1)x = -x$ for all $x \in X$.

**proof (a)** Consider

$$0x + 0x = (0 + 0)x = 0x.$$

Adding $-(0x)$ to both sides we get $0x = \mathbf{0}$.

**(b)** Now

$$\mathbf{0} = 0x = (1 + (-1))x = 1x + (-1)x = x + (-1)x;$$

adding $-x$ to both sides we get $-x = (-1)x$.

**7C Examples (a)** For any $n \in \mathbb{N}$, let $\mathbb{R}^n$ be the set of vectors $x = (\xi_1, \dots, \xi_n)$ with real coordinates; this is a linear space over $\mathbb{R}$ if we set

$$(\xi_1, \dots, \xi_n) + (\eta_1, \dots, \eta_n) = (\xi_1 + \eta_1, \dots, \xi_n + \eta_n),$$

$$\mathbf{0} = (0, \dots, 0),$$

$$-(\xi_1, \dots, \xi_n) = (-\xi_1, \dots, -\xi_n),$$

$$\alpha(\xi_1, \dots, \xi_n) = (\alpha\xi_1, \dots, \alpha\xi_n).$$

**(b)** Let $\mathbb{R}^{\mathbb{N}}$ be the set of all real sequences $x = (\xi_1, \xi_2, \dots)$; this is a linear space if we set

$$(\xi_1, \xi_2, \dots) + (\eta_1, \eta_2, \dots) = (\xi_1 + \eta_1, \xi_2 + \eta_2, \dots,$$

$$\mathbf{0} = (0, 0, \dots),$$

$$-(\xi_1,\xi_2,\dots)=(-\xi_1,-\xi_2,\dots),$$

$$\alpha(\xi_1,\xi_2,\dots)=(\alpha\xi_1,\alpha\xi_2,\dots).$$

**(c)** Both of these can be thought of as special cases of the following. If $I$ is any set, write $\mathbb{R}^I$ for the set of all functions $x:I\to\mathbb{R}$. Then this is a linear space over $\mathbb{R}$ if we write

$$(x+y)(i)=x(i)+y(i),$$

$$\mathbf{0}(i)=0,$$

$$(-x)(i)=-x(i),$$

$$(\alpha x)(i)=\alpha x(i)$$

for $x$, $y\in\mathbb{R}^I$, $\alpha\in\mathbb{R}$ and $i\in I$.

**7D Linear subspaces** The following construction is a very general method of finding new linear spaces. If $(X,+,0,-,.)$ is a linear space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, a set $Y\subseteq X$ is a **linear subspace** if

(i) $x+y\in Y$ for all $x$, $y\in Y$,

(ii) $0\in Y$,

(iii) $\alpha x\in Y$ for all $x\in Y$, $\alpha\in\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$.

The reason why this concept is important is that $(Y,+\restriction Y\times Y,0,-\restriction Y,.\restriction\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}\times Y)$ is now a linear space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$. **P** The condition (i) ensures that $+\restriction Y\times Y$ is a binary operation on $Y$; the condition (ii) ensures that $0\in Y$; the condition (iii) ensures that $-y=(-1)y\in Y$ for every $y\in Y$, so that $-\restriction Y$ is an operation on $Y$; and condition (iii) also ensures that $.\restriction\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}\times Y$ is a function from $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}\times Y$ to $Y$.

Now of course the required properties of these operations on $Y$ follow directly from the corresponding properties of the operations on $X$. **Q**

**7E More examples (a)** Let $\ell^1$ be the set of absolutely summable real sequences $x\in\mathbb{R}^{\mathbb{N}}$. Then $\ell^1$ is a linear subspace of $\mathbb{R}^{\mathbb{N}}$, therefore a linear space in its own right. **P** If $x=(\xi_1,\dots)$ and $y=(\eta_1,\dots)$ belong to $\ell^1$ and $\alpha\in\mathbb{R}$, then

$$\textstyle\sum_{n=1}^\infty|\xi_n+\eta_n|\le\sum_{n=1}^\infty|\xi_n|+|\eta_n|=\sum_{n=1}^\infty|\xi_n|+\sum_{n=1}^\infty|\eta_n|<\infty,$$

$$\textstyle\sum_{n=1}^\infty|\alpha\xi_n|=\sum_{n=1}^\infty|\alpha||\xi_n|=|\alpha|\sum_{n=1}^\infty|\xi_n|<\infty,$$

so $x+y$ and $\alpha x$ belong to $\ell^1$. And of course the constant sequence 0 belongs to $\ell^1$. **Q**

**(b)** Let $\ell^2$ be the set of square-summable real sequences, that is, the set of sequences $x=(\xi_1,\dots)$ such that $\sum_{n=0}^\infty\xi_n^2<\infty$. Then $\ell^2$ is a linear subspace of $\mathbb{R}^{\mathbb{N}}$. **P** If $x=(\xi_1,\dots)$ and $y=(\eta_1,\dots)$ belong to $\ell^2$, and $\alpha\in\mathbb{R}$, then

$$(\xi_n+\eta_n)^2\le(|\xi_n|+|\eta_n|)^2\le(2\max(|\xi_n|,|\eta_n|)^2=4\max(|\xi_n|^2,|\eta_n|^2)\le4(\xi_n^2+\eta_n^2)$$

for every $n$, so

$$\textstyle\sum_{n=1}^\infty(\xi_n+\eta_n)^2\le\sum_{n=1}^\infty4(\xi_n^2+\eta_n^2)=4\sum_{n=1}^\infty\xi_n^2+4\sum_{n=1}^\infty\eta_n^2<\infty,$$

while also

$$\textstyle\sum_{n=1}^\infty(\alpha\xi_n)^2=\alpha^2\sum_{n=1}^\infty\xi_n^2<\infty,$$

so $x+y$ and $\alpha x$ belong to $\ell^2$. Also, of course, the constant sequence 0 belongs to $\ell^2$. **Q**

**(c)** For any set $I$, write $\ell^\infty(I)$ for the set of bounded functions from $I$ to $\mathbb{R}$. Then $\ell^\infty(I)$ is a linear subspace of $\mathbb{R}^I$. **P** Take $x$, $y\in\ell^\infty(I)$ and $\alpha\in\mathbb{R}$. Then there are $M$, $M'\ge0$ such that $|x(i)|\le M$, $|y(i)|\le M'$ for every $i\in I$. Now

$$|(x+y)(i)|=|x(i)+y(i)|\le|x(i)|+|y(i)|\le M+M',$$

$$|(\alpha x)(i)|=|\alpha||x(i)|\le|\alpha|M$$

for every $i \in I$, so $x+y$, $\alpha x$ are bounded functions and belong to $\ell^\infty$. Also, of course, the constant function 0 is bounded. **Q**

**(d)** A special case of (c) is when $I = \mathbb{N}$; in this case I will usually write $\ell^\infty$ for $\ell^\infty(\mathbb{N})$, the linear space of all bounded real sequences.

**(e)** If $(X, \rho)$ is any metric space, then $C(X)$, the space of all continuous real-valued functions on $X$, is a linear subspace of $\mathbb{R}^X$; this is because $f+g$ and $\alpha f$ are continuous functions whenever $f$ and $g$ are and $\alpha \in \mathbb{R}$, and the constant function with value 0 is continuous.

**(f)** If $(X, \rho)$ is any metric space, then $C_b(X)$, the space of all bounded continuous real-valued functions on $X$, is a linear subspace of $\mathbb{R}^X$ (and also of $C(X)$ and of $\ell^\infty(X)$.)

(Generally it is the case that the intersection of linear subspaces is a linear subspace, and here we have $C_b(X) = C(X) \cap \ell^\infty(X)$.)

**(g)** The set $\boldsymbol{c}_0$ of sequences convergent to 0 is a linear subspace of $\ell^\infty = \ell^\infty(\mathbb{N})$. **P** Every convergent sequence in $\mathbb{R}$ is bounded, so $\boldsymbol{c}_0$ is a subset of $\ell^\infty$. If $x = (\xi_1, \dots)$ and $y = (\eta_1, \dots)$ belong to $\boldsymbol{c}_0$, then

$$\lim_{n\to\infty} \xi_n + \eta_n = \lim_{n\to\infty} \xi_n + \lim_{n\to\infty} \eta_n = 0 + 0 = 0,$$

$$\lim_{n\to\infty} \alpha\xi_n = \alpha \lim_{n\to\infty} \xi_n = \alpha 0 = 0,$$

so $x+y$, $\alpha x \in \boldsymbol{c}_0$. And of course the zero sequence is convergent to 0 and belongs to $\boldsymbol{c}_0$. **Q**

**7F Norms** Let $X$ be a linear space over $\substack{\mathbb{R}\\ \mathbb{C}}$. A **norm** on $X$ is a functional $\| \; \| : X \to [0, \infty[$ such that

$\|x+y\| \le \|x\| + \|y\|$ for all $x, y \in X$,

$\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$, $\alpha \in \substack{\mathbb{R}\\ \mathbb{C}}$,

$\|x\| = 0 \Rightarrow x = \mathbf{0}$.

**7G Useful facts (a)** If $\| \; \|$ is a norm on $X$, then

$$\|\mathbf{0}\| = \|0\mathbf{0}\| = |0|\|\mathbf{0}\| = 0,$$

so that we have $\|x\| = 0 \iff x = \mathbf{0}$.

**(b)** If $X$ is a linear space and $\| \; \| : X \to [0, \infty[$ is a functional such that $\|\alpha x\| \le |\alpha|\|x\|$ for all $x \in X$, $\alpha \in \substack{\mathbb{R}\\ \mathbb{C}}$ then $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$, $\alpha \in \substack{\mathbb{R}\\ \mathbb{C}}$. **P** If $\alpha = 0$ this is trivial, because we must have

$$0 \le \|0x\| \le |0|\|x\| = 0.$$

If $\alpha \ne 0$ then we have

$$\|\alpha x\| \le |\alpha|\|x\| = |\alpha|\|1x\| = |\alpha|\|(\alpha^{-1}\alpha)x\| = |\alpha|\|\alpha^{-1}(\alpha x)\| \le |\alpha||\alpha^{-1}|\|\alpha x\| = \|\alpha x\|. \textbf{ Q}$$

**7H Examples (a)** On $\mathbb{R}^n$ set

$$\|x\|_1 = \sum_{i=1}^n |\xi_i|$$

if $x = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. Then $\| \; \|_1$ is a norm on $\mathbb{R}^n$. **P** Of course $\|x\|_1$, being a sum of finitely many non-negative terms, is a non-negative real number. If $x = (\xi_1, \dots, \xi_n)$, $y = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ then

$$\|x+y\|_1 = \sum_{i=1}^n |\xi_i + \eta_i| \le \sum_{i=1}^n |\xi_i| + |\eta_i| = \sum_{i=1}^n |\xi_i| + \sum_{i=1}^n |\eta_i| = |x\|_1 + \|y\|_1,$$

$$\|\alpha x\|_1 = \sum_{i=1}^n |\alpha\xi_i| = \sum_{i=1}^n |\alpha||\xi_i| = |\alpha| \sum_{i=1}^n |\xi_i| = |\alpha|\|x\|_1.$$

If $\|x\|_1 = 0$ then every $|\xi_i|$ must be 0, so $x = 0$. **Q**

**(b)** On $\mathbb{R}^n$ set

$$\|x\|_\infty = \max_{i \le n} |\xi_i|$$

if $x = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. Then $\| \; \|_\infty$ is a norm on $\mathbb{R}^n$. **P** Of course $\|x\|_\infty$, being the maximum of finitely many non-negative terms, is a non-negative real number. If $x = (\xi_1, \dots, \xi_n)$, $y = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ then, for any $i \le n$,

$$|\xi_i + \eta_i| \le |\xi_i| + |\eta_i| \le \|x\|_\infty + \|y\|_\infty,$$

$$|\alpha\xi_i| = |\alpha||\xi_i| \le |\alpha|\|x\|_\infty;$$

as $i$ is arbitrary,

$$\|x+y\|_\infty = \max_{i\le n} |\xi_i+\eta_i| \le \|x\|_\infty + \|y\|_\infty, \quad \|\alpha x\|_\infty = \max_{i\le n}|\alpha\xi_i| \le |\alpha|\|x\|_\infty.$$

If $\|x\|_\infty = 0$ then every $\xi_i$ must be 0, so $x = 0$. By 7Gb, this is enough. **Q**

**(c)** On $\ell^1$ set

$$\|x\|_1 = \sum_{i=1}^\infty |\xi_i|$$

if $x = (\xi_1, \dots) \in \ell^1$. Then $\| \|_1$ is a norm on $\ell^1$. **P** By the definition of $\ell^1$, $\|x\|_1$ is defined in $\mathbb{R}$ for every $x \in \ell^1$; being a sum of non-negative terms, it is non-negative. If $x = (\xi_1, \dots)$, $y = (\eta_1, \dots) \in \ell^1$ and $\alpha \in \mathbb{R}$ then

$$\|x+y\|_1 = \sum_{i=1}^\infty |\xi_i+\eta_i| \le \sum_{i=1}^\infty |\xi_i| + |\eta_i| = \sum_{i=1}^\infty |\xi_i| + \sum_{i=1}^\infty |\eta_i| = \|x\|_1 + \|y\|_1,$$

$$\|\alpha x\|_1 = \sum_{i=1}^\infty |\alpha\xi_i| = \sum_{i=1}^\infty |\alpha||\xi_i| = |\alpha|\sum_{i=1}^\infty |\xi_i| = |\alpha|\|x\|_1.$$

If $\|x\|_1 = 0$ then every $|\xi_i|$ must be 0, so $x = 0$. **Q**

**(d)** For any non-empty set $I$, define

$$\|x\|_\infty = \sup_{i\in I} |x(i)|$$

if $x \in \ell^\infty(I)$. Then $\| \|_\infty$ is a norm on $\ell^\infty(I)$. **P** Of course $\|x\|_\infty$, being the supremum of a non-empty bounded set of non-negative real numbers, is a non-negative real number. If $x$, $y \in \ell^\infty(I)$ and $\alpha \in \mathbb{R}$ then, for any $i \in I$,

$$|x(i)+y(i)| \le |x(i)| + |y(i)| \le \|x\|_\infty + \|y\|_\infty,$$

$$|\alpha x(i)| = |\alpha||x(i)| \le |\alpha|\|x\|_\infty;$$

as $i$ is arbitrary,

$$\|x+y\|_\infty = \sup_{i\in I} |x(i)+y(i)| \le \|x\|_\infty + \|y\|_\infty,$$

$$\|\alpha x\|_\infty = \sup_{i\in I} |\alpha x(i)| \le |\alpha|\|x\|_\infty.$$

If $\|x\|_\infty = 0$ then every $x(i)$ must be 0, so $x = 0$. By 7Gb, this is enough. **Q**

