MA302 Complex Variables

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6. Trick Integrals I: trigonometric polynomials and the unit circle

In the next four chapters I describe methods of calculating certain integrals and infinite sums using the calculus of residues. The first gives a (fairly) quick method of dealing with certain integrals between 0 and 2π.

6A Example Consider the integral $\int_0^{2\pi}\frac{\sin\theta}{2+\sin\theta}d\theta$.

We can convert this into an integral around the unit circle, as follows:

$$\int_\Gamma \frac{\frac{1}{2i}(z-\frac{1}{z})}{2+\frac{1}{2i}(z-\frac{1}{z})}\frac{dz}{iz}=\int_0^{2\pi}\frac{\sin\theta}{2+\sin\theta}d\theta,$$

parametrizing Γ by $z=e^{i\theta}$ for $0\le\theta\le 2\pi$, $dz=ie^{i\theta}d\theta$, so that $\frac{1}{2i}(z-\frac{1}{z})=\frac{1}{2i}(e^{i\theta}-e^{-i\theta})=\sin\theta$ and $\frac{dz}{iz}=d\theta$. Accordingly we have

$$\int_0^{2\pi}\frac{\sin\theta}{2+\sin\theta}d\theta=\int_\Gamma \frac{\frac{1}{2i}(z-\frac{1}{z})}{2+\frac{1}{2i}(z-\frac{1}{z})}\frac{dz}{iz}=\int_\Gamma\frac{z^2-1}{iz(z^2+4iz-1)}dz.$$

Now this is a suitable case for the calculus of residues. Setting $f(z)=\frac{z^2-1}{iz(z^2+4iz-1)}$ we see that f has simple poles at 0 and $i(-2\pm\sqrt{3})$, of which only 0 and $i(-2+\sqrt{3})$ lie inside the unit circle; so we need to calculate

$$\begin{aligned}\operatorname*{Res}_0 f+\operatorname*{Res}_{i(-2+\sqrt{3})}f&=\frac{-1}{i(-1)}+\frac{(i(-2+\sqrt{3}))^2-1}{i(i(-2+\sqrt{3}))(i(-2+\sqrt{3})+2i+i\sqrt{3})}\\&=\frac{1}{i}+\frac{-7+4\sqrt{3}-1}{(2-\sqrt{3})2i\sqrt{3}}\\&=-i+\frac{4\sqrt{3}-8}{2i(2\sqrt{3}-3)}\\&=-i-2i\frac{(\sqrt{3}-2)(-2\sqrt{3}-3)}{-12+9}\\&=-i+\frac{2i}{3}\sqrt{3}=i(\frac{2\sqrt{3}}{3}-1).\end{aligned}$$

To get the integral, we must multiply by $2\pi i$, getting

$$2\pi(1-\frac{2\sqrt{3}}{3})\simeq -0.97.$$

6B Remarks (a) This method can, in principle, be applied to any integral of the form $\int_0^{2\pi}\frac{p(\sin\theta,\cos\theta)}{q(\sin\theta,\cos\theta)}d\theta$, where p and q are polynomials, using the translations $\cos\theta=\frac{1}{2}(e^{i\theta}+e^{-i\theta})=\frac{1}{2}(z+\frac{1}{z})$, $\sin\theta=\frac{1}{2i}(z-\frac{1}{z})$, $d\theta=\frac{dz}{iz}$. Of course the method will work only when q is such that $q(\sin\theta,\cos\theta)$ is never zero, because we must be sure that the function f we integrate has no singularities lying on the unit circle.

(b) A $\sin k\theta$ or $\cos k\theta$ can be treated as $\frac{1}{2i}(e^{ik\theta}-e^{-ik\theta})=\frac{1}{2i}(z^k-\frac{1}{z^k})$ or $\frac{1}{2}(e^{ik\theta}+e^{-ik\theta})=\frac{1}{2}(z^k+\frac{1}{z^k})$.

(c) Note however that every integral of this type can also be done with the substitution $t=\tan\frac{\theta}{2}$, $\sin\theta=\frac{2t}{1+t^2}$, $\cos\theta=\frac{1-t^2}{1+t^2}$, $d\theta=\frac{2}{1+t^2}dt$, if we first change the range of integration to $\int_{-\pi}^{\pi}$, so that $\tan\frac{\theta}{2}$ is well-defined on the whole (open) interval.

