Complex Variable II Problems

1. For each of the following complex functions (regarded as defined on the natural domain associated with the formula given), find its isolated singularities, classify them (i.e., state whether they are removable singularities, poles, or essential singularities, and in the case of poles give their orders), and give the corresponding residues.
(a) $\frac{5}{z} + \frac{4i}{z^3} + \frac{1}{(2z+i)^3} + \frac{2i}{2z+i} - \frac{3}{(3z-1)^4}$.
(b) $\frac{z}{(z^2+1)^2}$.
(c) $\frac{\exp(z)}{z^4}$.
(d) $\sin(\frac{1}{z})$.
(e) $\frac{1}{\exp(z)-1}$.
(f) $\frac{1}{z} - \frac{1}{\exp(z)-1}$.
(g) $\frac{\exp(i\lambda z)}{(z^2+a^2)^2}$ where λ, a are real.
(h) $\exp(\frac{1}{1-\cos z})$. [A]

2. Find the following line integrals:
(a) $\int_{\Gamma_1} \frac{5}{z} + \frac{4i}{z^3} + \frac{1}{(2z+i)^3} + \frac{2i}{2z+i} - \frac{3}{(3z-1)^4} dz$, where Γ_1 is the unit circle, taken anticlockwise.
(b) $\int_{\Gamma_2} \frac{z}{(z^2+1)^2} dz$, where Γ_2 is the straight line from -10 to 10, followed by the upper half of the circle centre 0 radius 10.
(c) $\int_{\Gamma_2} \exp(-z^3) dz$.
(d) $\int_{\Gamma_2} \frac{\exp(i\lambda z)}{(z^2+a^2)^2} dz$ where λ, a are real. [B]

3. By integrating appropriate functions around the unit circle find
(a) $\int_0^{2\pi} \frac{1}{k+\cos\theta}$ for $k > 1$;
(b) $\int_0^{2\pi} \frac{\cos(n\theta)}{2-\cos\theta} d\theta$ for integers $n \geq 0$. (*Hint.* Find $\int_0^{2\pi} \frac{e^{in\theta}}{2-\cos\theta} d\theta$ and take the real part.) What happens if $n \leq 0$? [C]

4. (a) Find the isolated singularities of $\frac{e^{imz}}{1+z^4}$, where $m \in \mathbb{R}$, and the residue at each.
(b) Writing Γ_R for the semicircle from R to $-R$ in the upper half-plane, where $R > 1$, show that

$$\lim_{R\to\infty} \left| \int_{\Gamma_R} \frac{1}{1+z^4} dz \right| = 0.$$

(c) By considering the integral of $\frac{1}{1+z^4}$ around Γ_R and its diameter, find $\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx$.
(d) Find $\int_0^{\infty} \frac{\cos mx}{1+x^4} dx$ for $m \in \mathbb{R}$. Explain why you need different arguments for the cases $m > 0$, $m = 0$ and $m < 0$. [D]

5. (a) Find the isolated singularities of $\frac{e^{imz}}{z(1+z^2)}$ and the residue at each.
(b) Give the first three non-zero terms of the Taylor series centered at 0 for the functions $\frac{1}{1+z^2}$, $\frac{e^{imz}}{1+z^2}$.
(c) By integrating $\frac{e^{iz}}{z(1+z^2)}$ around a curve made up of a semicircle in the upper half-plane from R to $-R$, a straight-line segment from $-R$ to $-\epsilon$, another semicircle from $-\epsilon$ to ϵ, and a straight-line segment back to R, where $0 < \epsilon < 1 < R$, find $\int_{-\infty}^{\infty} \frac{\sin x}{x(1+x^2)} dx$.
[E]

6. (a) For each of the following formulae, state the natural domain on which it defines an analytic function, and give its set of values:

(i) $\mathrm{Ln}(-z)$ (ii) $\mathrm{Ln}(z) - \mathrm{Ln}(-z)$ (iii) $1 + z^2$
(iv) $\mathrm{Ln}(1 + z^2)$ (v) $\mathrm{Ln}(1 - \frac{1}{z^2})$ (vi) $\frac{1+z}{1-z}$
(vii) $\frac{1}{2}\mathrm{Ln}(\frac{1+z}{1-z})$ (viii) $z^{-\frac{1}{3}}$ (ix) $(z^2)^{\frac{1}{2}}$
(x) z^i.