**(e)** For $x = (\xi_1, \dots) \in \ell^2$, set $\|x\|_2 = \sqrt{\sum_{i=1}^\infty |\xi_i|^2}$. Then $\| \|_2$ is a norm on $\ell^2$. **P (i)** By the definition of $\ell^2$, the sum $\sum_{i=1}^\infty |\xi_i^2$ is finite, and of course it is non-negative; so $\|x\|_2$ is defined and non-negative. **(ii)** It is worth noting at this point that if $\|x\|_2 = 0$, where $x = (\xi_1, \dots)$, then $\sum_{i=1}^\infty \xi_i^2 = 0$ so every $\xi_i$ is zero and $x = 0$. **(iii)** If $x = (\xi_1, \dots)$ and $y = (\eta_1, \dots)$ belong to $\ell^2$, set $(x|y) = \sum_{i=1}^\infty \xi_i\eta_i$; this is well-defined because

$$\sum_{i=1}^\infty |\xi_i\eta_i| \le \sum_{i=1}^\infty \xi_i^2 + \eta_i^2 = \sum_{i=1}^\infty \xi_i^2 + \sum_{i=1}^\infty \eta_i^2 < \infty,$$

so we have an absolutely summable series which must be summable. **(iv)** If $x = (\xi_1, \dots)$ and $y = (\eta_1, \dots)$ belong to $\ell^2$, and $\alpha$, $\beta \in \mathbb{R}$, then

$$\begin{aligned} 0 &\le \sum_{i=1}^\infty (\beta\xi_i - \alpha_i\eta_i)^2 \\ &= \sum_{i=1}^\infty \beta^2\xi_i^2 - 2\alpha\beta\xi_i\eta_i + \alpha_i^2\eta_i^2 \\ &= \beta^2\sum_{i=1}^\infty \xi_i^2 - 2\alpha_i\beta_i\sum_{i=1}^\infty \xi_i\eta_i + \alpha_i^2\sum_{i=1}^\infty \eta_i^2 \\ &= \beta^2\|x\|_2^2 - 2\alpha\beta(x|y) + \alpha^2\|y\|_2^2, \end{aligned}$$

so that

$$2\alpha\beta(x|y) \le \beta^2\|x\|_2^2 + \alpha^2\|y\|_2^2.$$

**(v)** Setting $\alpha = \|x\|_2$, $\beta = \|y\|_2$ then we get

$$2\alpha\beta(x|y) \le \beta^2\alpha^2 + \alpha^2\beta^2 = 2\alpha^2\beta^2;$$

so if $\alpha$, $\beta$ are both non-zero then

$$(x|y) \le \alpha\beta = \|x\|_2\|y\|_2.$$

On the other hand, if $\alpha = 0$ then $\xi_i = 0$ for every $i$ and $(x|y) = 0$, while if $\beta = 0$ then $\eta_i = 0$ for every $i$ and again $(x|y) = 0$. Thus we have

$$(x|y) \le \|x\|_2\|y\|_2$$

for all $x$, $y \in \ell^2$. **(vi)** If $x = (\xi_1, \dots)$ and $y = (\eta_1, \dots)$ belong to $\ell^2$, then

$$\begin{aligned}\|x+y\|_2^2 &= \sum_{i=1}^\infty (\xi_i + \eta_i)^2 \\ &= \sum_{i=1}^\infty \xi_i^2 + 2\xi_i\eta_i + \eta_i^2 \\ &= \sum_{i=1}^\infty \xi_i^2 + 2\sum_{i=1}^\infty \xi_i\eta_i + \sum_{i=1}^\infty \eta_i^2 \\ &= \|x\|_2^2 + 2(x|y) + \|y\|_2 \\ &\le \|x\|_2^2 + 2\|x\|_2\|y\|_2 + \|y\|_2^2 \\ &= (\|x\|_2 + \|y\|_2)^2;\end{aligned}$$

taking the square roots,

$$\|x+y\|_2 \le \|x\|_2 + \|y\|_2.$$

**(vii)** Finally, if $x = (\xi_1, \dots) \in \ell^2$ and $\alpha \in \mathbb{R}$, then

$$\|\alpha x\|_2 = \sqrt{\textstyle\sum_{i=1}^\infty (\alpha\xi_i^2)} = \sqrt{\alpha^2 \textstyle\sum_{i=1}^\infty \xi_i^2} = |\alpha|\sqrt{\textstyle\sum_{i=1}^\infty \xi_i^2} = |\alpha|\|x\|_2.$$

Thus $\|\,\|_2$ is a norm on $\ell^2$. **Q**

**(f)** Take $a < b$ in $\mathbb{R}$. On the linear space $C([a,b])$ of continuous real-valued functions on $[a,b]$, set

$$\|f\|_1 = \textstyle\int_a^b |f(t)|dt.$$

Then $\|\,\|_1$ is a norm on $C([a,b])$. **P** We have to know the following results:

($\alpha$) continuous functions are integrable over closed bounded intervals;

($\beta$) $\int_a^b f(t) + g(t)dt = \int_a^b f(t)dt + \int_a^b g(t)dt$ for all integrable functions $f$, $g$;

($\gamma$) $\int_a^b \alpha f(t)dt = \alpha \int_a^b f(t)dt$ for all integrable functions $f$ and all scalars $\alpha$;

($\delta$) $\int_a^b f(t)dt \le \int_a^b g(t)dt$ whenever $f$, $g$ are integrable functions and $f(t) \le g(t)$ for every $t \in [a,b]$.

Armed with these, we can argue as follows. **(i)** If $f \in C([a,b])$ then $|f| \in C([a,b])$ so $\|f\|_1 = \int_a^b |f(t)|dt$ is well-defined. **(ii)** If $f$, $g \in C([a,b])$, then $|f(t) + g(t)| \le |f(t)| + |g(t)|$ for every $t \in [a,b]$, so putting ($\delta$) and ($\beta$) together)

$$\|f+g\|_1 = \textstyle\int_a^b |f(t)+g(t)|dt \le \int_a^b |f(t)| + |g(t)|dt = \int_a^b |f(t)|dt + \int_a^b |g(t)|dt = \|f\|_1 + \|g\|_1.$$

**(iii)** If $f \in C([a,b])$ and $\alpha \in \mathbb{R}$, then, using ($\gamma$),

$$\|\alpha f\|_1 = \textstyle\int_a^b |\alpha f(t)|dt = |\alpha| \int_a^b |f(t)|dt = |\alpha|\|f\|_1.$$

**(iv)** Now suppose that $f \in C([a,b]$ and that $f \ne 0$. In this case there is a $t_0 \in [a,b]$ such that $f(t_0) \ne 0$. Set $\epsilon = \frac{1}{2}|f(t_0)| > 0$. Then there is a $\delta > 0$ such that $|f(t) - f(t_0)| \le \epsilon$ whenever $t \in [a,b]$ and $|t - t_0| \le \delta$. (This is where we use the fact that we are dealing with continuous functions.) Set $a' = \max(a, t_0 - \delta)$, $b' = \min(b, t_0 + \delta)$; then $a' < b'$ (because $a < b$). If $t \in [a', b']$, then

$$|f(t)| \ge |f(t_0)| - |f(t) - f(t_0)| \ge 2\epsilon - \epsilon \ge \epsilon.$$

So

$$\|f\|_1 = \int_a^b |f(t)|dt \ge \int_{a'}^{b'} |f(t)|dt \ge \int_{a'}^{b'} \epsilon dt = \epsilon(b' - a') > 0.$$

Turning this round, we see that if $\|f\|_1 = 0$ then $f$ must be the zero function, so $\| \|_1$ is a norm. **Q**

**7I Linear subspaces** If $(X, \| \|)$ is a normed space and $Y$ is a linear subspace of $X$, then $(Y, \| \|_Y)$ is a normed space, where $\| \|_Y$ is the restriction of the norm to $Y$; this is immediate from the definition in 7F.

**7J Examples (a)** If $(X, \rho)$ is any metric space, then $C_b(X)$ is a normed space under the norm $\| \|_\infty$ inherited from $\ell^\infty(X)$.

**(b)** The space $\boldsymbol{c}_0$ is a normed space under the norm $\| \|_\infty$ inherited from $\ell^\infty$.

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## 8. Equivalent norms

**8A The metric associated with a norm** Let $(X, \| \|)$ be a normed space. Set $\rho(x, y) = \|x - y\|$ for $x$, $y \in X$. Then $\rho$ is a metric on $X$. **P** (i) $\rho$ takes values in $[0, \infty[$ because $\| \|$ does. (ii) If $x$, $y$, $z \in X$ then

$$\rho(x, z) = \|x - z\| = \|x - y + y - z\| \le \|x - y\| + \|y - z\| = \rho(x, y) + \rho(y, z).$$

(iii) If $x$, $y \in X$ then

$$\rho(x, y) = \|x - y\| = \|(-1)(y - x)\| = |-1|\|y - x\| = \|y - x\| = \rho(y, x).$$

(iv) If $x$, $y \in X$ then

$$\rho(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = 0 \iff x = y. \text{ } \mathbf{Q}$$

**8B A useful fact** If $\rho$ is the metric associated with a norm $\| \|$ on a linear space $X$, then $\| \| : X \to \mathbb{R}$ is uniformly continuous. **P** For any $x$, $y \in X$,

$$\|x\| - \|y\| = \|y + (x - y)\| - \|y\| \le \|y\| + \|x - y\| - \|y\| = \rho(x, y),$$

and similarly $\|y\| - \|x\| \le \rho(y, x) = \rho(x, y)$, so $|\|x\| - \|y\|| \le \rho(x, y)$. **Q**

**8C Equivalent norms** If $X$ is a linear space and $\| \|$, $\| \|'$ are two norms on $X$, they are called **equivalent** if there is an $M \ge 0$ such that

$$\|x\| \le M\|x\|', \quad \|x\|' \le M\|x\|$$

for all $x \in X$.

**8D Proposition** Let $X$ be a linear space with two norms $\| \|$, $\| \|'$ and corresponding metrics $\rho$, $\rho'$ defined as in 8A. Then $\| \|$ and $\| \|'$ are equivalent norms iff $\rho$ and $\rho'$ are equivalent metrics.

**proof (a)** Suppose that $\| \|$ and $\| \|'$ are equivalent. Then there is an $M \ge 0$ such that $\|x\| \le M\|x\|'$ and $\|x\|' \le M\|x\|$ for every $x \in X$. Take any $x_0 \in X$ and $\epsilon > 0$. Set $\delta = \epsilon/(1 + M) > 0$. Then

$$\rho(x, x_0) \le \delta \Rightarrow \rho'(x, x_0) = \|x - x_0\|' \le M\|x - x_0\| = M\rho(x, x_0) \le M\delta \le \epsilon,$$

$$\rho'(x, x_0) \le \delta \Rightarrow \rho(x, x_0) = \|x - x_0\| \le M\|x - x_0\|' = M\rho'(x, x_0) \le M\delta \le \epsilon.$$

As $x_0$, $\epsilon$ are arbitrary, $\rho$ and $\rho'$ are equivalent.

**(b)** If $\rho$, $\rho'$ are equivalent, then there must be $\delta_1$, $\delta_2 > 0$ such that

$$\rho(x, 0) \le \delta_1 \Rightarrow \rho'(x, 0) \le 1,$$

$$\rho'(x, 0) \le \delta_2 \Rightarrow \rho(x, 0) \le 1.$$

Set $M = \max(1/\delta_1, 1/\delta_2)$. Take any $x \in X$. If $x = 0$, then

$$\|x\| = 0 = M\|x\|', \quad \|x\|' = 0 = M\|x\|.$$

Otherwise, $\gamma = \|x\| > 0$. Consider $x_1 = M^{-1}\gamma^{-1}x$. Then

$$\|x_1\| = \frac{1}{M\gamma}\|x\| \le \frac{1}{M} \le \delta_1$$

so $\|x_1\|' \le 1$ and

$$\|x\|' = M\gamma\|x_1\|' \le M\gamma = M\|x\|.$$

Similarly (because $M$ was chosen by a procedure symmetric between $\| \,\|$ and $\| \,\|'$) $\|x\| \le M\|x\|'$. Accordingly $\| \,\|$ and $\| \,\|'$ are equivalent.

**8E Corollary** (a) Two norms on a linear space $X$ are equivalent iff they give rise to the same open sets on $X$; thus in this case they will give rise to the same closed sets, compact sets, continuous functions etc.

(b) 'Equivalence' is an equivalence relation on the set of all norms on a given linear space.

**proof (a)** We know from 1L that two metrics are equivalent iff they give rise to the same open sets.

**(b)** This follows from (a).

**8F Theorem** Let $X$ be a finite-dimensional real linear space. Then any two norms on $X$ are equivalent.

**proof (a)** If $X$ is zero-dimensional, then $X = \{0\}$, so there is only one norm on $X$ (because $\|0\|$ must be 0), and the result is trivial.

**(b)** For any $n \ge 1$, any norm $\| \,\|$ on $\mathbb{R}^n$ is equivalent to $\| \,\|_\infty$. **P** **(i)** Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$, so that the vector $(\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ is just $\sum_{i=1}^n \xi_i e_i$. Set $M_1 = \|e_1\| + \ldots + \|e_n\|$. Then for any $x = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$,

$$\begin{aligned}\|x\| &= \|\sum_{i=1}^n \xi_i e_i\| \\ &\le \sum_{i=1}^n \|\xi_i e_i\| \\ &= \sum_{i=1}^n |\xi_i|\|e_i\| \\ &\le \sum_{i=1}^n \|x\|_\infty \|e_i\| \\ &= M_1\|x\|_\infty.\end{aligned}$$

**(ii)** **?** Suppose, if possible, that there is no $M \ge M_1$ such that $\|x\|_\infty \le M\|x\|$ for every $x \in \mathbb{R}^n$. Then for every $k \in \mathbb{N}$ we can find $x_k \in \mathbb{R}^n$ such that $\|x_k\|_\infty > k\|x_k\|$. Set $\gamma_k = \|x_k\|_\infty > 0$, $y_k = \gamma_k^{-1}x_k$ for each $k$; then $\|y_k\|_\infty = 1$ while $\|y_k\| \le \frac{1}{k} \to 0$. Now $\langle y_k\rangle_{k\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}^n$, so has a subsequence $\langle y_{k(i)}\rangle_{i\in\mathbb{N}}$ which is convergent for the product metric in $\mathbb{R}^n$, which is just the metric $\rho_\infty$ derived from $\| \,\|_\infty$. Let $y$ be the limit of $\langle y_{k(i)}\rangle_{i\in\mathbb{N}}$ for $\| \,\|_\infty$. Because $\| \,\|_\infty$ is $\rho_\infty$-continuous (8B), $\|y\|_\infty = \lim_{i\to\infty} \|y_{k(i)}\|_\infty = 1$ and $y \ne 0$. On the other hand, we now have

$$\|y\| \le \|y_{k(i)}\| + \|y - y_{k(i)}\| \le \|y_{k(i)}\| + M\|y - y_{k(i)}\|_\infty \to 0$$

as $i \to \infty$, so $\|y\| = 0$ and $y = 0$. **X**

**(iii)** This contradiction shows that there is an $M \ge M_1$ such that $\|x\|_\infty \le M\|x\|$ for every $x \in \mathbb{R}^n$. Since we surely have $\|x\| \le M\|x\|_\infty$ for every $x$, $\| \,\|$ and $\| \,\|_\infty$ are equivalent. **Q**

**(c)** Now let $X$ be any $n$-dimensional linear space over $\mathbb{R}$, and $\| \,\|$, $\| \,\|'$ two norms on $X$. Let $T : X \to \mathbb{R}^n$ be a linear space isomorphism. Then we have two norms $\| \,\|^*$, $\| \,\|'^*$ on $\mathbb{R}^n$ given by writing

$$\|z\|^* = \|T^{-1}z\|, \quad \|z\|'^* = \|T^{-1}z\|'$$

for $z \in \mathbb{R}^n$. By (b), both $\| \,\|^*$ and $\| \,\|'^*$ are equivalent to $\| \,\|_\infty$, so there are $M_1$, $M_2 \ge 0$ such that

$$\|z\|^* \le M_1\|z\|_\infty, \quad \|z\|_\infty \le M_1\|z\|^*, \quad \|z\|'^* \le M_2\|z\|_\infty, \quad \|z\|_\infty \le M_2\|z\|'^*$$

for every $z \in \mathbb{R}^n$. Set $M = M_1M_2 \ge 0$. Then for any $x \in X$,

$$\|x\| = \|Tx\|^* \le M_1\|Tx\|_\infty \le M_2M_1\|Tx\|'^* = M\|x\|',$$

$$\|x\|' = \|Tx\|'^* \le M_2\|Tx\|_\infty \le M_1M_2\|Tx\|^* = M\|x\|,$$

so $\| \|$, $\| \|'$ are equivalent.

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## 9. Banach spaces

**9A Definition** A **Banach space** is a normed space which is complete under the metric associated with its norm (see 4Ac, 8A).

**9B Definition** If $(X, \| \|)$ is a normed space, a set $A \subseteq X$ is **bounded** if there is an $M \ge 0$ such that $\|x\| \le M$ for every $x \in A$.