(d) As suggested in (c), note that all integrals which can be dealt with by this method at all are of functions which are periodic with period 2π, so that $\int_0^{2\pi} f(\theta)d\theta=\int_{-\pi}^{\pi} f(\theta)d\theta==\int_{\alpha}^{\alpha+2\pi} f(\theta)d\theta$ for all real α.

(e) It is also well worth while looking for elementary substitutions or other tricks which will simplify the algebra you will need. For example:

6C Example To calculate $\int_0^{2\pi}\frac{a}{a^2+\sin^2 t}dt$, where $a>0$.

Start with the substitution $\theta=2t$, so that $\sin^2 t=\frac{1-\cos 2t}{2}=\frac{1-\cos\theta}{2}$, $dt=\frac{d\theta}{2}$ and the integral becomes

$$\int_0^{4\pi}\frac{a}{a^2+\frac{1}{2}(1-\cos\theta)}\frac{d\theta}{2}=2\int_0^{2\pi}\frac{a}{2a^2+1-\cos\theta}d\theta.$$

Taking Γ to be the unit circle as usual, this is

$$2\int_\Gamma\frac{a}{2a^2+1-\frac{1}{2}(z+\frac{1}{z})}\frac{dz}{iz}=\int_\Gamma\frac{4a}{i((4a^2+2)z-z^2-1}dz=\int_\Gamma\frac{4ia}{z^2-(4a^2+2)z+1}dz.$$

Setting $f(z)=\frac{4ia}{z^2-(4a^2+2)z+1}$, f has simple poles at $z_0=2a^2+1+2a\sqrt{a^2+1}$, $z_1=2a^2+1-2a\sqrt{a^2+1}$, of which only z_1 lies within the unit circle; the residue there is $\frac{4ia}{z_1-z_0}=\frac{4ia}{-4a\sqrt{a^2+1}}=-\frac{i}{\sqrt{a^2+1}}$. Multiplying by $2\pi i$ we get $\frac{2\pi}{\sqrt{a^2+1}}$ as the value of the integral.

6D Another trick worth remembering is the real- or imaginary-part method:

Example To find $\int_0^{2\pi}\frac{\cos n\theta}{3+\cos\theta}d\theta$.

Express this as

$$\int_0^{2\pi}\frac{\mathcal{R}\mathrm{e}(e^{in\theta})}{3+\cos\theta}d\theta=\mathcal{R}\mathrm{e}\int_0^{2\pi}\frac{e^{in\theta}}{3+\cos\theta}d\theta.$$

Now if we have $z=e^{i\theta}$ as usual, $e^{in\theta}$ becomes z^n, so we find ourselves with

$$\mathcal{R}\mathrm{e}\int_\Gamma\frac{z^n}{3+\frac{1}{2}(z+\frac{1}{z})}\frac{dz}{iz}=\mathcal{R}\mathrm{e}\int_\Gamma\frac{2z^n}{i(z^2+6z+1)}dz.$$

Setting $f(z)=\frac{2z^n}{i(z^2+6z+1)}$, then, *provided* $n\geq 0$, f has singularities only at $z_0=-3+2\sqrt{2}$, $z_1=-3-2\sqrt{2}$, of which only z_0 lies within the unit circle, and the residue at z_0 is $\frac{2z_0^n}{i(z_0-z_1)}=\frac{-2i}{4\sqrt{2}}z_0^n=-\frac{i\sqrt{2}}{4}z_0^n$. Multiplying by $2\pi i$, the integral is $\frac{\pi\sqrt{2}}{2}(-3+2\sqrt{2})^n$. Of course this is real, so this is also $\int_0^{2\pi}\frac{\cos n\theta}{3+\cos\theta}d\theta$.

This argument works only for $n\geq 0$. But of course cos is an even function, so for general n we have

$$\int_0^{2\pi}\frac{\cos n\theta}{3+\cos\theta}d\theta=\int_0^{2\pi}\frac{\cos|n|\theta}{3+\cos\theta}d\theta=\frac{\pi\sqrt{2}}{2}(-3+2\sqrt{2})^{|n|}.$$

Note that discovering that

$$\int_\Gamma\frac{z^n}{3+\frac{1}{2}(z+\frac{1}{z})}\frac{dz}{iz}=\int_0^{2\pi}\frac{e^{in\theta}}{3+\cos\theta}d\theta$$

was real implied that its imaginary part was zero, that is, that

$$\int_0^{2\pi}\frac{\sin n\theta}{3+\cos\theta}d\theta=0$$

for $n\geq 0$ (and therefore for all integers n). This is of course true, because $\frac{\sin n\theta}{3+\cos\theta}$ is an odd function, so its integral over $[0,2\pi]$, which must be equal to its integral over $[-\pi,\pi]$, is surely 0.