(b) Which of the functions in (a) are injective, and which are not? For the injective ones, give formulae for their inverses.

[F]

7. (a) Find the isolated singularities of $\frac{\mathrm{Ln}(z)}{1+z^2}$ and the residue at each.

(b) By integrating $\frac{\mathrm{Ln}(z)}{1+z^2}$ around a curve approximating to that made up of a semicircle in the upper half-plane from R to $-R$, a straight-line segment from $-R$ to $-\epsilon$, another semicircle from $-\epsilon$ to ϵ, and a straight-line segment back to R, where $0 < \epsilon < 1 < R$, find $\int_0^\infty \frac{\ln x}{1+x^2}dx$.

[G]

8. (a) Using `MAPLE`, or otherwise, find the integrals

$$\int_0^\infty \frac{1}{1+x^4}dx, \quad \int_0^\infty \frac{x}{1+x^4}dx, \quad \int_0^\infty \frac{x^2}{1+x^4}dx.$$

(b) By integrating $\frac{z^\alpha}{1+z^4}$ around a curve approximating to that made up of an anticlockwise circle, centre 0, starting and finishing at $-R$, a straight-line segment from $-R$ to $-\epsilon$, a clockwise circle centre 0 starting and finishing at $-\epsilon$, and a straight-line segment back to $-R$, where $0 < \epsilon < 1 < R$, show that

$$\int_0^\infty \frac{x^\alpha}{1+x^4}dx = -\frac{\pi}{2\sin\pi\alpha}(\cos\frac{\pi(\alpha+1)}{4} + \cos\frac{3\pi(\alpha+1)}{4})$$

whenever $-1 < \alpha < 3$ and α is not an integer.

(c) Explain exactly where in your arguments you used each of the hypotheses '$\alpha > -1$', '$\alpha < 3$' and 'α is not an integer'.

(d) Find

$$\lim_{\alpha\to k} \frac{\pi}{2\sin\pi\alpha}(\cos\frac{\pi(\alpha+1)}{4} + \cos\frac{3\pi(\alpha+1)}{4})$$

for $k = 0, 1, 2$, and compare the limits with your answers in (a).

[H]

9. Find $\int_0^\infty \frac{x^\alpha}{x^2+2x+2}dx$. For which values of α is your solution valid? [HA]

10. Find $J(\alpha) = \int_0^\infty \frac{x^\alpha}{(x+1)^2}dx$. For which α is your solution valid? Find $\lim_{\alpha\to 0} J(\alpha)$ and show that it is equal to $\int_0^\infty \frac{1}{(x+1)^2}dx$. [HB]

11. Show that $\sum_{n=1}^\infty \frac{1}{n^2+a^2} = \frac{1}{2}(\frac{\pi\cosh\pi a}{a\sinh\pi a} - \frac{1}{a^2})$ for real $a \neq 0$, and find the limit of this expression as $a \to 0$. [I]

12. Find $\sum_{n=0}^\infty \frac{1}{(n+\frac{1}{2})^2}$. [IA]

13. (a) Find the inverse of the Möbius transformation

$$w = \frac{2z+1}{3z+4}.$$

(b) For this transformation, find the images of (i) the unit circle (ii) the real axis (iii) the imaginary axis (iv) the upper half-plane (v) the open unit disc. [J]

14. (a) Find a Möbius transformation $w = \frac{az+b}{cz+d}$ which moves the real axis to the unit circle and 0 to 1.
(b) Find a Möbius transformation which moves the real axis to the unit circle and the unit circle to the real axis. [K]