**Remark** In $\mathbb{R}^n$, with any of the standard norms (e.g., the norms $\| \|_1$, $\| \|_\infty$ of 7Ha) a set is bounded iff it is bounded in the ordinary sense, that is, bounded in each coordinate separately. Indeed, using 8G, we see that all norms on a finite-dimensional space give the same notion of 'bounded' set.

**9C Useful fact** If $X$ is any normed space and $\langle x_n\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in $X$, then $\{x_n : n \in \mathbb{N}\}$ is bounded. **P** Let $n_0$ be such that $\|x_m - x_n\| = \rho(x_m, x_n) \le 1$ for all $m$, $n \ge n_0$. Then for any $n \ge n_0$

$$\|x_n\| = \|x_{n_0} + (x_n - x_{n_0})\| \le \|x_{n_0}\| + \|x_n - x_{n_0}\| \le \|x_{n_0}\| + 1,$$

so setting

$$M = \max(\|x_1\|, \|x_2\|, \dots, \|x_{n_0}\|, \|x_{n_0}\| + 1)$$

we shall have $\|x_n\| \le M$ for every $n$. **Q**

**9D Example** $\ell^1$ is a Banach space.

**proof** We have seen in 7Hc that $\ell^1$ is a normed space under its usual norm $\| \|_1$. Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be a Cauchy sequence in $\ell^1$. Express each $x_n$ as $(\xi_1^{(n)}, \dots)$.

**(i)** For each $i \in \mathbb{N}$,

$$|\xi_i^{(m)} - \xi_i^{(n)}| \le \sum_{j=1}^\infty |\xi_j^{(m)} - \xi_j^{(n)}| = \|x_m - x_n\|_1 \to 0$$

as $m$, $n \to \infty$, that is, $\langle \xi_i^{(n)}\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Because $\mathbb{R}$ is complete, it has a limit $\xi_i \in \mathbb{R}$. Set

$$x = (\xi_1, \xi_2, \dots) \in \mathbb{R}^{\mathbb{N}}.$$

**(ii)** By 9C, there is an $M \ge 0$ such that $\|x_n\|_1 \le M$ for every $n \in \mathbb{N}$. Now

$$\sum_{i=1}^k |\xi_i^{(n)}| \le \sum_{i=1}^\infty |\xi_i^{(n)}| = \|x_n\|_1 \le M \text{ for all } k,\ n \in \mathbb{N};$$

letting $n \to \infty$,

$$\sum_{i=1}^k |\xi_i| = \lim_{n\to\infty} \sum_{i=1}^k |\xi_i^{(n)}| \le M \text{ for every } k \in \mathbb{N};$$

letting $k \to \infty$,

$$\sum_{i=1}^\infty |\xi_i| = \lim_{n\to\infty} \sum_{i=1}^k |\xi_i| \le M < \infty.$$

So $x \in \ell^1$.

**(iii)** Let $\epsilon > 0$. Then there is an $n_0 \in \mathbb{N}$ such that $\|x_m - x_n\|_1 \le \epsilon$ for all $m$, $n \ge n_0$. Accordingly

$$\sum_{i=1}^k |\xi_i^{(m)} - \xi_i^{(n)}| \le \sum_{i=1}^\infty |\xi_i^{(m)} - \xi_i^{(n)}| = \|x_m - x_n\|_1 \le \epsilon \text{ for all } k \in \mathbb{N},\ m,\ n \ge n_0;$$

letting $n \to \infty$,

$$\sum_{i=1}^k |\xi_i^{(m)} - \xi_i| = \lim_{n\to\infty} \sum_{i=1}^k |\xi_i^{(m)} - \xi_i^{(n)}| \le \epsilon \text{ for every } k \in \mathbb{N},\ m \ge n_0;$$

letting $k \to \infty$,

$$\|x_m - x\|_1 = \sum_{i=1}^\infty |\xi_i^{(m)} - \xi_i| = \lim_{k\to\infty} \sum_{i=1}^k |\xi_i^{(m)} - \xi_i| \le \epsilon \text{ for every } m \ge n_0.$$

As $\epsilon$ is arbitrary, $\langle x_n\rangle_{n\in\mathbb{N}}$ converges to $x$. As $\langle x_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\ell^1$ is complete.

**9E Example** For any non-empty set $I$, $\ell^\infty(I)$ is a Banach space.

**proof** We have seen in 7Hd that $\ell^\infty(I)$ is a normed space under its usual norm $\| \|_\infty$. Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be a Cauchy sequence in $\ell^\infty(I)$.

**(i)** For each $i\in\mathbb{N}$,

$$|x_m(i)-x_n(i)|\le\|x_m-x_n\|_\infty\to 0$$

as $m$, $n\to\infty$, that is, $\langle x_n(i)\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Because $\mathbb{R}$ is complete, it has a limit $x(i)\in\mathbb{R}$. This defines a function $x\in\mathbb{R}^I$.

**(ii)** By 9C, there is an $M\ge 0$ such that $\|x_n\|_\infty\le M$ for every $n\in\mathbb{N}$. Now

$$|x_n(i)|\le\|x_n\|_\infty\le M \text{ for all } i\in I,\ n\in\mathbb{N};$$

letting $n\to\infty$,

$$|x(i)|=\lim_{n\to\infty}|x_n(i)|\le M \text{ for every } i\in I,$$

and $x\in\ell^\infty(I)$.

**(iii)** Let $\epsilon>0$. Then there is an $n_0\in\mathbb{N}$ such that $\|x_m-x_n\|_\infty\le\epsilon$ for all $m$, $n\ge n_0$. Accordingly

$$|x_m(i)-x_n(i)|\le\|x_m-x_n\|_\infty\le\epsilon \text{ for all } i\in I,\ m,\ n\ge n_0;$$

letting $n\to\infty$,

$$|x_m(i)-x(i)|=\lim_{n\to\infty}|x_m(i)-x_n(i)|\le\epsilon \text{ for every } i\in I,\ m\ge n_0.$$

As $i$ is arbitrary,

$$\|x_m-x\|_\infty\le\epsilon \text{ for every } m\ge n_0.$$

As $\epsilon$ is arbitrary, $\langle x_n\rangle_{n\in\mathbb{N}}$ converges to $x$. As $\langle x_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\ell^\infty$ is complete.

**9F Example** $\ell^2$ is a Banach space.

**proof** We have seen in 7He that $\ell^2$ is a normed space under its usual norm $\| \|_2$. Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be a Cauchy sequence in $\ell^2$. Express each $x_n$ as $(\xi_1^{(n)},\dots)$.

**(i)** For each $i\in\mathbb{N}$,

$$|\xi_i^{(m)}-\xi_i^{(n)}|\le\sqrt{\sum_{j=1}^\infty|\xi_j^{(m)}-\xi_j^{(n)}|^2}=\|x_m-x_n\|_2\to 0$$

as $m$, $n\to\infty$, that is, $\langle\xi_i^{(n)}\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Because $\mathbb{R}$ is complete, it has a limit $\xi_i\in\mathbb{R}$. Set

$$x=(\xi_1,\xi_2,\dots)\in\mathbb{R}^{\mathbb{N}}.$$

**(ii)** By 9C, there is an $M\ge 0$ such that $\|x_n\|_2\le M$ for every $n\in\mathbb{N}$. Now

$$\sum_{i=1}^k(\xi_i^{(n)})^2\le\sum_{i=1}^\infty(\xi_i^{(n)})^2=\|x_n\|_2^2\le M^2 \text{ for all } k,\ n\in\mathbb{N};$$

letting $n\to\infty$,

$$\sum_{i=1}^k\xi_i^2=\lim_{n\to\infty}\sum_{i=1}^k(\xi_i^{(n)})^2\le M^2 \text{ for every } k\in\mathbb{N};$$

letting $k\to\infty$,

$$\sum_{i=1}^\infty\xi_i^2=\lim_{n\to\infty}\sum_{i=1}^k\xi_i^2\le M^2<\infty.$$

So $x\in\ell^2$.

**(iii)** Let $\epsilon>0$. Then there is an $n_0\in\mathbb{N}$ such that $\|x_m-x_n\|_2\le\epsilon$ for all $m$, $n\ge n_0$. Accordingly

$$\sum_{i=1}^k(\xi_i^{(m)}-\xi_i^{(n)})^2\le\sum_{i=1}^\infty(\xi_i^{(m)}-\xi_i^{(n)})^2=\|x_m-x_n\|_2^2\le\epsilon^2 \text{ for all } k\in\mathbb{N},\ m,\ n\ge n_0;$$

letting $n\to\infty$,

$$\sum_{i=1}^k(\xi_i^{(m)}-\xi_i)^2=\lim_{n\to\infty}\sum_{i=1}^k(\xi_i^{(m)}-\xi_i^{(n)})^2\le\epsilon^2 \text{ for every } k\in\mathbb{N},\ m\ge n_0;$$

letting $k\to\infty$,

$$\|x_m-x\|_2^2=\sum_{i=1}^\infty(\xi_i^{(m)}-\xi_i)^2=\lim_{k\to\infty}\sum_{i=1}^k(\xi_i^{(m)}-\xi_i)^2\le\epsilon^2 \text{ for every } m\ge n_0,$$

and

$$\|x_m-x\|_2\le\epsilon \text{ for every } m\ge n_0.$$

As $\epsilon$ is arbitrary, $\langle x_n\rangle_{n\in\mathbb{N}}$ converges to $x$. As $\langle x_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\ell^2$ is complete.

**9G Closed linear subspaces** If $(X,\|\,\|)$ is a Banach space and $Y$ is a linear subspace which is closed (with respect to the metric $\rho$ defined from the norm of $X$), then $Y$, with the induced norm $\|\,\|_Y$, is a normed space (7I). Now the metric $\rho_Y$ on $Y$ associated with $\|\,\|_Y$ has

$$\rho_Y(y,y')=\|y-y'\|_Y=\|y-y'\|=\rho(y,y')$$

for every $y$, $y'\in Y$; that is, $\rho_Y$ is just the subspace metric on $Y$ induced by $\rho$ (1K). So $Y$ is complete for $\rho_Y$, because any $\rho_Y$-Cauchy sequence in $Y$ is also $\rho$-Cauchy, so has a limit in $X$, which belongs to $Y$ because $Y$ is closed. Thus $Y$ is also a Banach space.

**9H Example** Let $(X,\rho)$ be any metric space. Then $C_b(X)$, with its usual norm $\|\,\|_\infty$, is a Banach space.

**proof** I have already noted (7Ef) that $C_b(X)$ is a linear subspace of $\ell^\infty(X)$. Now it is closed. **P** If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in $C_b(X)$ converging to $f\in\ell^\infty(X)$, then

for every $\epsilon>0$ there is an $n_0\in\mathbb{N}$ such that $|f_n(x)-f(x)|\le\|f_n-f\|_\infty\le\epsilon$ for every $n\ge n_0$, $x\in X$,

that is, $\langle f_n\rangle_{n\in\mathbb{N}}$ converges uniformly to $f$. But it follows that $f$ is continuous (2G), so $f\in C_b(X)$. **Q**

**9I Example** The space $\boldsymbol{c}_0$, with its usual norm $\|\,\|_\infty$, is a Banach space.

**proof** I have already noted (7Eg) that $\boldsymbol{c}_0$ is a linear subspace of $\ell^\infty$. To see that it is a closed linear subspace, let $x=(\xi_1,\dots)$ be any member of its closure. For any $\epsilon>0$, the ball $N_{\epsilon/2}(x)$ must meet $\boldsymbol{c}_0$, that is, there is a $y=(\eta_1,\dots)\in\boldsymbol{c}_0$ such that $\|x-y\|_\infty\le\frac{\epsilon}{2}$. There is an $r_0\in\mathbb{N}$ such that $|\eta_i|\le\frac{\epsilon}{2}$ for every $i\ge r_0$. Now if $i\ge r_0$,

$$|\xi_i|\le|\xi_i-\eta_i|+|\eta_i|\le\|x-y\|_\infty+\frac{\epsilon}{2}\le\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

As $\epsilon$ is arbitrary, $\lim_{i\to\infty}\xi_i=0$, that is, $x\in\boldsymbol{c}_0$. As $x$ is arbitrary, $\boldsymbol{c}_0$ is closed in $\ell^\infty$, therefore a Banach space.

**9J Example** The space $C([0,1])$ with $\|\,\|_1$ (7Hf) is *not* a Banach space.

**proof (i)** For each $n\in\mathbb{N}$, define $f_n:[0,1]\to[0,1]$ by setting $f_n(t)=2^nt^n$ if $t\in[0,\frac12]$, 1 if $t\in[\frac12,1]$. Then for $m\le n$,

$$\begin{aligned}\|f_m-f_n\|_1&=\int_0^1|f_m(t)-f_n(t)|dt\\&=\int_0^{1/2}(2t)^m-(2t)^n dt\\&=\frac{1}{2(m+1)}-\frac{1}{2(n+1)}\\&\le\frac{1}{2(m+1)}\\&\to 0\end{aligned}$$

as $m\to\infty$. So $\langle f_n\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,1])$.

**(ii) ?** Suppose, if possible, that $\langle f_n\rangle_{n\in\mathbb{N}}$ has limit $f\in C([0,1])$. Then for any $x\in[0,1]$,

$$
\begin{aligned}
|\int_0^x f(t)dt - \int_0^x f_n(t)dt| &= |\int_0^x f(t) - f_n(t)dt| \\
&\le \int_0^x |f(t) - f_n(t)|dt \\
&\le \int_0^1 |f(t) - f_n(t)|dt \\
&= \|f - f_n\|_1 \\
&\to 0
\end{aligned}
$$

as $n \to \infty$, and $\int_0^x f(t)dt = \lim_{n\to\infty} \int_0^x f_n(t)dt$. But for $x \le \frac{1}{2}$

$$\int_0^x f_n(t)dt \le \int_0^{1/2} f_n(t)dt = \frac{1}{2(n+1)} \to 0,$$

while for $x \ge \frac{1}{2}$

$$\int_0^x f_n(t)dt = \int_0^{1/2} f_n(t)dt + \int_{1/2}^x 1\,dt = \frac{1}{2(n+1)} + x - \frac{1}{2} \to x - \frac{1}{2}.$$

Thus we have

$$F(x) = \int_0^x f(t)dt = 0 \text{ for } x \le \frac{1}{2},\ x - \frac{1}{2} \text{ for } x \ge \frac{1}{2}.$$

However, $f$ is supposed to be continuous, so by the Fundamental Theorem of Calculus $F'(x)$ exists and is equal to $f(x)$ for every $x \in [0,1]$; in particular, $F'(\frac{1}{2})$ must be defined; which of course it is not. **X**

Thus $\langle f_n \rangle_{n\in\mathbb{N}}$ has no limit in $C([0,1])$ and $C([0,1])$ is not complete.

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## 10. Picard's theorem

**10A Ordinary differential equations** Consider the equation

$$\frac{dx}{dt} = f(t,x), \quad x(t_0) = x_0.$$

We ordinarily expect such equations to have unique solutions; the purpose of this chapter is to state and prove the fundamental theorem of this type.

We start by converting to an integral equation

$$x(t) = x_0 + \int_{t_0}^t f(u, x(u))du,$$

and try to solve 'by iteration', that is, by setting

$$(Tx)(t) = x_0 + \int_{t_0}^t f(u, x(u))du,$$

and hoping that a sequence $z, Tz, T^2z, T^3z, \ldots$ will converge with reasonable rapidity to the solution.

This is reminiscent of the contraction mapping theorem, and the theorem I will give will be based on $T$ being a contraction mapping. The idea is to set $T$ up as an operator form $C([a',b'])$ to itself for some interval $[a',b']$. In this case

$$
\begin{aligned}
\|Tx - Ty\|_\infty &= \sup_{t\in[a',b']} |\int_{t_0}^t f(u,x(u)) - f(u,y(u))du| \\
&\le \sup_{t\in[a',b']} \int_{\min(t,t_0)}^{\max(t,t_0)} |f(u,x(u)) - f(u,y(u))|du.
\end{aligned}
$$

In order to get a control over this we need $f$ to satisfy a **Lipschitz condition**

$$|f(u,\xi) - f(u,\eta)| \le K|\xi - \eta| \text{ for all relevant } u,\ \xi,\ \eta;$$

then we shall have

$$\|Tx - Ty\|_\infty \le K\|x - y\|_\infty \sup_{t\in[a',b']} |t - t_0|,$$

so that things will work smoothly if

$$K \max(a' - t_0, t_0 - b) < 1.$$

Now for the details.