Note also that $\lim_{n\to\infty}\int_0^{2\pi}\frac{\cos n\theta}{3+\cos\theta}d\theta = 0$. It is generally true that for any integrable function h on $[0,2\pi]$, $\sum_{n=0}^{\infty}|\int_0^{2\pi}h(\theta)\cos n\theta\,d\theta|^2<\infty$, so that $\lim_{n\to\infty}\int_0^{2\pi}h(\theta)\cos n\theta\,d\theta = 0$.

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7. Trick integrals II: Big Semicircle Method

The next method is a very much more powerful and useful technique which handles integrals of the type $\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}\cos\lambda x\,dx$, $\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}\sin\lambda x\,dx$ where p and q are polynomials with real coefficients.

The idea is to express these as the real and imaginary parts of $f(x)=\frac{p(x)}{q(x)}e^{i\lambda x}$, and to integrate f around a large contour which runs from $-R$ to R along the axis and then back from R to $-R$ along a semi-circle, centre 0, in the upper half-plane. The integral along the straight-line portion will be $\int_{-R}^{R}f(x)dx$, which will approximate the integral $\int_{-\infty}^{\infty}f(x)dx$ which we seek; the integral along the whole contour will, we hope, be calculable using the residues of the complex function $f(z)=\frac{p(z)}{q(z)}e^{i\lambda z}$; and it remains only to deal with the integral along the curved part. Typically, there will be no hope of getting an exact expression for this, and the best we can do is to find an adequate approximation as $R\to\infty$; in fact, in all the cases I shall look at here, the limit as $R\to\infty$ of the integral along the curved section is zero.

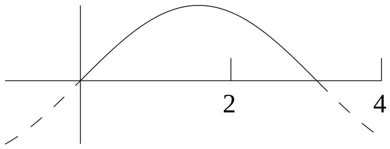
A basic tool in dealing with this is the following.

7A Lemma (a) If $\lambda>0$, then $\int_0^{\pi}e^{-\lambda\sin\theta}d\theta\le\frac{\pi}{\lambda}$.

(b) If λ, $R>0$ and f is a continuous complex-valued function defined (at least) on the semicircle Γ_R running from R to $-R$ in the upper half-plane, then

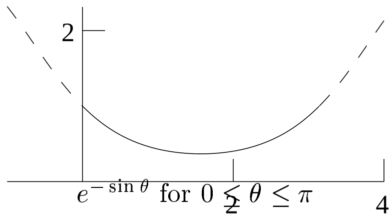
$$|\int_{\Gamma_R}f(z)e^{i\lambda z}dz|\le\frac{\pi}{\lambda}\sup_{0\le\theta\le\pi}|f(Re^{i\theta})|.$$

proof (a) Because $\sin\theta$ is symmetric about $\frac{\pi}{2}$,



$\sin\theta$ for $0\le\theta\le\pi$

so is $e^{-\lambda\sin\theta}$:

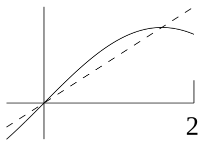


$e^{-\sin\theta}$ for $0\le\theta\le\pi$

Accordingly

$$\int_0^{\pi}e^{-\lambda\sin\theta}d\theta = 2\int_0^{\pi/2}e^{-\lambda\sin\theta}d\theta.$$

Now for $0\le\theta\le\frac{\pi}{2}$, $\sin\theta\ge\frac{2}{\pi}\theta$ – see diagram –



$\sin\theta$ and $\frac{2}{\pi}\theta$ for $0 \le \theta \le \frac{\pi}{2}$

so

$$e^{-\lambda\sin\theta} \le e^{-2\lambda\theta/\pi},$$

and

$$\int_0^{\pi/2} e^{-\lambda\sin\theta}d\theta \le \int_0^{\pi/2} e^{-2\lambda\theta/\pi}d\theta = \Big[-\frac{\pi}{2\lambda}e^{-2\lambda\theta/\pi}\Big]_0^{\pi/2} = \frac{\pi}{2\lambda}(1-e^{-\lambda}) \le \frac{\pi}{2\lambda}.$$