15. (a) Find a Möbius transformation $w = \frac{az+b}{cz+d}$ which takes the points 1, $\omega = \exp(\frac{2\pi i}{3})$, ω^2 to 0, 1, -1.
(b) Find a Möbius transformation $w = \frac{az+b}{cz+d}$ which takes the line $\{z : \mathcal{R}\mathrm{e}(z) = \mathcal{I}\mathrm{m}(z)\}$ to the line $\{w : \mathcal{R}\mathrm{e}(w) + \mathcal{I}\mathrm{m}(w) = 0\}$ and the point 1 to itself.
(c) Find a Möbius transformation $w = \frac{az+b}{cz+d}$ which takes the half-plane $\{z : \mathcal{R}\mathrm{e}((1+i)z) > 1\}$ to the open unit disc $\{w : |w| < 1\}$ and the point 0 to the point 2.
(d) Let z^* be a complex number of modulus less than 1. Find a Möbius transformation $w = \frac{az+b}{cz+d}$ which takes the disc $\{z : |z - z^*| < 1\}$ to the upper half-plane $\{w : \mathcal{I}\mathrm{m}(w) > 0\}$ and the point 0 to the point i. [L]

16. (a) Sketch the image of the square with corners 0, 1, $1+i$ and i under the mapping $z \mapsto z^2$, and give formulae for its curved sides, showing that they are parts of certain conic sections.
(b) By considering the equation $(x+iy)^2 = a+ib$ find formulae for the real and imaginary parts of the square roots of a complex number $a+ib$.
(c) Sketch the image of the square with corners 0, 1, $1+i$ and i under the mapping $z \mapsto z^{1/2}$, and give formulae for its curved sides, showing that they are parts of certain conic sections.

(Recall that a *conic section* is a plane curve obtainable as a section of a double cone; that is, a circle, ellipse, parabola or double hyperbola, or (conventionally) a pair of straight lines, though normally a single point is disallowed.) [M]

17. Define an analytic function f by saying that $\mathrm{dom}(f) = \{z : |z| < 1\}$, $f(z) = \frac{2z}{z^2+1}$ for $|z| < 1$. Show that f is injective; find its set of values and its inverse. Find $f^{-1}(i)$ and give the first two non-zero terms of the Taylor series for f^{-1} about 0.

(This seems to be fairly hard.) [N]

18. (a) Use Z-transforms to solve the difference equation
$$a_{n+1} - 2a_n = 3,\ a_0 = 1.$$
(b) Use Laplace transforms to solve the differential equation
$$y' - 2y = 3,\ y(0) = 1.$$
[O]

19. (a) Use Z-transforms to solve the difference equation
$$a_{n+2} + 3a_{n+1} + 2a_n = 3,\ a_0 = 1,\ a_1 = -1.$$
(b) Use Laplace transforms to solve the differential equation
$$y'' + 3y' + 2y = 3,\ y(0) = 1,\ y'(0) = -1.$$
[Q]

20. (a) Use Z-transforms to solve the difference equation
$$a_{n+2} + 2a_{n+1} + a_n = n,\ a_0 = 0,\ a_1 = 0.$$
(b) Use Laplace transforms to solve the differential equation
$$y'' + 2y' + y = t,\ y(0) = 0,\ y'(0) = 0.$$
[R]

21. (a) Use Z-transforms to solve the difference equation
$$a_{n+2} + a_n = \cos \omega n,\ a_0 = 0,\ a_1 = 0,$$
where $\cos\omega \neq 0$; find the limit of your solution as $\omega \to \frac{\pi}{2}$.
(b) Use Laplace transforms to solve the differential equation
$$y'' + y = \cos \omega t,\ y(0) = 0,\ y'(0) = 0,$$
where $\omega \neq \pm 1$; find the limit of your solution as $\omega \to 1$. [S]

22. (a) Solve the difference equation
$$a_{n+2} + 2a_{n+1} + 2a_n = b_n, \quad a_0 = 1,\ a_1 = 0,$$
where $\langle b_n \rangle_{n\geq 0}$ is an arbitrary sequence, giving your answer in the form
$$a_n = c_n + \sum_{k=0}^{n} b_k d_{n-k}.$$
(b) Solve the differential equation
$$y'' + 2y' + 2y = f, \quad y(0) = 1,\ y'(0) = 0,$$
where $f : [0, \infty[\to \mathbb{R}$ is an arbitrary locally integrable function, giving your answer in the form
$$y(t) = g(t) + \int_0^t f(u)h(t-u)du.$$
[T]