**10B Picard's theorem** Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function which is Lipschitz in the second variable, that is, there is a $K \ge 0$ such that $|f(t, \xi) - f(t, \eta)| \le K|\xi - \eta|$ whenever $t \in [a, b]$ and $\xi$, $\eta \in [c, d]$. If $t_0 \in [a, b]$ and $x_0 \in ]c, d[$, then there is a $\delta > 0$ such that the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(u, x(u))du$$

has a unique solution as a continuous function from $[a, b] \cap [t_0 - \delta, t_0 + \delta]$ to $[c, d]$.

**proof (a)** Because $f$ is continuous and $[a, b] \times [c, d]$ is compact, $f$ is bounded; say $|f(t, \xi)| \le M$ for every $t \in [a, b]$, $\xi \in [c, d]$. Take $\delta > 0$ such that $\delta K < 1$ and $c \le x_0 - \delta M \le x_0 + \delta M \le d$.

**(b)** Set $a' = \max(a, t_0 - \delta)$, $b' = \min(b, t_0 + \delta)$. Let $X$ be the set of continuous functions from $[a', b']$ to $[c, d]$. Then $X$ is a closed subset of $C([a', b'])$. **P** If $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in $X$ converging to $x \in C([a', b'])$, then for any $t \in [a', b']$

$$|x(t) - x_n(t)| \le \|x - x_n\|_\infty \to 0,$$

so

$$x(t) = \lim_{n\to\infty} x_n(t) \in [c, d]. \ \mathbf{Q}$$

Consequently $X$ is a complete metric space.

**(c)** Now we have a function $T : X \to X$ defined by writing

$$(Tx)(t) = x_0 + \int_{t_0}^{t} f(u, x(u))du$$

for every $t \in [a', b']$. **P** Note first that the function $u \mapsto (u, x(u)) : [a', b'] \to \mathbb{R}^2$ is continuous; because $f$ is continuous, $u \mapsto f(u, x(u)) : [a', b'] \to \mathbb{R}$ is continuous, therefore integrable. So $(Tx)(t)$ is defined as a real number. Next, if $t$, $t' \in [a', b']$ then

$$\begin{aligned}|(Tx)(t) - (Tx)(t')| &= |\int_{\min(t,t')}^{\max(t,t')} f(u, x(u))du| \\ &\le \int_{\min(t,t')}^{\max(t,t')} |f(u, x(u))|du \\ &\le \int_{\min(t,t')}^{\max(t,t')} M\, du \\ &= M|t - t'|,\end{aligned}$$

so $Tx$ is continuous. Moreover,

$$|(Tx)(t) - x_0| = |(Tx)(t) - (Tx)(t_0)| \le M|t - t_0| \le \delta M$$

so

$$c \le x_0 - \delta M \le (Tx)(t) \le x_0 + \delta M \le d$$

for every $t \in [a', b']$, so $Tx \in X$. **Q**

**(d)** Now $T : X \to X$ is a contraction. **P** If $x$, $y \in X$ and $t \in [a', b']$, then

$$\begin{aligned}|(Tx)(t)-(Ty)(t)| &= |\int_{t_0}^{t} f(u,x(u))-f(u,y(u))du| \\ &\le \int_{\min(t,t_0)}^{\max(t,t_0)} |f(u,x(u))-f(u,y(u))|du \\ &\le \int_{\min(t,t_0)}^{\max(t,t_0)} K|x(u)-y(u)|du \\ &\le \int_{\min(t,t_0)}^{\max(t,t_0)} K\|x-y\|_\infty du \\ &\le K|t-t_0|\|x-y\|_\infty \\ &\le K\delta\|x-y\|_\infty .\end{aligned}$$

As $t$ is arbitrary, $\|Tx-Ty\|_\infty \le K\delta\|x-y\|_\infty$. But $\delta K<1$, by the choice of $\delta$, so $T$ is a contraction. **Q**

**(e)** By the contraction mapping theorem (6A), there is a unique $x\in X$ such that $Tx=x$, that is,

$$x(t)=x_0+\int_{t_0}^{t} f(u,x(u))du$$

for every $t\in[a',b']=[a,b]\cap[t_0-\delta,t_0+\delta]$.

**10C Notes (a)** Most of the functions we apply this result to have partial derivatives. If $f$ is such that $\frac{\partial f}{\partial x}$ is bounded on $[a,b]\times[c,d]$, say $|\frac{\partial f}{\partial x}(u,\xi)|\le K$ for $u\in[a,b]$, $\xi\in[c,d]$, then by the mean value theorem we shall have $|f(u,\xi)-f(u,\eta)|\le K|\xi-\eta|$ for $u\in[a,b]$ and $\xi$, $\eta\in[c,d]$.

**(b)** The theorem tells us only that we can find a solution in a small interval around $t_0$. However, if $f$ is well-behaved in a long interval, we can use the theorem to find a solution in an interval $[t_0-\delta_0,t_0+\delta_0]$; now set $t_1=t_0+\delta_0$, and repeat the theorem to find a solution in $[t_1,t_1+\delta_1]$ with the right value at $t_1$ to match the first section; set $t_2=t_1+\delta_1$ and repeat.

**(c)** The trouble with this step-by-step extension is that even for 'good' functions $f$ the $\delta_i$ may tend to 0, so that the $t_i$ may tend to a finite limit. Indeed, this is sometimes inevitable. If we start with the equation

$$\frac{dx}{dt}=x^2, \quad x(0)=1,$$

corresponding to the equation

$$x(t)=1+\int_0^t x(u)^2du,$$

then we can find an exact solution by separating the variables:

$$\frac{dx}{x^2}=dt, \quad -\frac{1}{x}=t+c, \quad -\frac{1}{1}=0+c, \quad c=-1,$$

so that $x=\frac{1}{1-t}$. Thus the solution has a singularity at $t=1$ and no procedure can pass this barrier.

**(d)** However we do have important cases in which $f$ is defined everywhere on $[a,b]\times\mathbb{R}$ and is Lipschitz in the second variable over this whole infinite strip. In this case there is a simplification in the argument (since we no longer have to worry about keeping within an interval $[c,d]$, and can work with $X=C([a',b'])$), and we can take every $\delta_i$ to be (for instance) $1/2K$, where $K$ is the Lipschitz constant; thus in finitely many steps, both up and down, we can cover the whole interval $[a,b]$.

**(e)** The same method can be used for vector-valued equations

$$\frac{d\boldsymbol{x}}{dt}=\boldsymbol{f}(t,\boldsymbol{x}),$$

where $\boldsymbol{x}$ is to be a function taking values in $\mathbb{R}^n$. In this case the Lipschitz condition we need will be

$$\|\boldsymbol{f}(t,\boldsymbol{\xi})-\boldsymbol{f}(t,\boldsymbol{\eta})\|\le K\|\boldsymbol{\xi}-\boldsymbol{\eta}\|$$

for $t\in[a,b]$, $\boldsymbol{\xi}$, $\boldsymbol{\eta}\in N_\epsilon(\boldsymbol{x}_0)$, where $\|\ \|$ is some convenient norm on $\mathbb{R}^n$.

**(f)** The importance of this is that it enables us to deal with higher-order equations

$$\frac{d^n x}{dt^n} = f(t, x, \frac{dx}{dt}, \dots, \frac{d^{n-1}x}{dt^{n-1}});$$

we set $x_1 = x$, $x_2 = \frac{dx}{dt}$, $x_3 = \frac{d^2x}{dt^2}$ and so on, so that we get a system of equations

$$\begin{aligned}\frac{dx_1}{dt} &= x_2,\\ \frac{dx_2}{dt} &= x_3,\\ \dots &= \dots\\ \frac{dx_n}{dt} &= f(t, x_1, \dots, x_n),\end{aligned}$$

corresponding to the vector equation

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(t, \boldsymbol{x})$$

if we set

$$\boldsymbol{f}(t, \xi_1, \dots, \xi_n) = (\xi_2, \dots, \xi_n, f(t, \xi_1, \dots, \xi_n)).$$

**10D Example** To see that we really do need the Lipschitz condition, consider the equation

$$\frac{dx}{dt} = \frac{3}{2}\sqrt[3]{x}.$$

This has solutions $x = \pm(t-c)^{3/2}$ for $t \geq c$. But note that it also has the constant solution 0. This means that any function of the form

$$x(t) = 0 \text{ for } t \leq c, \quad x(t) = (t-c)^{3/2} \text{ for } t \geq c,$$

or

$$x(t) = 0 \text{ for } t \leq c, \quad x(t) = -(t-c)^{3/2} \text{ for } t \geq c,$$

will be a solution. If our initial value is $x(0) = 0$, we can be sure that $x(t) = 0$ for all $t \leq 0$, but for $t > 0$ we don't know whether $x$ is ever non-zero, nor whether it leaves the axis upwards or downwards, nor when it leaves the axis if it does.

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## 11. Linear Operators

**11A The space $Y^X$** Suppose that $Y$ is any linear space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$ and that $X$ is any set. Write $Y^X$ for the set of all functions from $X$ to $Y$. For $f$, $g \in Y^X$ and $\alpha \in \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$ write

$$(f+g)(x) = f(x) + g(x), \quad (-f)(x) = -f(x), \quad (\alpha f)(x) = \alpha f(x)$$

for every $x \in X$; this defines $f+g$, $\alpha f$ and $-f$ as functions from $X$ to $Y$. If we also define $\mathbf{0}(x) = 0$ for every $x \in X$, then $(Y^X, +, \mathbf{0}, -, .)$ is a linear space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$. **P** We have to check the following. If $f$, $g$, $h \in Y^X$ and $\alpha$, $\beta \in \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$ then

$$\begin{aligned}((f+g)+h)(x) &= (f+g)(x) + h(x)\\ &= (f(x) + g(x)) + h(x)\\ &= f(x) + (g(x) + h(x))\\ &= f(x) + (g+h)(x)\\ &= (f + (g+h))(x),\end{aligned}$$

$$\begin{aligned}(f+\mathbf{0})(x) &= f(x)+\mathbf{0}(x)\\ &= f(x)+0\\ &= f(x)\\ &= 0+f(x)\\ &= \mathbf{0}(x)+f(x)\\ &= (\mathbf{0}+f)(x),\end{aligned}$$

$$\begin{aligned}(f+(-f))(x) &= f(x)+(-f)(x)\\ &= f(x)+(-f(x))\\ &= 0\\ &= \mathbf{0}(x)\\ &= (-f(x))+f(x)\\ &= ((-f)+f)(x),\end{aligned}$$

$$(f+g)(x)=f(x)+g(x)=g(x)+f(x)=(g+f)(x),$$

$$(\alpha(f+g))(x)=\alpha(f+g)(x)=\alpha(f(x)+g(x))=\alpha f(x)+\alpha g(x)=(\alpha f)(x)+(\alpha g)(x)=(\alpha f+\alpha g)(x),$$

$$((\alpha+\beta)f)(x)=(\alpha+\beta)f(x)=\alpha f(x)+\beta f(x)=(\alpha f)(x)+(\beta f)(x)=(\alpha f+\beta f)(x),$$

$$(\alpha(\beta f))(x)=\alpha(\beta f)(x)=\alpha(\beta f(x))=(\alpha\beta)f(x)=((\alpha\beta)f)(x),$$

$$(1f)(x)=1.f(x)=f(x)$$

for every $x\in X$, so that

$$(f+g)+h=f+(g+h),$$

$$f+\mathbf{0}=\mathbf{0}+f=f,$$

$$f+(-f)=\mathbf{0}=(-f)+f,$$

$$f+g=g+f,$$

$$\alpha(f+g)=\alpha f+\alpha g,$$

$$(\alpha+\beta)f=\alpha f+\beta f,$$

$$\alpha(\beta f)=(\alpha\beta)f,$$

$$1f=f.\ \mathbf{Q}$$

(Compare 7C.)

**11B The space** $L(X;Y)$ Now let $X$ and $Y$ be two linear spaces over $\begin{smallmatrix}\mathbb{R}\\ \mathbb{C}\end{smallmatrix}$. A function $T:X\to Y$ is **linear** if

$$T(x+y)=Tx+Ty \text{ for all } x,\,y\in X,$$

$$T(\alpha x)=\alpha Tx \text{ for all } x\in X,\ \alpha\in\begin{smallmatrix}\mathbb{R}\\ \mathbb{C}\end{smallmatrix}.$$

I write $L(X;Y)$ for the space of all linear operators from $X$ to $Y$.

**11C Proposition** For any linear spaces $X$ and $Y$, $L(X;Y)$ is a linear subspace of $Y^X$, and is therefore a linear space in its own right.

**proof (i)** Let $S,\,T\in L(X;Y)$. Then for any $x,\,y\in X$ and $\alpha\in\begin{smallmatrix}\mathbb{R}\\ \mathbb{C}\end{smallmatrix}$,

$$\begin{aligned}(S+T)(x+y) &= S(x+y)+T(x+y)\\ &= Sx+Sy+Tx+Ty\end{aligned}$$

(because $S$ and $T$ are linear)

$$\begin{aligned}&= Sx+Tx+Sy+Ty\\ &= (S+T)(x)+(S+T)(y),\end{aligned}$$

$$\begin{aligned}(S+T)(\alpha x) &= S(\alpha x)+T(\alpha x)\\ &= \alpha Sx+\alpha Tx\end{aligned}$$

(because $S$ and $T$ are linear)

$$\begin{aligned}&= \alpha(Sx+Tx)\\ &= \alpha(S+T)(x),\end{aligned}$$

so $S+T$ is linear, and belongs to $L(X;Y)$.

**(ii)** Let $T \in L(X;Y)$ and $\gamma \in {\mathbb{R} \atop \mathbb{C}}$. Then for any $x$, $y \in X$ and $\alpha \in {\mathbb{R} \atop \mathbb{C}}$,

$$\begin{aligned}(\gamma T)(x+y) &= \gamma T(x+y)\\ &= \gamma(Tx+Ty)\end{aligned}$$

(because $T$ is linear)

$$\begin{aligned}&= \gamma Tx+\gamma Ty\\ &= (\gamma T)(x)+(\gamma T)(y),\end{aligned}$$

$$\begin{aligned}(\gamma T)(\alpha x) &= \gamma T(\alpha x)\\ &= \gamma(\alpha Tx)\end{aligned}$$

(because $T$ is linear)

$$\begin{aligned}&= (\gamma\alpha)Tx\\ &= (\alpha\gamma)Tx\\ &= \alpha(\gamma Tx)\\ &= \alpha(\gamma T)(x),\end{aligned}$$

so $\gamma T \in L(X;Y)$.

**(iii)** Finally, if $x$, $y \in X$ and $\alpha \in {\mathbb{R} \atop \mathbb{C}}$,

$$\mathbf{0}(x+y) = 0 = 0+0 = \mathbf{0}(x)+\mathbf{0}(y),$$

$$\mathbf{0}(\alpha x) = 0 = \alpha 0 = \alpha\mathbf{0}(x),$$

so $\mathbf{0} \in L(X;Y)$.