Multiplying by 2,

$$\int_0^{\pi} e^{-\lambda\sin\theta}d\theta \le \frac{\pi}{\lambda}.$$

(b) Set $M = \sup_{0\le\theta\le\pi}|f(Re^{i\theta}|$. Parametrizing Γ_R by $z = Re^{i\theta}$ for $0 \le \theta \le \pi$, $dz = iRe^{i\theta}d\theta$, we get

$$\begin{aligned}
|\int_{\Gamma_R} f(z)e^{i\lambda z}dz| &= |\int_0^{\pi} f(Re^{i\theta})e^{i\lambda R(\cos\theta+i\sin\theta)}iRe^{i\theta}d\theta| \\
&\le \int_0^{\pi} |f(Re^{i\theta})e^{i\lambda R(\cos\theta+i\sin\theta)}iRe^{i\theta}|d\theta \\
&= \int_0^{\pi} |f(Re^{i\theta})||e^{i\lambda R\cos\theta-\lambda R\sin\theta}||iRe^{i\theta}|d\theta \\
&= \int_0^{\pi} |f(Re^{i\theta})|e^{-\lambda R\sin\theta}R\,d\theta \\
&\le \int_0^{\pi} Me^{-\lambda R\sin\theta}R\,d\theta \\
&= MR\int_0^{\pi} e^{-\lambda R\sin\theta}d\theta \\
&\le MR\frac{\pi}{\lambda R} \\
&= \frac{\pi}{\lambda}M,
\end{aligned}$$

as claimed.

7B Corollary (‘**Jordan’s Lemma**’) Let f be a continuous complex-valued function such that $\operatorname{dom} f$ includes $\{z : |z| \ge R_0, \mathcal{I}\mathrm{m}\, z \ge 0\}$ for some $R_0 \ge 0$. Set $M_R = \sup_{0\le\theta\le\pi}|f(Re^{i\theta})|$ for $R \ge R_0$, and suppose that $\lim_{R\to\infty} M_R = 0$. Then, for any $\lambda > 0$,

$$\lim_{R\to\infty}\int_{\Gamma_R} f(z)e^{i\lambda z}dz = 0,$$

where Γ_R is the semicircle running from R to $-R$ in the upper half-plane.

proof By 7Ab,

$$|\int_{\Gamma_R} f(z)e^{i\lambda z}dz| \le \frac{\pi}{\lambda}M_R \to 0.$$

7C Example To find $\int_{-\infty}^{\infty}\frac{x\sin\lambda x}{x^2+1}$ for real λ.

I will try to set the main argument out in a standard form with four sections: (a) the idea; (b) the parametrizations, with a little algebra; (c) residues and the calculation of the whole integral; (d) the assembly of parts, with appropriate estimates of the limits required.

(a) The idea is to consider the integral of $\frac{z}{z^2+1}e^{i\lambda z}$ around the contour consisting of the straight-line segment $[-R,R]$ and the semicircle Γ_R from R to $-R$ in the upper half-plane, where R is large.

(b) We can use the standard parametrization of $[-R,R]$ by $z=x$ for $-R\le x\le R$, $dz=dx$ so the first part of the integral is

$$\begin{aligned}
I_1(R) &= \int_{-R}^{R}\frac{xe^{i\lambda x}}{x^2+1}dx\\
&= \int_{-R}^{R}\frac{x\cos\lambda x}{x^2+1}dx + i\int_{-R}^{R}\frac{x\sin\lambda x}{x^2+1}dx\\
&= i\int_{-R}^{R}\frac{x\sin\lambda x}{x^2+1}dx
\end{aligned}$$

because $\frac{x\cos\lambda x}{x^2+1}$ is an odd function. Writing $I_2(R)$ for the other part we have

$$I_2(R)=\int_{\Gamma_R}\frac{ze^{i\lambda z}}{z^2+1}dz.$$

(c) The singularities of $\frac{z}{z^2+1}e^{i\lambda z}$ are when $z^2+1=0$, that is, $z=\pm i$; of these, only i lies within the contour (taking $R>1$), so we get