23. Consider the function $\cot = \frac{\cos}{\sin}$.
(a) What type of singularity does cot have at 0? with what residue?
(b) What is the largest R such that cot is defined in the region $\{z : 0 < |z| < R\}$?
(c) Suppose that the Laurent series for cot in this region is
$$\cot z = \sum_{n=0}^{\infty} c_n z^n + \sum_{n=1}^{\infty} d_n z^{-n}.$$
Find the coefficients d_n for $n \geq 1$. Find the coefficients c_0, c_2, c_4, c_6. By equating coefficients in the product
$$\begin{aligned}(\ldots + d_2 z^{-2} + d_1 z^{-1} + c_0 + c_1 z + c_2 z^2 + \ldots)(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots)\\ = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots,\end{aligned}$$
or otherwise, find c_1, c_3, c_5. [U]

24. Let Γ be the circle centre 0 radius 2, taken anticlockwise. Find
$$\text{(i) } \int_\Gamma \operatorname{Ln}(1 + \tfrac{1}{2z})dz \quad \text{(ii) } \int_\Gamma z^2 \arctan(\tfrac{1}{z})dz \quad \text{(iii) } \int_\Gamma \frac{\exp(\frac{1}{z})}{z^2-1}dz.$$
(*Hint:* find the relevant terms in the appropriate Laurent series.) [V]

25. Set $f(z) = \frac{\exp\frac{1}{z}}{z-1}$. Find the coefficients of $\frac{1}{z}$ or $\frac{1}{z-1}$, as appropriate, in the Laurent series for f in each of the regions
(i) $\{z : 0 < |z| < 1\}$,
(ii) $\{z : |z| > 1\}$
(*Hint:* find the appropriate Laurent series for $\exp\frac{1}{z}$ and $\frac{1}{z-1}$)
(iii) $\{z : 0 < |z-1| < 1\}$
(*Hint:* find the appropriate coefficient in the Taylor series for $\exp(\frac{1}{w+1})$)
(iv) $\{z : |z-1| > 1\}$
(*Hint:* consider the substitution $\zeta = \frac{1}{z-1}$ and the Taylor series for $\exp(\frac{\zeta}{\zeta+1})$.) [W]

26. (a) Set $f(x) = 1$ for $-\frac{1}{2} \leq x \leq \frac{1}{2}$, 0 otherwise. Find the Fourier transform $\hat{f}$ of f.
(b) Find the convolution $f * f$.
(c) Set $h_1(x) = 1 - x$ for $0 \leq x \leq 1$, 0 otherwise. Find the Fourier transform of h_1.
(d) Set $h_2(x) = h_1(-x)$ for every x. What is the Fourier transform of h_2?
(e) How are $\hat{h}_1 + \hat{h}_2$ and $\hat{f}$ related, and why?
(f) Use your answers to parts (a)-(e) to find $\int_{-\infty}^{\infty} \frac{1}{y^2} \sin^2 \frac{y}{2} dy$. [XA]

27. (a) Set $f(x) = e^{-a|x|}$ for $x \in \mathbb{R}$, where $a > 0$. Find the Fourier transform of f.

(b) Set $g(x) = \frac{1}{x^2+b^2}$ for $x \in \mathbb{R}$, where $b > 0$. Find the Fourier transform of g (i) by using part (a) (ii) by an appropriate contour integration.

(c) Set $h(x) = \frac{1}{x^2+c^2}$ for $x \in \mathbb{R}$, where $c > 0$. Find the convolution $g * h$ (i) by using part (b) (*ii) by *either* an appropriate contour integration *or* a partial fraction expression. [XB]

28. (a) Set $f(x) = xe^{-x^2/2\sigma^2}$ for $x \in \mathbb{R}$, where $\sigma > 0$. Find the Fourier transform of f.
(b) Set $g(x) = e^{-x^2}$. Using Fourier transforms, find the convolution $g * g$.
[You may assume that the Fourier transform of $e^{-x^2/2}$ is $e^{-y^2/2}$.] [XC]

29. Using the answer to question 4, find the Fourier transform of $f(x) = \frac{1}{1+x^4}$. [XD]