**11D Useful facts** Let $X$ and $Y$ be linear spaces and $T : X \to Y$ a linear operator. Then

**(a)** $T(0_X) = 0_Y$ and $T(-x) = -Tx$ for every $x \in X$. **P**

$$T(0_X) = T(0_X+0_X) = T(0_X)+T(0_X),$$

$$Tx+T(-x) = T(x+(-x)) = T(0_X) = 0_Y. \text{ } \mathbf{Q}$$

**(b)** $T$ is injective iff $T^{-1}[\{0_Y\}] = \{0_X\}$. **P** **(i)** By (a), $T^{-1}[\{0_Y\}] \supseteq \{0_X\}$; if $T$ is injective, they must be equal. **(ii)** If $T^{-1}[\{0_Y\}] = \{0_X\}$, that is, $Tx = 0_Y \Rightarrow x = 0_X$, and $Tx = Ty$, then

$$T(x-y) = T(x+(-y)) = Tx + T(-y) = Tx + (-Ty) = Tx + (-Tx) = 0_Y,$$

so $x - y = 0_X$ and $x = y$; thus $T$ is injective. **Q**

**(c)** If $T$ is a bijection, then $T^{-1} : Y \to X$ is a linear operator. **P** If $u, v \in Y$ and $\alpha \in \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, then

$$T^{-1}u + T^{-1}v = T^{-1}T(T^{-1}u + T^{-1}v) = T^{-1}(TT^{-1}u + TT^{-1}v) = T^{-1}(u+v),$$

$$\alpha T^{-1}u = T^{-1}T(\alpha T^{-1}u) = T^{-1}(\alpha TT^{-1}u) = T^{-1}(\alpha u).$$ **Q**

**11E Composition of linear operators** If $X$, $Y$ and $Z$ are linear spaces over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$ and $S : X \to Y$, $T : Y \to Z$ are linear operators, then $TS : X \to Z$ is a linear operator.

**proof** If $x, y \in X$ and $\alpha \in \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, then

$$(TS)(x+y) = T(S(x+y)) = T(Sx) + T(Sy) = (TS)(x) + (TS)(y),$$

$$(TS)(\alpha x) = T(S(\alpha x)) = T(\alpha Sx) = \alpha T(Sx) = \alpha(TS)(x).$$

**11F Proposition** Let $X$ and $Y$ be normed spaces and $T : X \to Y$ a linear operator. Then the following are equivalent:
(i) $T$ is continuous at 0;
(ii) $T$ is continuous;
(iii) $T$ is uniformly continuous.
(iv) there is an $M \geq 0$ such that $\|Tx\| \leq M$ whenever $\|x\| \leq 1$;
(v) there is an $M \geq 0$ such that $\|Tx\| \leq M\|x\|$ for every $x \in X$;

**proof (i)⇒(v)** Suppose that $T$ is continuous at 0. Then there must be a $\delta > 0$ such that the distance $\|Tx\|$ of $Tx$ from 0 in $Y$ is at most 1 whenever the distance $\|x\|$ from 0 in $X$ is at most $\delta$ (because $T(0) = 0$). Set $M = 1/\delta$, and consider any $x \in X$. If $x = 0$, then $Tx = 0$ so $\|Tx\| = 0 = M\|x\|$. If $x \neq 0$, then $\gamma = \|x\| > 0$; consider $x' = \delta\gamma^{-1}x$, so that $\|x'\| = \delta\gamma^{-1}\|x\| = \delta$. Then

$$\|Tx\| = \|T(\delta^{-1}\gamma x')\| = \delta^{-1}\gamma\|Tx'\| \leq M\gamma = M\|x\|.$$

as $x$ is arbitrary, $T$ satisfies (v).

**(v)⇒(iv)** is trivial.

**(iv)⇒(iii)** Assume (iv), and take any $\epsilon > 0$. Set $\delta = \epsilon/(1+M) > 0$. If $\|x - y\| \leq \delta$, consider $x' = \delta^{-1}(x-y)$. We have $\|x'\| = \delta^{-1}\|x-y\| \leq 1$ so $\|Tx'\| \leq M$; consequently

$$\|Tx - Ty\| = \|T(x-y)\| = \|T(\delta x')\| = \|\delta Tx'\| = \delta\|Tx'\| \leq \delta M \leq \epsilon.$$

As $\epsilon$ is arbitrary, $T$ is uniformly continuous.

**(iii)⇒(ii)⇒(i)** is trivial.

**11G Definition** A linear operator satisfying the conditions of 11F is called **bounded** (because it is bounded on the ball $\{x : \|x\| \leq 1\}$; of course it is not normally bounded on the whole space $X$).

For a bounded linear operator $T : X \to Y$, we write $\|T\| = \sup\{\|Tx\| : x \in X, \|x\| \leq 1\}$.

For any normed spaces $X$ and $Y$, I will write $B(X;Y)$ for the space of bounded linear operators from $X$ to $Y$.

**11H Useful fact** If $X$ and $Y$ are normed spaces and $T \in B(X;Y)$, then $\|Tx\| \leq \|T\|\|x\|$ for every $x \in X$.

**proof** If $x = 0$ then $\|Tx\| = \|0\| = 0 = \|T\|\|x\|$. If $x \neq 0$, set $\gamma = \|x\| > 0$, $x' = \gamma^{-1}x$. Then $\|x'\| = \gamma^{-1}\|x\| = 1$, so

$$\|Tx\| = \|T(\gamma x')\| = \|\gamma Tx'\| = \gamma\|Tx'\| \leq \gamma\|T\| = \|T\|\|x\|.$$

**11I Theorem** Let $X$ and $Y$ be any normed spaces over $\substack{\mathbb{R}\\ \mathbb{C}}$. Then $B(X;Y)$ is a linear subspace of $L(X;Y)$, therefore a linear space in its own right, and is a normed space when given the norm described in 11G.

**proof (i)** If $S$, $T \in B(X;Y)$ then, for any $x \in X$ such that $\|x\| \le 1$,

$$\|(S+T)(x)\| = \|Sx + Tx\| \le \|Sx\| + \|Tx\| \le \|S\| + \|T\|;$$

as $x$ is arbitrary, $S + T \in B(X;Y)$ and $\|S+T\| \le \|S\| + \|T\|$.

**(ii)** If $T \in B(X;Y)$ and $\alpha \in \substack{\mathbb{R}\\ \mathbb{C}}$, then, for any $x \in X$ such that $\|x\| \le 1$,

$$\|(\alpha T)(x)\| = \|\alpha Tx\| = |\alpha|\|Tx\| \le |\alpha|\|T\|;$$

as $x$ is arbitrary, $\alpha T \in B(X;Y)$ and $\|\alpha T\| \le |\alpha|\|T\|$.

**(iii)** Writing $\mathbf{0}$ for the zero operator in $L(X;Y)$, we have $\|\mathbf{0}(x)\| = \|0\| = 0$ for every $x$, so $\mathbf{0} \in B(X;Y)$. Thus $B(X;Y)$ is a linear subspace of $L(X;Y)$.

**(iv)** If $T \in B(X;Y)$ and $\|T\| = 0$, then 11H tells us that $\|Tx\| \le \|T\|\|x\| = 0$ for every $x \in X$, so $Tx = 0$ for every $x$ and $T = \mathbf{0}$, the zero operator. By 7Gb, $\|\ \|$ is a norm on $B(X;Y)$.

**11J Theorem** Let $X$ be a normed space over $\substack{\mathbb{R}\\ \mathbb{C}}$ and $Y$ a Banach space over $\substack{\mathbb{R}\\ \mathbb{C}}$. Then $B(X;Y)$ is a Banach space.

**proof** We know from 11I that it is a normed space. Let $\langle T_n\rangle_{n\in\mathbb{N}}$ be any Cauchy sequence in $B(X;Y)$.

**(a)** For any $x \in X$,

$$\|T_m x - T_n x\| = \|(T_m - T_n)(x)\| \le \|T_m - T_n\|\|x\| \to 0$$

as $m$, $n \to \infty$, so $\langle T_n x\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in $Y$; because $Y$ is complete, it has a limit $Tx$ say in $Y$. This defines a function $T : X \to Y$.

**(b)(i)** $T : X \to Y$ is linear. **P** Let $x$, $y \in X$ and $\alpha \in \substack{\mathbb{R}\\ \mathbb{C}}$. Then for any $n \in \mathbb{N}$

$$\begin{aligned}\|T(x+y) - Tx - Ty\| &= \|T(x+y) - T_n(x+y) - Tx + T_n x - Ty + T_n y\|\\ &\le \|T(x+y) - T_n(x+y)\| + \|Tx - T_n x\| + \|Ty - T_n y\|\\ &\to 0 \text{ as } n \to \infty,\end{aligned}$$

$$\begin{aligned}\|T(\alpha x) - \alpha Tx\| &= \|T(\alpha x) - T_n(\alpha x) - \alpha Tx + \alpha T_n x\|\\ &\le \|T(\alpha x) - T_n(\alpha x)\| + \|\alpha Tx - \alpha T_n x\|\\ &= \|T(\alpha x) - T_n(\alpha x)\| + |\alpha|\|Tx - T_n x\|\\ &\to 0 \text{ as } n \to \infty,\end{aligned}$$

so

$$\|T(x+y) - Tx - Ty\| = 0 \text{ and } T(x+y) = Tx + Ty,$$

$$\|T(\alpha x) - \alpha Tx\| = 0 \text{ and } T(\alpha x) = \alpha Tx. \text{ } \mathbf{Q}$$

**(ii)** $T$ is bounded. **P** Because $\langle T_n\rangle_{n\in\mathbb{N}}$ is Cauchy, it is bounded; say $\|T_n\| \le M$ for every $n \in \mathbb{N}$. If $x \in X$ and $\|x\| \le 1$, then (because $\|\ \|$ is continuous)

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \le \sup_{n\in\mathbb{N}} \|T_n x\| \le M.$$

As $x$ is arbitrary, $T$ is bounded. **Q**

**(c)** Now let $\epsilon > 0$. Then there is an $n_0 \in \mathbb{N}$ such that $\|T_m - T_n\| \le \epsilon$ whenever $m$, $n \ge n_0$. So

$$\begin{aligned}\|T_m x - Tx\| &\le \|T_m x - T_n x\| + \|T_n x - Tx\| \le \|T_m - T_n\| + \|T_n x - Tx\|\\ &\le \epsilon + \|T_n x - Tx\|\end{aligned}$$

whenever $m, n \geq n_0$ and $\|x\| \leq 1$. Letting $n \to \infty$,

$$\|T_m x - Tx\| \leq \epsilon \text{ whenever } m \geq n_0,\ \|x\| \leq 1.$$

As $x$ is arbitrary,

$$\|T_m - T\| \leq \epsilon \text{ whenever } m \geq n_0.$$

As $\epsilon$ is arbitrary,

$$T = \lim_{n\to\infty} T_n.$$

**(d)** Thus every Cauchy sequence in $B(X;Y)$ has a limit in $B(X;Y)$, and $B(X;Y)$ is complete, therefore a Banach space.

**11K Theorem** If $X$, $Y$ and $Z$ are normed spaces over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, and $S \in B(X;Y)$, $T \in B(Y;Z)$ then $TS \in B(X;Z)$ and $\|TS\| \leq \|T\|\|S\|$.

**proof** By 11E, $TS \in L(X;Z)$. If $x \in X$ and $\|x\| \leq 1$, then

$$\|(TS)x\| = \|T(Sx)\| \leq \|T\|\|Sx\| \leq \|T\|\|S\|,$$

using 11H. So $TS$ is bounded and $\|TS\| \leq \|T\|\|S\|$.

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## 12. Dual spaces

**12A Definition** Let $X$ be any normed space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$. Write $X^*$ for $B(X;\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}})$, the space of bounded linear operators from $X$ to $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$. By 11J, this is a Banach space.

**12B Identification of duals:** $(\ell^1)^*$ For $x = (\xi_1, \xi_2, \dots) \in \ell^1$, $y = (\eta_1, \eta_2, \dots) \in \ell^\infty$ set $f_y(x) = \sum_{i=1}^\infty \xi_i \eta_i$. Then $y \mapsto f_y$ is a normed space isomorphism between $\ell^\infty$ and $(\ell^1)^*$.

**proof (a)** The first thing to check is that $f_y(x)$ is always defined, that is, that $\langle \xi_i \eta_i \rangle_{i\in\mathbb{N}}$ is always summable. But

$$\textstyle\sum_{i=1}^\infty |\xi_i \eta_i| = \sum_{i=1}^\infty |\xi_i||\eta_i| \leq \|y\|_\infty \sum_{i=1}^\infty |\xi_i| = \|y\|_\infty \|x\|_1 < \infty,$$

so in fact the series is absolutely summable. We also see that

$$|f_y(x)| = |\textstyle\sum_{i=1}^\infty \xi_i \eta_i| \leq \sum_{i=1}^\infty |\xi_i \eta_i| = \|y\|_\infty \|x\|_1.$$

**(b)** Thus $f_y : \ell^1 \to \mathbb{R}$ is defined for each $y \in \ell^\infty$. Now each $f_y$ is linear. **P** If $x$, $z \in \ell^1$ and $\alpha \in \mathbb{R}$, then

$$\begin{aligned} f_y(x+z) &= \sum_{i=1}^\infty \eta_i(\xi_i + \zeta_i) = \sum_{i=1}^\infty \eta_i \xi_i + \eta_i \zeta_i \\ &= \sum_{i=1}^\infty \eta_i \xi_i + \sum_{i=1}^\infty \eta_i \zeta_i = f_y(x) + f_y(z), \end{aligned}$$

$$\begin{aligned} f_y(\alpha x) &= \sum_{i=1}^\infty \eta_i(\alpha \xi_i) \\ &= \alpha \sum_{i=1}^\infty \eta_i \xi_i = \alpha f_y(x). \textbf{ Q} \end{aligned}$$

Also, because $f_y(x) \leq \|y\|_\infty \|x\|_1$ for every $x$ (see part (a)), $f_y$ is a bounded operator, and $f_y \in (\ell^1)^*$, with $\|f_y\| \leq \|y\|_\infty$.

**(c)** The map $y \mapsto f_y : \ell^\infty \to (\ell^1)^*$ is linear. **P** Let $y$, $z \in \ell^\infty$, $\alpha \in \mathbb{R}$. Then, for any $x \in \ell^1$,

$$f_{y+z}(x) = \sum_{i=0}^{\infty}(\eta_i + \zeta_i)\xi_i = \sum_{i=0}^{\infty}\eta_i\xi_i + \sum_{i=0}^{\infty}\zeta_i\xi_i$$
$$= f_y(x) + f_z(x) = (f_y + f_z)(x),$$

$$f_{\alpha y}(x) = \sum_{i=0}^{\infty}(\alpha\eta_i)\xi_i = \alpha\sum_{i=0}^{\infty}\eta_i\xi_i$$
$$= \alpha f_y(x) = (\alpha f_y)(x).$$

As $x$ is arbitrary, $f_{y+z} = f_y + f_z$ and $f_{\alpha y} = \alpha f_y$; as $y$, $z$ and $\alpha$ are arbitrary, $y \mapsto f_y$ is linear. **Q**

**(d)** Now $\|f_y\| = \|y\|_\infty$ for every $y \in \ell^\infty$. **P** We already know (see part (b)) that $\|f_y\| \le \|y\|_\infty$. Now, for $i \in \mathbb{N}$ define a sequence $d_i = \langle\delta_{ij}\rangle_{j\in\mathbb{N}}$ by writing $\delta_{ii} = 1$, $\delta_{ij} = 0$ if $j \ne i$. Then $d_i \in \ell^1$ and $\|d_i\|_1 = \sum_{j=1}^{\infty}|\delta_{ij}| = 1$. So $\|f_y\| \ge |f_y(d_i)|$ for every $i$. But $f_y(d_i) = \sum_{j=1}^{\infty}\eta_j\delta_{ij} = \eta_i$, so

$$\|f_y\| \ge \sup_{i\in\mathbb{N}}|f_y(d_i)| = \sup_{i\in\mathbb{N}}|\eta_i| = \|y\|_\infty.$$

Thus $\|f_y\| = \|y\|_\infty$. **Q**

Accordingly the map $y \mapsto f_y : \ell^\infty \to (\ell^1)^*$ is a norm-preserving linear operator.