$$I_1(R)+I_2(R)=2\pi i\operatorname*{Res}_{z=i}\frac{z}{z^2+1}e^{i\lambda z}.$$

Now $\frac{z}{z^2+1}=\frac{z}{(z+i)(z-i)}$, so

$$\begin{aligned}
\operatorname*{Res}_{z=i}\frac{z}{z^2+1}e^{i\lambda z} &= \lim_{z\to i}(z-i)\frac{z}{(z+i)(z-i)}e^{i\lambda z}\\
&= \lim_{z\to i}\frac{z}{z+i}e^{i\lambda z}\\
&= \frac{i}{2i}e^{-\lambda}\\
&= \frac{1}{2}e^{-\lambda},
\end{aligned}$$

and

$$I_1(R)+I_2(R)=\pi i e^{-\lambda}.$$

(d) Next, if $|z|=R>1$, then

$$\left|\frac{z}{z^2+1}\right|=\frac{|z|}{|z^2+1|}\le\frac{|z|}{|z^2|-1}=\frac{R}{R^2-1}\to 0 \text{ as } R\to\infty,$$

so by Jordan's Lemma

$$\lim_{R\to\infty}I_2(R)=0$$

provided $\lambda>0$. For the moment, therefore, let us take it that $\lambda>0$. In this case

$$\begin{aligned}
i\int_{-\infty}^{\infty}\frac{x\sin\lambda x}{x^2+1} &= \lim_{R\to\infty}I_1(R)\\
&= \pi i e^{-\lambda},
\end{aligned}$$

that is,

$$\int_{-\infty}^{\infty}\frac{x\sin\lambda x}{x^2+1}dx=\pi e^{-\lambda}.$$

(e) The argument in (d) required us to assume $\lambda > 0$. What about other λ? If $\lambda = 0$ then we have

$$\int_{-\infty}^{\infty} \frac{x \sin \lambda x}{x^2+1} dx = \int_{-\infty}^{\infty} 0\, dx = 0,$$

while if $\lambda < 0$ then

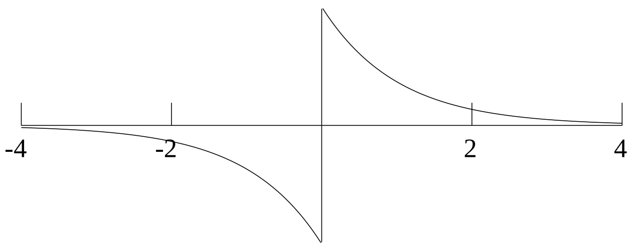
$$\int_{-\infty}^{\infty} \frac{x \sin \lambda x}{x^2+1} dx = -\int_{-\infty}^{\infty} \frac{x \sin(-\lambda) x}{x^2+1} dx = -\pi e^{-(-\lambda)} = -\pi e^{\lambda}.$$

7D Notes (a) Observe that for $\lambda \le 0$ the integral $\int_{-\infty}^{\infty} \frac{x \sin \lambda x}{x^2+1}$ is no longer $2\pi i \operatorname{Res}_{z=i} \frac{z}{z^2+1} e^{i\lambda z}$; this is because $\lim_{R\to\infty} \int_{\Gamma_R} \frac{z}{z^2+1} e^{i\lambda z} dz$ is no longer 0. When using Jordan's Lemma you *must* remember to check that you have $\lambda > 0$.

(b) Because $\frac{x \sin \lambda x}{x^2+1}$ is an even function of x, we have

$$\begin{aligned}\int_0^{\infty} \frac{x \sin \lambda x}{x^2+1} dx &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin \lambda x}{x^2+1} dx \\ &= \frac{\pi}{2} e^{-\lambda} \text{ if } \lambda > 0, \\ &= 0 \text{ if } \lambda = 0, \\ &= -\frac{\pi}{2} e^{\lambda} \text{ if } \lambda < 0.\end{aligned}$$

Plotting this as a function of λ, we have



$\int_0^{\infty} \frac{x \sin \lambda x}{x^2+1} dx$

(This is of course an odd function of λ.)

(c) The method calculated $\int_{-\infty}^{\infty} \frac{x e^{i\lambda x}}{x^2+1} dx$ (for $\lambda > 0$). The fact that the integral came out purely imaginary shows that $\int_{-\infty}^{\infty} \frac{x \cos \lambda x}{x^2+1} dx = 0$. This is obvious, because $\frac{x \cos \lambda x}{x^2+1}$ is an odd function of x.

(d) To my surprise, MAPLE doesn't seem to be able to do this one. It can do $\int_0^{\infty} \frac{\sin x}{x} dx$ (below).