**(e)** (The real point.) Every $f \in (\ell^1)^*$ is expressible as $f_y$ for some $y \in \ell^\infty$. **P** Take $d_i$ from (d) above, and set $\eta_i = f(d_i)$ for each $i \in \mathbb{N}$. Then

$$|\eta_i| = |f(d_i)| \le \|f\|\|d_i\|_1 = \|f\|$$

for each $i$, so $y = (\eta_1, \eta_2, \dots)$ belongs to $\ell^\infty$. Consider $f_y \in (\ell^1)^*$. We have $f_y(d_i) = \eta_i = f(d_i)$ for every $i$. Now let $x = (\xi_1, \xi_2, \dots)$ be any member of $\ell^1$. For each $n \in \mathbb{N}$, define $x^{(n)}$ by setting $x^{(n)} = \sum_{i=1}^{n}\xi_i d_i$, that is, $x^{(n)}$ is the sequence $(\xi_1, \dots, \xi_n, 0, 0, \dots)$. Then, because $f$ and $f_y$ are both linear,

$$f(x^{(n)}) = \sum_{i=0}^{n}\xi_i f(d_i) = \sum_{i=0}^{n}\xi_i\eta_i = f_y(x^{(n)}).$$

And this is true for every $n$. On the other hand, for each $n$,

$$\|x - x^{(n)}\|_1 = \sum_{i=n+1}^{\infty}|\xi_i| \to 0$$

as $n \to \infty$. (In fact

$$\sum_{i=n+1}^{\infty}|\xi_i| = \sum_{i=1}^{\infty}|\xi_i| - \sum_{i=1}^{n}|\xi_i| \to 0$$

by the definition of

$$\sum_{i=1}^{\infty}|\xi_i| = \lim_{n\to\infty}\sum_{i=1}^{n}|\xi_i|.)$$

So $x = \lim_{n\to\infty}x^{(n)}$ in $\ell^1$. Since $f$ and $f_y$ are both bounded linear operators, therefore continuous (11F),

$$f(x) = \lim_{n\to\infty}f(x^{(n)}) = \lim_{n\to\infty}f_y(x^{(n)}) = f_y(x).$$

As $x$ is arbitrary, $f = f_y$. **Q**

So the map $y \mapsto f_y : \ell^\infty \to (\ell^1)^*$ is surjective and is a normed space isomorphism.

**12C Identification of duals: $(\boldsymbol{c}_0)^*$** For $x = (\xi_1, \xi_2, \dots) \in \boldsymbol{c}_0$, $y = (\eta_1, \eta_2, \dots) \in \ell^1$ set $f_y(x) = \sum_{i=1}^{\infty}\xi_i\eta_i$. Then $y \mapsto f_y$ is a normed space isomorphism between $\ell^1$ and $(\boldsymbol{c}_0)^*$.

**proof (a)** The first thing to check is that $f_y(x)$ is always defined, that is, that $\langle\xi_i\eta_i\rangle_{i\in\mathbb{N}}$ is always summable. But

$$\sum_{i=1}^{\infty}|\xi_i\eta_i| = \sum_{i=1}^{\infty}|\xi_i||\eta_i| \le \|x\|_\infty\sum_{i=1}^{\infty}|\eta_i| = \|x\|_\infty\|y\|_1 < \infty,$$

so in fact the series is absolutely summable. We also see that

$$|f_y(x)| = |\sum_{i=1}^{\infty}\xi_i\eta_i| \le \sum_{i=1}^{\infty}|\xi_i\eta_i| = \|y\|_1\|x\|_\infty.$$

**(b)** Thus $f_y : \boldsymbol{c}_0 \to \mathbb{R}$ is defined for each $y \in \ell^1$. Now each $f_y$ is linear. **P** If $x$, $z \in \boldsymbol{c}_0$ and $\alpha \in \mathbb{R}$, then

$$\begin{aligned} f_y(x+z) &= \sum_{i=1}^{\infty} \eta_i(\xi_i + \zeta_i) = \sum_{i=1}^{\infty} \eta_i\xi_i + \eta_i\zeta_i \\ &= \sum_{i=1}^{\infty} \eta_i\xi_i + \sum_{i=1}^{\infty} \eta_i\zeta_i = f_y(x) + f_y(z), \end{aligned}$$

$$\begin{aligned} f_y(\alpha x) &= \sum_{i=1}^{\infty} \eta_i(\alpha\xi_i) \\ &= \alpha \sum_{i=1}^{\infty} \eta_i\xi_i = \alpha f_y(x). \ \mathbf{Q} \end{aligned}$$

Also, because $f_y(x) \leq \|y\|_1\|x\|_\infty$ for every $x$ (see part (a)), $f_y$ is a bounded operator, and $f_y \in (\boldsymbol{c}_0)^*$, with $\|f_y\| \leq \|y\|_\infty$.

**(c)** The map $y \mapsto f_y : \ell^1 \to (\boldsymbol{c}_0)^*$ is linear. **P** Let $y$, $z \in \ell^1$, $\alpha \in \mathbb{R}$. Then, for any $x \in \boldsymbol{c}_0$,

$$\begin{aligned} f_{y+z}(x) &= \sum_{i=0}^{\infty} (\eta_i + \zeta_i)\xi_i = \sum_{i=0}^{\infty} \eta_i\xi_i + \sum_{i=0}^{\infty} \zeta_i\xi_i \\ &= f_y(x) + f_z(x) = (f_y + f_z)(x), \end{aligned}$$

$$\begin{aligned} f_{\alpha y}(x) &= \sum_{i=0}^{\infty} (\alpha\eta_i)\xi_i = \alpha \sum_{i=0}^{\infty} \eta_i\xi_i \\ &= \alpha f_y(x) = (\alpha f_y)(x). \end{aligned}$$

As $x$ is arbitrary, $f_{y+z} = f_y + f_z$ and $f_{\alpha y} = \alpha f_y$; as $y$, $z$ and $\alpha$ are arbitrary, $y \mapsto f_y$ is linear. **Q**

**(d)** Now $\|f_y\| = \|y\|_1$ for every $y \in \ell^1$. **P** We already know (see part (b)) that $\|f_y\| \leq \|y\|_1$. Now, for $n \in \mathbb{N}$ define a sequence $e^{(n)} = \langle \epsilon_j^{(n)} \rangle_{j\in\mathbb{N}}$ by setting $\epsilon_j^{(n)} = |\eta_j|/\eta_j$ if $j \leq n$ and $\eta_j \neq 0$, 0 otherwise. Then $e^{(n)} \in \boldsymbol{c}_0$ and $\|e^{(n)}\|_\infty = 1$ for every $n$. So $\|f_y\| \geq |f_y(e^{(n)})|$ for every $i$. But

$$f_y(e^{(n)}) = \textstyle\sum_{j=1}^{\infty} \eta_j\epsilon_j^{(n)} = \sum_{j=1}^{n} |\eta_j|,$$

so

$$\|y\|_1 = \lim_{n\to\infty} \textstyle\sum_{j=1}^{n} |\eta_j| = \lim_{n\to\infty} f_y(e^{(n)}) \leq \|f_y\|.$$

Thus $\|f_y\| = \|y\|_1$. **Q**

Accordingly the map $y \mapsto f_y : \ell^1 \to (\boldsymbol{c}_0)^*$ is a norm-preserving linear operator.

**(e)** (The real point.) Every $f \in (\boldsymbol{c}_0)^*$ is expressible as $f_y$ for some $y \in \ell^1$. **P** Take $d_i$ from (d) in the proof of 12B above, and set $\eta_i = f(d_i)$ for each $i \in \mathbb{N}$. If $n \in \mathbb{N}$, set $\epsilon_j^{(n)} = |\eta_j|/\eta_j$ if $j \leq n$ and $\eta_j \neq 0$, 0 otherwise; set $e^{(n)} = \langle \epsilon_j^{(n)} \rangle_{j\in\mathbb{N}}$; then

$$\textstyle\sum_{j=1}^{n} |\eta_i| = f(e^{(n)}) \leq \|f\|$$

for every $n$. Taking the limit as $n \to \infty$,

$$\|y\|_1 = \lim_{n\to\infty} \textstyle\sum_{j=1}^{n} |\eta_j| \leq \|f\| < \infty$$

and $y \in \ell^1$.

Consider $f_y \in (\boldsymbol{c}_0)^*$. We have $f_y(d_i) = \eta_i = f(d_i)$ for every $i$. Now let $x = (\xi_1, \xi_2, \dots)$ be any member of $\boldsymbol{c}_0$. For each $n \in \mathbb{N}$, define $x^{(n)}$ by setting $x^{(n)} = \sum_{i=1}^{n} \xi_i d_i$, that is, $x^{(n)}$ is the sequence $(\xi_1, \dots, \xi_n, 0, 0, \dots)$. Then, because $f$ and $f_y$ are both linear,

$$f(x^{(n)}) = \textstyle\sum_{i=0}^{n} \xi_i f(d_i) = \sum_{i=0}^{n} \xi_i\eta_i = f_y(x^{(n)}).$$

And this is true for every $n$. On the other hand, for each $n$,

$$\|x - x^{(n)}\|_\infty = \sup_{i>n} |\xi_i| \to 0$$

as $n \to \infty$, because $\lim_{i\to\infty} \xi_i = 0$. (Given $\epsilon > 0$, there is an $n_0 \in \mathbb{N}$ such that $|\xi_i| \le \epsilon$ for every $i \ge n_0$, and now $\|x - x^{(n)}\|_\infty \le \epsilon$ for every $n \ge n_0$.) So $x = \lim_{n\to\infty} x^{(n)}$ in $\boldsymbol{c}_0$. Since $f$ and $f_y$ are both bounded linear operators, therefore continuous,

$$f(x) = \lim_{n\to\infty} f(x^{(n)}) = \lim_{n\to\infty} f_y(x^{(n)}) = f_y(x).$$

As $x$ is arbitrary, $f = f_y$. **Q**

So the map $y \mapsto f_y : \ell^1 \to (\boldsymbol{c}_0)^*$ is surjective and is a normed space isomorphism.

**12D Identification of duals:** $(\ell^2)^*$ For $x = (\xi_1, \xi_2, \dots) \in \ell^2$, $y = (\eta_1, \eta_2, \dots) \in \ell^2$ set $f_y(x) = \sum_{i=1}^{\infty} \xi_i \eta_i$. Then $y \mapsto f_y$ is a normed space isomorphism between $\ell^2$ and $(\ell^2)^*$.

**proof (a)** The first thing to check is that $f_y(x)$ is always defined, that is, that $\langle \xi_i \eta_i \rangle_{i\in\mathbb{N}}$ is always summable. But in fact, for each $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} |\xi_i \eta_i| \le \sqrt{\sum_{i=1}^{n} |\xi_i|^2} \sqrt{\sum_{i=1}^{n} |\eta_i|^2} \le \|x\|_2 \|y\|_2$$

by Cauchy's inequality. Letting $n \to \infty$,

$$\sum_{i=1}^{\infty} |\xi_i \eta_i| = \lim_{n\to\infty} \sum_{i=1}^{n} |\xi_i \eta_i| \le \|x\|_2 \|y\|_2,$$

and the series is absolutely summable. Moreover,

$$|f_y(x)| = |\sum_{i=1}^{\infty} \xi_i \eta_i| \le \sum_{i=1}^{\infty} |\xi_i \eta_i| \le \|y\|_2 \|x\|_2.$$

**(b)** Thus $f_y : \ell^2 \to \mathbb{R}$ is defined for each $y \in \ell^2$. Now each $f_y$ is linear. **P** If $x$, $z \in \ell^2$ and $\alpha \in \mathbb{R}$, then

$$\begin{aligned} f_y(x+z) &= \sum_{i=1}^{\infty} \eta_i(\xi_i + \zeta_i) = \sum_{i=1}^{\infty} \eta_i \xi_i + \eta_i \zeta_i \\ &= \sum_{i=1}^{\infty} \eta_i \xi_i + \sum_{i=1}^{\infty} \eta_i \zeta_i = f_y(x) + f_y(z), \end{aligned}$$

$$\begin{aligned} f_y(\alpha x) &= \sum_{i=1}^{\infty} \eta_i(\alpha \xi_i) \\ &= \alpha \sum_{i=1}^{\infty} \eta_i \xi_i = \alpha f_y(x). \ \textbf{Q} \end{aligned}$$

Also, because $f_y(x) \le \|y\|_2 \|x\|_2$ for every $x$ (see part (a)), $f_y$ is a bounded operator, and $f_y \in (\ell^2)^*$, with $\|f_y\| \le \|y\|_2$.

**(c)** The map $y \mapsto f_y : \ell^2 \to (\ell^2)^*$ is linear. **P** Let $y$, $z \in \ell^2$, $\alpha \in \mathbb{R}$. Then, for any $x \in \ell^2$,

$$\begin{aligned} f_{y+z}(x) &= \sum_{i=0}^{\infty} (\eta_i + \zeta_i) \xi_i = \sum_{i=0}^{\infty} \eta_i \xi_i + \sum_{i=0}^{\infty} \zeta_i \xi_i \\ &= f_y(x) + f_z(x) = (f_y + f_z)(x), \end{aligned}$$

$$\begin{aligned} f_{\alpha y}(x) &= \sum_{i=0}^{\infty} (\alpha \eta_i) \xi_i = \alpha \sum_{i=0}^{\infty} \eta_i \xi_i \\ &= \alpha f_y(x) = (\alpha f_y)(x). \end{aligned}$$

As $x$ is arbitrary, $f_{y+z} = f_y + f_z$ and $f_{\alpha y} = \alpha f_y$; as $y$, $z$ and $\alpha$ are arbitrary, $y \mapsto f_y$ is linear. **Q**

**(d)** Now $\|f_y\| = \|y\|_2$ for every $y \in \ell^2$. **P** We already know (see part (b)) that $\|f_y\| \le \|y\|_2$. If $y = 0$, then $f_y = 0$ so $\|f_y\| = 0 = \|y\|_2$. Otherwise, look at

$$\|y\|_2^2 = \sum_{i=i}^{\infty} \eta_i^2 = f_y(y) \le \|f_y\| \|y\|_2.$$

Dividing both sides by $\|y\|_2$, $\|f_y\| \geq \|y\|_2$ and the two norms are equal. **Q**

Accordingly the map $y \mapsto f_y : \ell^2 \to (\ell^2)^*$ is a norm-preserving linear operator.

**(e)** (The real point.) Every $f \in (\ell^2)^*$ is expressible as $f_y$ for some $y \in \ell^2$. **P** Take $d_i$ from (d) in the proof of 12B above, and set $\eta_i = f(d_i)$ for each $i \in \mathbb{N}$. If $n \in \mathbb{N}$, set $\epsilon_j^{(n)} = \eta_j$ if $j \leq n$, 0 otherwise; set $e^{(n)} = \langle \epsilon_j^{(n)} \rangle_{j \in \mathbb{N}}$; then

$$\sum_{j=1}^{n} \eta_i^2 = f(e^{(n)}) \leq \|f\| \|e^{(n)}\|_2 = \|f\| \sqrt{\sum_{j=1}^{n} \eta_i^2}.$$

But this means that $\sqrt{\sum_{j=1}^{n} \eta_j^2} \leq \|f\|$. Taking the limit as $n \to \infty$,

$$\|y\|_2^2 = \lim_{n \to \infty} \sum_{j=1}^{n} \eta_j^2 \leq \|f\|^2 < \infty$$

and $y \in \ell^2$.

Consider $f_y \in (\ell^2)^*$. We have $f_y(d_i) = \eta_i = f(d_i)$ for every $i$. Now let $x = (\xi_1, \xi_2, \dots)$ be any member of $\ell^2$. For each $n \in \mathbb{N}$, define $x^{(n)}$ by setting $x^{(n)} = \sum_{i=1}^{n} \xi_i d_i$, that is, $x^{(n)}$ is the sequence $(\xi_1, \dots, \xi_n, 0, 0, \dots)$. Then, because $f$ and $f_y$ are both linear,

$$f(x^{(n)}) = \sum_{i=0}^{n} \xi_i f(d_i) = \sum_{i=0}^{n} \xi_i \eta_i = f_y(x^{(n)}).$$

And this is true for every $n$. On the other hand, for each $n$,

$$\|x - x^{(n)}\|_2^2 = \sum_{i=n+1}^{\infty} \xi_i^2 \to 0$$

as $n \to \infty$. So $x = \lim_{n \to \infty} x^{(n)}$ in $\ell^2$. Since $f$ and $f_y$ are both bounded linear operators, therefore continuous,

$$f(x) = \lim_{n \to \infty} f(x^{(n)}) = \lim_{n \to \infty} f_y(x^{(n)}) = f_y(x).$$

As $x$ is arbitrary, $f = f_y$. **Q**

So the map $y \mapsto f_y : \ell^2 \to (\ell^2)^*$ is surjective and is a normed space isomorphism.

**12E The Hahn-Banach theorem: normed space form** Let $X$ be a normed space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, and $Y$ a linear subspace of $X$. Let $f_0 : Y \to \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$ be a bounded linear functional. Then there is a bounded linear functional $f : X \to \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, extending $f_0$, such that $\|f\| = \|f_0\|$.

**Remark** I do not attempt to give a proof of this theorem. Any proof must be heavily dependent on the Axiom of Choice.

**12F Corollary** Let $X$ be a normed space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, and $x \in X$. Then there is an $f \in X^*$ such that $\|f\| \leq 1$ and $f(x) = \|x\|$.

**proof** If $x = 0$ this is trivial; set $f = 0$. Otherwise, set $Y = \{\alpha x : \alpha \in \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}\}$. Then $Y$ is a linear subspace of $X$. **P** $\alpha x + \beta x = (\alpha + \beta)x$, so $Y$ is closed under addition; $\alpha(\beta x) = (\alpha\beta)x$, so $Y$ is closed under scalar multiplication; $0x = 0$, so $0 \in Y$. **Q**

If $\alpha \neq \beta$, then $\alpha x \neq \beta x$. **P** $(\alpha - \beta)^{-1}(\alpha - \beta)x = 1x \neq 0$ so $\alpha x - \beta x \neq 0$ and $\alpha x \neq \beta x$. **Q** Consequently we have a well-defined function $f_0 : Y \to \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$ given by setting $f_0(\alpha x) = \alpha \|x\|$. Now $f_0 \in Y^*$ and $\|f_0\| \leq 1$. **P**

$$f_0(\alpha x + \beta x) = f_0((\alpha + \beta)x) = (\alpha + \beta)\|x\| = \alpha\|x\| + \beta\|x\| = f_0(\alpha x) + f_0(\beta x),$$

$$f_0(\alpha(\beta x)) = f_0((\alpha\beta)x) = \alpha\beta\|x\| = \alpha f_0(\beta x),$$

$$|f_0(\alpha x)| = |\alpha\|x\|| = |\alpha|\|x\| = \|\alpha x\|. \text{ } \mathbf{Q}$$

By the Hahn-Banach theorem, there is an $f \in X^*$, extending $f_0$, with $\|f\| = \|f_0\| \leq 1$, and

$$f(x) = f_0(1x) = 1\|x\| = \|x\|.$$

**Remark** If $x \neq 0$ then $\|f\| = 1$ exactly.

**12G Bidual spaces** Let $X$ be any normed space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$. Then $X^*$ is a Banach space so $X^{**} = (X^*)^* = B(X^*; \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}})$ is a Banach space.

For $x \in X$, we have a functional $\hat{x} : X^* \to {}^{\mathbb{R}}_{\mathbb{C}}$ given by setting

$$\hat{x}(f) = f(x) \text{ for every } f \in X^*.$$

**12H Theorem** For any normed space $X$, the map $x \mapsto \hat{x}$ is a norm-preserving linear map from $X$ to $X^{**}$.

**proof (a)** We must check first that $\hat{x} \in X^{**}$ for every $x \in X$. **P** If $f$, $g \in X^*$ and $\alpha \in {}^{\mathbb{R}}_{\mathbb{C}}$, then

$$\hat{x}(f+g) = (f+g)(x) = f(x) + g(x) = \hat{x}(f) + \hat{x}(g),$$

$$\hat{x}(\alpha f) = (\alpha f)(x) = \alpha f(x) = \alpha \hat{x}(f),$$

$$|\hat{x}(f)| = |f(x)| \le \|f\|\|x\|.$$

Thus $\hat{x} : X^* \to {}^{\mathbb{R}}_{\mathbb{C}}$ is a bounded linear operator. **Q**

Note also that the formulae above show that $\|\hat{x}\| \le \|x\|$.

**(b)** The map $x \mapsto \hat{x} : X \to X^{**}$ is linear. **P** If $x$, $y \in X$ and $\alpha \in {}^{\mathbb{R}}_{\mathbb{C}}$, then

$$\widehat{x+y}(f) = f(x+y) = f(x) + f(y) = \hat{x}(f) + \hat{y}(f) = (\hat{x} + \hat{y})(f),$$

$$\widehat{\alpha x}(f) = f(\alpha x) = \alpha f(x) = \alpha \hat{x}(f) = (\alpha \hat{x})(f)$$

for every $f \in X^*$; but this means that

$$\widehat{x+y} = \hat{x} + \hat{y}, \quad \widehat{\alpha x} = \alpha \hat{x},$$

which is what we need to know. **Q**

**(c)** Finally, $\|\hat{x}\| = \|x\|$ for every $x$. **P** We have already seen that $\|\hat{x}\| \le \|x\|$. On the other hand, if $x \in X$, there is an $f \in X^*$ such that $f(x) = \|x\|$ and $\|f\| \le 1$ (12E), so that

$$\|\hat{x}\| \ge |\hat{x}(f)| = |f(x)| = \|x\|. \textbf{ Q}$$

**12I Remark** Because the map $x \mapsto \hat{x}$ is linear and norm-preserving, it is an isomorphism between $X$ and its image in $X^{**}$; as normed space, we can regard $X$ as embedded in $X^{**}$.

In general, $X^{**}$ is very much larger, and more complex, than $X$. However, there are important cases in which the map $x \mapsto \hat{x}$ is surjective, so that it is an isomorphism between $X^{**}$ and $X$; such spaces $X$ are called **reflexive**.

**12J Adjoint operators** Let $X$ and $Y$ be normed spaces over ${}^{\mathbb{R}}_{\mathbb{C}}$, and $T : X \to Y$ a bounded linear operator. If $g \in Y^*$, the composition $gT$ is a bounded linear operator from $X$ to ${}^{\mathbb{R}}_{\mathbb{C}}$, by 11K; so we have a map $g \mapsto gT : Y^* \to X^*$. Set

$$T^\times(g) = gT \text{ for every } g \in Y^*;$$

this defines an operator $T^\times : Y^* \to X^*$, the **adjoint** of $T$. Note the basic formula

$$(T^\times g)(x) = (gT)(x) = g(Tx) \text{ for every } g \in Y^*, x \in X.$$

**12K Theorem** Let $X$ and $Y$ be any normed spaces over ${}^{\mathbb{R}}_{\mathbb{C}}$. Then the map $T \mapsto T^\times$ is a norm-preserving linear operator from $B(X;Y)$ to $B(Y^*;X^*)$.

**proof (a)** The first step is to show that $T^\times \in B(Y^*;X^*)$ for every $T \in B(X;Y)$. **P** If $g$, $h \in Y^*$ and $\alpha \in {}^{\mathbb{R}}_{\mathbb{C}}$, then

$$(T^\times(g+h))(x) = (g+h)(Tx) = g(Tx) + h(Tx) = (T^\times g)(x) + (T^\times h)(x) = (T^\times g + T^\times h)(x),$$

$$(T^\times(\alpha g))(x) = (\alpha g)(Tx) = \alpha g(Tx) = \alpha(T^\times g)(x) = (\alpha T^\times g)(x)$$

for every $x \in X$, so

$$T^\times(g+h) = T^\times g + T^\times h, \quad T^\times(\alpha g) = \alpha T^\times g$$

and $T^\times$ is linear. We know also from 11K that $\|T^\times g\| = \|gT\| \le \|T\|\|g\|$ for every $g$, so $T^\times$ is bounded and $\|T^\times\| \le \|T\|$. **Q**

**(b)** Next, the map $T \mapsto T^\times : B(X;Y) \to B(Y^*;X^*)$ is linear. **P** If $S$, $T \in B(X;Y)$ and $\alpha \in \substack{\mathbb{R}\\ \mathbb{C}}$, then

$$\begin{aligned}((S+T)^\times g)(x) &= g((S+T)(x))\\ &= g(Sx+Tx)\\ &= g(Sx)+g(Tx)\\ &= (S^\times g)(x)+(T^\times g)(x)\\ &= (S^\times g+T^\times g)(x)\\ &= ((S^\times+T^\times)(g))(x),\end{aligned}$$

$$\begin{aligned}((\alpha T)^\times g)(x) &= g((\alpha T)(x))\\ &= g(\alpha Tx)\\ &= \alpha g(Tx)\\ &= \alpha(T^\times g)(x)\\ &= (\alpha T^\times g)(x)\\ &= ((\alpha T^\times)(g))(x),\end{aligned}$$

for every $g \in Y^*$, $x \in X$. As $x$ is arbitrary,

$$(S+T)^\times g = (S^\times+T^\times)(g), \quad (\alpha T)^\times g = (\alpha T^\times)(g)$$

for every $g \in Y^*$. As $g$ is arbitrary,

$$(S+T)^\times = S^\times+T^\times, \quad (\alpha T)^\times = \alpha T^\times. \ \mathbf{Q}$$

**(c)** Finally, $\|T^\times\| = \|T\|$ for every $T \in B(X;Y)$. **P**

$$\begin{aligned}\|T\| &= \sup_{x\in X,\|x\|\le 1} \|Tx\|\\ &= \sup_{x\in X,\|x\|\le 1}\ \sup_{g\in Y^*,\|g\|\le 1} |g(Tx)|\end{aligned}$$

(using 12E yet again)

$$\begin{aligned}&= \sup_{g\in Y^*,\|g\|\le 1}\ \sup_{x\in X,\|x\|\le 1} |g(Tx)|\\ &= \sup_{g\in Y^*,\|g\|\le 1}\ \sup_{x\in X,\|x\|\le 1} |(T^\times g)(x)|\\ &= \sup_{g\in Y^*,\|g\|\le 1} \|T^\times g\|\\ &= \|T^\times\|. \ \mathbf{Q}\end{aligned}$$

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## 13. Three Theorems

**13A Uniform Boundedness Theorem** Let $X$ be a Banach space over $\substack{\mathbb{R}\\ \mathbb{C}}$, $Y$ a normed space over $\substack{\mathbb{R}\\ \mathbb{C}}$, and $A \subseteq B(X;Y)$ a set such that $\{Tx : T \in A\}$ is bounded in $Y$ for each $x \in X$. Then $A$ is a bounded subset of the normed space $B(X;Y)$.

**proof (a)** For each $n \in \mathbb{N}$, set

$$F_n = \{x : x \in X,\ \|Tx\| \le n \text{ for every } T \in A\}.$$

Then $F_n$ is a closed subset of $X$. **P** For each $T \in A$, the functions $x \mapsto Tx : X \to Y$ and $y \mapsto \|y\| : Y \to \mathbb{R}$ are continuous (see 11F, 8B), so their composition $x \mapsto \|Tx\| : X \to \mathbb{R}$ is continuous. Now $[-n, n]$ is a closed subset of $\mathbb{R}$, so its inverse image

$$F_{nT} = \{x : x \in X,\ \|Tx\| \in [-n, n]\}$$

must be closed in $X$. Consequently

$$F_n = \{x : x \in X,\ x \in F_{nT} \text{ for every } T \in A\}$$

is an intersection of closed sets, and therefore closed. **Q**

**(b)** Now the hypothesis '$\{Tx : T \in A\}$ is bounded for every $x \in X$' means just that

for every $x \in X$ there is an $n \in \mathbb{N}$ such that $\|Tx\| \leq n$ for every $T \in A$,

that is, that $\bigcup_{n\in\mathbb{N}} F_n = X$.

**By Baire's theorem** (6E), or rather by its corollary 6H, there is an $n \in \mathbb{N}$ such that $F_n$ includes a non-trivial ball; that is, there are an $x_0 \in X$ and a $\delta > 0$ such that $N_\delta(x_0) \subseteq F_n$.

**(c)** Now take any $T \in A$, $x \in X$ such that $\|x\| \leq 1$. Then $x_0$ and $x_0 + \delta x$ both belong to $N_\delta(x_0) \subseteq F_n$. So

$$\begin{aligned}\|Tx\| &= \frac{1}{\delta}\|T(\delta x)\| = \frac{1}{\delta}\|T(x_0 + \delta x) - Tx_0\| \\ &\leq \frac{1}{\delta}(\|T(x_0 + \delta x)\| + \|Tx_0\|) \leq \frac{1}{\delta}(n + n) = \frac{2n}{\delta}.\end{aligned}$$

As $x$ is arbitrary, $\|T\| \leq 2n/\delta$. As $T$ is arbitrary, $A$ is bounded.

**13B Corollary** Let $X$ and $Y$ be normed spaces and $T : X \to Y$ a linear operator. Suppose that the composition $gT : X \to {}^{\mathbb{R}}_{\mathbb{C}}$ is continuous for every $g \in Y^*$. Then $T \in B(X;Y)$.

**proof** For each $x \in X$, consider $\widehat{Tx} \in Y^{**}$ (12E). Set

$$A = \{\widehat{Tx} : x \in X,\ \|x\| \leq 1\} \subseteq Y^{**} = B(Y^*; {}^{\mathbb{R}}_{\mathbb{C}}).$$

For any $g \in Y^*$,

$$|\widehat{Tx}(g)| = |g(Tx)| = |(gT)(x)| \leq \|gT\|$$

whenever $\|x\| \leq 1$, so that $|\phi(g)| \leq \|gT\|$ for every $\phi \in A$ and $\{\phi(g) : \phi \in A\}$ is bounded. Moreover, $Y^*$ is a Banach space (11J). So we may use 13A to see that $A$ is bounded in $Y^{**}$, that is, there is an $M \geq 0$ such that $\|\phi\| \leq M$ for every $\phi \in A$.

But this means that

$$\|Tx\| = \|\widehat{Tx}\| \leq M$$

whenever $\|x\| \leq 1$, using 12E. So $T$ is bounded.

**13C Corollary** Let $X$ be a Banach space, $Y$ a normed space, and $\langle T_n\rangle_{n\in\mathbb{N}}$ a sequence in $B(X;Y)$. If $Tx = \lim_{n\to\infty} T_n x$ is defined in $Y$ for every $x \in X$, then $T \in B(X;Y)$.

**proof (a)** I check first that $T$ is linear. **P** (Compare 11J.) Let $x, y \in X$ and $\alpha \in {}^{\mathbb{R}}_{\mathbb{C}}$. Then for any $n \in \mathbb{N}$

$$\begin{aligned}\|T(x+y) - Tx - Ty\| &= \|T(x+y) - T_n(x+y) - Tx + T_n x - Ty + T_n y\| \\ &\leq \|T(x+y) - T_n(x+y)\|\|\|Tx - T_n x\| + \|T_y - T_n y\| \\ &\to 0 \text{ as } n \to \infty,\end{aligned}$$

$$\begin{aligned}\|T(\alpha x) - \alpha Tx\| &= \|T(\alpha x) - T_n(\alpha x) - \alpha Tx + \alpha T_n x\| \\ &\leq \|T(\alpha x) - T_n(\alpha x)\| + \|\alpha Tx - \alpha T_n x\| \\ &= \|T(\alpha x) - T_n(\alpha x)\| + |\alpha|\|Tx - T_n x\| \\ &\to 0 \text{ as } n \to \infty,\end{aligned}$$

so

$$\|T(x+y)-Tx-Ty\|=0 \text{ and } T(x+y)=Tx+Ty,$$

$$\|T(\alpha x)-\alpha Tx\|=0 \text{ and } T(\alpha x)=\alpha Tx. \ \mathbf{Q}$$

**(b)** Now for any $x\in X$, the set $\{T_nx:n\in\mathbb{N}\}$ is bounded, being the set of terms of a Cauchy sequence. By 13A (because $X$ is a Banach space), $\{T_n:n\in\mathbb{N}\}$ is bounded in $B(X;Y)$; say $\|T_n\|\le M$ for every $n\in\mathbb{N}$. Then whenever $\|x\|\le 1$

$$\|Tx\|=\|\lim_{n\to\infty}T_nx\|=\lim_{n\to\infty}\|T_nx\|\le\sup_{n\in\mathbb{N}}\|T_nx\|\le\sup_{n\in\mathbb{N}}\|T_n\|\le M.$$

As $x$ is arbitrary, $T$ is bounded.

**13D Corollary** Let $X$ be a normed space and $A\subseteq X$ a set. If $\{f(x):x\in A\}$ is bounded for every $f\in X^*$, then $A$ is bounded.

**proof** For $x\in X$, write $\hat{x}$ for the corresponding member of $X^{**}=B(X^*,{}^{\mathbb{R}}_{\mathbb{C}})$ (12G), so that $\hat{(x)}(f)=f(x)$ for every $f\in X^*$. Our hypothesis is that $\{\hat{x}(f):x\in A\}$ is bounded for every $f\in X^*$; as $X^*$ is a Banach space (11J), 13A tells us that $\{\hat{(x)}:x\in A\}$ is bounded, that is, that there is some $M$ such that $\|\hat{x}\|\le M$ for every $x\in A$. But we know also that $\|x\|=\|\hat{x}\|$ for every $x$ (12H), so $\|x\|\le M$ for every $x\in A$, and $A$ is bounded.

**13E Theorem** Let $X$ and $Y$ be Banach spaces, and $T:X\to Y$ a surjective bounded linear operator. Then there is an $M\ge 0$ such that

for every $y\in Y$ there is an $x\in X$ such that $Tx=y$ and $\|x\|\le M\|y\|$.

**proof (a)** The first step is to show that there is a $K\ge 0$ such that

$y\in\overline{\{Tx:x\in X,\ \|x\|\le K\|y\|\}}$ for every $y\in Y$.

$\mathbf{P}$ For each $n\in\mathbb{N}$, set

$$F_n=\overline{\{Tx\in X,\ \|x\|\le n\}}\subseteq Y.$$

Then each $F_n$ is a closed set and $\bigcup_{n\in\mathbb{N}}F_n$ includes $T[X]=Y$. By 6G, there are $n\in\mathbb{N}$, $y_0\in Y$ and $\delta>0$ such that $N_\delta(y_0)\subseteq F_n$. Try $K=2n/\delta$.

Take any $y\in Y$. If $y=0$, then $y=T0\in\overline{\{Tx:\|x\|\le K\|y\|\}}$; so suppose that $y\ne 0$. Set $\gamma=\|y\|$. Let $\epsilon>0$ and set $\eta=\delta\epsilon/2\gamma>0$. Then $y_0$ and $y_0+\delta\gamma^{-1}y$ both belong to $N_\delta(y_0)\subseteq F_n$, so there are $x_0$, $x_1\in X$ such that $\|x_0\|\le n$, $\|x_1\|\le n$, $\|y_0-Tx_0\|\le\eta$ and $\|y_0+\delta\gamma^{-1}y-Tx_1\|\le\eta$. Consider $x=\gamma\delta^{-1}(x_1-x_0)$. Then

$$\|x\|=\tfrac{\gamma}{\delta}\|x_1-x_0\|\le\tfrac{\gamma}{\delta}(\|x_1\|+\|x_0\|)\le\tfrac{\gamma}{\delta}(n+n)=K\gamma=K\|y\|,$$

while

$$\|y-Tx\|=\|\gamma\delta^{-1}(y_1-y_0-T(x_1-x_0))\|\le\tfrac{\gamma}{\delta}(\|y_1-Tx_1\|+\|y_0-Tx_0\|)\le\tfrac{2\eta\gamma}{\delta}=\epsilon.$$

As $\epsilon$ is arbitrary,

$$y\in\overline{\{Tx:\|x\|\le K\|y\|\}},$$

as required. $\mathbf{Q}$

**(b)** Now set $M=2K$ and take any $y\in Y$. If $y=0$ then $y=T0\in\{Tx:\|x\|\le M\|y\|\}$; so suppose that $y\ne 0$. Choose $x_0$, $x_1,\dots$ inductively, as follows. $x_0\in X$ is to be such that $\|x_0\|\le K\|y\|$ and $\|y-Tx_0\|\le\frac{1}{2}\|y\|$. Given $x_0,\dots,x_n$, $x_{n+1}\in X$ is to be such that $\|x_{n+1}\|\le K\|y-Tx_0-\ldots-Tx_n)\|$ and $\|y-Tx_0-\ldots-Tx_n-Tx_{n+1}\|\le 2^{-n-2}\|y\|$.

Set $z_n=x_0+\ldots+x_n$ for each $n$, so that $y-Tx_0-\ldots-Tx_n=y-Tz_n$; we have $\|y-Tz_n\|\le 2^{-n-1}\|y\|$ for every $n$, and $\|x_{n+1}\|\le K\|y-Tz_n\|\le 2^{-n-1}K\|y\|$ for every $n$. Also $\|x_0\|\le K\|y\|$, so in fact $\|x_n\|\le 2^{-n}K\|y\|$ for every $n\ge 0$.

It follows that $z=\lim_{n\to\infty}z_n$ exists in $X$ and that $\|z\|\le M\|y\|$. $\mathbf{P}$ For $m<n$ we have

$$\begin{aligned}\|z_n-z_m\|&=\|x_{m+1}+x_{m+1}+\ldots+x_m\|\le\|x_{m+1}\|+\ldots+\|x_n\|\\&\le 2^{-m-1}K\|y\|+\ldots+2^{-n}K\|y\|\le 2^{-m}K\|y\|;\end{aligned}$$

so for all $m$, $n \in \mathbb{N}$ we have $\|z_m - z_n\| \le 2^{-\min(m,n)}K\|y\| \to 0$ as $m$, $n \to \infty$, and $\langle z_n\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence. *Because $X$ is complete*, it has a limit $z \in X$. Moreover, for any $n \in \mathbb{N}$,

$$\|z_n\| = \|x_0 + \ldots + x_n\| \le \|x_0\| + \ldots + \|x_n\| \le K\|y\| = \frac{1}{2}K\|y\| + \ldots + 2^{-n}K\|y\| \le 2K\|y\| = M\|y\|,$$

so $\|z\| = \lim_{n\to\infty}\|z_n\| \le M\|y\|$. **Q**

But, because $T$ is continuous (this is the point at which we use the hypothesis that $T$ is a bounded operator!),

$$Tz = \lim_{n\to\infty} Tz_n = y.$$

Thus $y = Tz \in \{Tx : \|x\| \le M\|y\|\}$, as required.

**13F Corollary: the Open Mapping Theorem** Let $X$ and $Y$ be Banach spaces, and $T : X \to Y$ a surjective bounded linear operator. Then $T[G]$ is an open subset of $Y$ for every open set $G \subseteq X$.

**proof** Let $M \ge 0$ be such that

for every $y \in Y$ there is an $x \in X$ such that $Tx = y$ and $\|x\| \le M\|y\|$.

Let $y_0 \in T[G]$. Then there is an $x_0 \in G$ such that $Tx_0 = y_0$. Because $G$ is open, there is a $\delta > 0$ such that $N_\delta(x_0) \subseteq G$. Set $\eta = \delta/(M+1) > 0$. If $y \in N_\eta(y_0)$, take $z \in X$ such that $\|z\| \le M\|y - y_0\|$ and $Tz = y - y_0$; then $\|z\| \le M\eta \le \delta$, so $x_0 + z \in N_\delta(x_0) \subseteq G$ and

$$y = y_0 + Tz = T(x_0 + z) \in T[G].$$

As $y$ is arbitrary, $N_\eta(y_0) \subseteq T[G]$. As $y_0$ is arbitrary, $T[G]$ is open.

**13G Corollary** Let $X$ and $Y$ be Banach spaces, and $T : X \to Y$ a continuous linear bijection. Then $T^{-1} : Y \to X$ is a continuous linear operator.

**proof** By 11Dc, $T^{-1}$ is linear. By 13E, there is an $M \ge 0$ such that

for every $y \in Y$ there is an $x \in X$ such that $Tx = y$ and $\|x\| \le M\|y\|$;

but as $T$ is injective, the only $x$ such that $Tx = y$ is $T^{-1}y$, so this tells us just that $\|T^{-1}y\| \le M\|y\|$ for every $y$, so that $T^{-1}$ is a bounded operator.

**13H Remarks** It is commonplace in *algebra* that the inverse of a 'good' bijective function should be 'good'; thus the inverse of a bijective group homomorphism is a group homomorphism, the inverse of a bijective linear operator is a linear operator, and so forth. In *analysis* this is relatively rare and correspondingly remarkable. For instance, the function $x \mapsto x^3 : \mathbb{R} \to \mathbb{R}$ is a bijective differentiable function, but its inverse $x \mapsto \sqrt[3]{x}$ is not differentiable at 0; the function $\theta \mapsto e^{i\theta} : \,]-\pi,\pi] \to S^1$, the unit circle, is a continuous bijective function, but its inverse $z \mapsto \arg z : S^1 \to \,]-\pi,\pi]$ is not continuous at $-1$.

Thus 13G, stating that (between Banach spaces!) the inverse of a bijective continuous linear operator is continuous, is something to take note of.

A similar phenomenon has already appeared in Chapter 3. If $X$ is a *compact* metric space, $Y$ is a metric space, and $f : X \to Y$ is a bijective continuous function, then $f^{-1} : Y \to X$ is continuous.

**13I Closed Graphs** If $X$ and $Y$ are arbitrary sets and $f : X \to Y$ is any function, its **graph** is the set $\Gamma = \{(x, f(x)) : x \in X\} \subseteq X \times Y$. If $X$ and $Y$ are metric spaces, we may say that $f$ has **closed graph** if $\Gamma$ is closed for the product metric of $X \times Y$ (see Chapter 5).

**13J The Closed Graph Theorem** Let $X$ and $Y$ be Banach spaces and $T : X \to Y$ a linear operator. If the graph of $T$ is closed, then $T \in B(X;Y)$.

**proof (a)** For $x \in X$, set $\|x\|' = \|x\| + \|Tx\|$. Then $\| \|'$ is a norm on $X$. **P** If $x$, $y \in X$ and $\alpha \in \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, then

$$\begin{aligned}\|x+y\|' &= \|x+y\| + \|T(x+y)\| = \|x+y\| + \|Tx+Ty\| \\ &\le \|x\| + \|y\| + \|Tx\| + \|Ty\| = \|x\|' + \|y\|',\end{aligned}$$

$$\|\alpha x\|' = \|\alpha x\| + \|T(\alpha x)\| = \|\alpha x\| + \alpha Tx\| = |\alpha|\|x\| + |\alpha|\|Tx\| = |\alpha|\|x\|',$$

$$\|x\|' = 0 \Rightarrow \|x\| = 0 \Rightarrow x = 0. \ \mathbf{Q}$$

**(b)** $X$ is complete under $\| \|'$. **P** Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be a $\| \|'$-Cauchy sequence in $X$. Then for any $m$, $n \in \mathbb{N}$

$$\|x_m - x_n\| \le \|x_m - x_n\|' \to 0 \text{ as } m,\, n \to \infty,$$

so $\langle x_n\rangle_{n\in\mathbb{N}}$ is a $\| \|$-Cauchy sequence and has a $\| \|$-limit $x \in X$.

Moreover,

$$\|Tx_m - Tx_n\| = \|T(x_m - x_n)\| \le \|x_m - x_n\|' \to 0 \text{ as } m,\, n \to \infty,$$

so $\langle Tx_n\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in $Y$ and has a limit $y \in Y$. But now, writing $\Gamma \subseteq X \times Y$ for the graph of $T$, $\langle (x_n, Tx_n)\rangle_{n\in\mathbb{N}}$ is a sequence in $\Gamma$ converging to $(x, y)$ in $X \times Y$. Because $\Gamma$ is closed, $(x, y) \in \Gamma$ and $y = Tx$.

Now

$$\begin{aligned}\|x - x_n\|' &= \|x - x_n\| + \|T(x - x_n)\| = \|x - x_n\| + \|Tx - Tx_n\| \\ &= \|x - x_n\| + \|Y - Tx_n\| \to 0\end{aligned}$$

as $n \to \infty$, and $x$ is the $\| \|'$-limit of $\langle x_n\rangle_{n\in\mathbb{N}}$. As $\langle x_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $X$ is $\| \|$-complete. **Q**

**(c)** Now consider the identity operator $I : X \to X$, writing $I(x) = x$, as an operator from the Banach space $(X, \| \|)$ to the Banach space $(X, \| \|)$. We have

$$\|Ix\| = \|x\| \le \|x\|'$$

for every $x \in X$, so $I$ is bounded. Also $I$ is surely linear and bijective. So its inverse is bounded, by 13G; that is, there is an $M \ge 0$ such that $\|x\|' \le M\|x\|$ for every $x \in X$. But this means that

$$\|Tx\| \le \|x\|' \le M\|x\|$$

for every $x \in X$, and $T$ is bounded.

**13K Corollary** Suppose that $x \in \mathbb{R}^{\mathbb{N}}$ is such that $x \times y \in \ell^1$ for every $y \in \ell^2$, writing $x \times y = (\xi_1\eta_1, \xi_2\eta_2, \dots)$ if $x = (\xi_1, \dots)$, $y = (\eta_1, \dots)$. Then $x \in \ell^2$.

**proof** Define $T : \ell^2 \to \ell^1$ by setting $Ty = x \times y$ for every $y \in \ell^2$. Then $T$ is linear. **P** If $y$, $z \in \ell^2$ and $\alpha \in \mathbb{R}$, then

$$T(y + z) = (\xi_1(\eta_1 + \zeta_1), \dots) = (\xi_1\eta_1 + \xi_1\zeta_1, \dots) = (\xi_1\eta_1, \dots) + (\xi_1\zeta_1, \dots) = Ty + Tz,$$

$$T(\alpha y) = (\xi_1\alpha\eta_1, \dots) = \alpha(\xi_1\eta_1, \dots) = \alpha Ty. \ \mathbf{Q}$$

Also $T$ has closed graph. **P** Suppose that $(y, z)$ belongs to the closure of the graph $\Gamma$ of $T$ in $\ell^2 \times \ell^1$. Then for any $n \in \mathbb{N}$, $\epsilon > 0$ there must be $(y', z') \in \Gamma$ such that $\|y - y'\|_2 \le \epsilon$ and $\|z - z'\|_1 \le \epsilon$. Now, writing $y' = (\eta_1', \dots)$, etc., we have $\zeta_n' = \xi_n\eta_n'$, because $z' = Ty' = x \times y'$, and also

$$|\eta_n - \eta_n'| \le \|y - y'\|_2 \le \epsilon,$$

$$|\zeta_n - \zeta_n'| \le \|z - z\|_1 \le \epsilon,$$

so that

$$|\zeta_n - \xi_n\eta_n| \le |\zeta_n - \zeta_n'| + |\xi_n||\eta_n - \eta_n'| \le \epsilon(1 + |\xi_n|).$$

As $\epsilon$ is arbitrary, $\zeta_n = \xi_n\eta_n$. As $n$ is arbitrary, $z = x \times y$ and $(y, z) \in \Gamma$. As $(y, z)$ is arbitrary, $\Gamma$ is closed. **Q**

Consequently, by 13J, $T$ is bounded; say $\|Ty\|_1 \le M\|y\|_2$ for every $y \in \ell^2$.

Now take any $n \in \mathbb{N}$. Then $x_n = (\xi_1, \dots, \xi_n, 0, 0, \dots) \in \ell^2$, and

$$Tx_n = (\xi_1^2, \dots, \xi_n^2, 0, \dots).$$

So we must have

$$\textstyle\sum_{i=1}^n \xi_i^2 = \|Tx_n\|_1 \le M\|x_n\|_2 = M\sqrt{\sum_{i=1}^n \xi_i^2}.$$

But this means that $\sum_{i=1}^n \xi_i^2 \le M^2$. As $n$ is arbitrary, $\sum_{i=1}^n \xi_i^2 \le M^2$ and $x \in \ell^2$, as claimed.