(e) In the form above, the method works for integrals of the form $\int_{-\infty}^{\infty} f(x) \sin \lambda x\, dx$, $\int_{-\infty}^{\infty} f(x) \cos \lambda x\, dx$ where $f(x)$ is of the form $\frac{p(x)}{q(x)}$, both p and q being real polynomials and the degree of q being greater than the degree of p (so that $\lim_{R\to\infty} \sup_{0\le\theta\le\pi} |f(Re^{i\theta})| = 0$); we also need to have $q(x) \ne 0$ for all real x, as otherwise we can have a singularity arising on the contours we wish to integrate around. All other singularities can be avoided by taking the semicircle large enough to enclose all those in the upper half-plane.

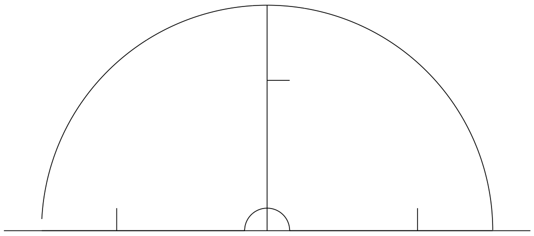
To actually calculate the residues we need, of course, we normally have to be able to factorize q.

Variants of the method, however, can give further information. I show how (sometimes!) we can cope with a singularity on the real axis.

7E Example To find $\int_0^{\infty} \frac{\sin x}{x} dx$.

(a) Begin by noting that the singularity at 0 is not necessarily disastrous, because $\lim_{x\to 0} \frac{\sin x}{x}$ is finite; accordingly this will not give rise to any difficulty in defining the real integral, though it will certainly affect the complex integral we shall need to consider.

This time, we integrate $\frac{e^{iz}}{z}$ (*not* $\frac{\sin z}{z}$!) around a more complex contour, designed to avoid the singularity at 0:



Dodging 0

In this contour, we first proceed along the straight line segment $[-R, -\epsilon]$ (where $0 < \epsilon \ll 1 \ll R$), next along the semicircle Δ_ϵ from $-\epsilon$ to ϵ in the upper half-plane, then along the straight-line segment $[\epsilon, R]$, and finally return along the semicircle Γ_R in the upper half-plane from R to $-R$.

(b) Now we consider the four pieces:

(i) ($[-R, -\epsilon]$) Parametrize with $z = x$ for $-R \le x \le -\epsilon$, $dz = dx$,

$$\begin{aligned} I_1(\epsilon, R) &= \int_{-R}^{-\epsilon} \frac{e^{ix}}{x} dx \\ &= \int_{R}^{\epsilon} \frac{e^{-iy}}{-y}(-dy) \end{aligned}$$

(substituting $x = -y$, $dx = -dy$)

$$\begin{aligned} &= -\int_{\epsilon}^{R} \frac{e^{-iy}}{y} dy \\ &= -\int_{\epsilon}^{R} \frac{e^{-ix}}{x} dx. \end{aligned}$$

(ii) (Δ_ϵ) For reasons which will appear later, it's not yet helpful to parametrize this, so let us just write

$$I_2(\epsilon) = \int_{\Delta_\epsilon} \frac{e^{iz}}{z} dz.$$

(iii) ($[\epsilon, R]$) Parametrize with $z = x$ for $\epsilon \le x \le R$, $dz = dx$,

$$I_3(\epsilon, R) = \int_{\epsilon}^{R} \frac{e^{ix}}{x} dx.$$

(iv) (Γ_R)

$$I_4(R) = \int_{\Gamma_R} \frac{e^{iz}}{z} dz.$$

(c) The integrand $\frac{e^{iz}}{z}$ has no singularities inside the contour, so by Cauchy's theorem

$$I_1(\epsilon, R) + I_2(\epsilon) + I_3(\epsilon, R) + I_4(R) = 0.$$

(d) Now consider the various parts of this sum. We have

$$\begin{aligned} I_1(\epsilon,R)+I_3(\epsilon,R) &= -\int_\epsilon^R \frac{e^{-ix}}{x}dx+\int_\epsilon^R \frac{e^{ix}}{x}dx \\ &= \int_\epsilon^R \frac{e^{ix}-e^{-ix}}{x}dx \\ &= 2i\int_\epsilon^R \frac{\sin x}{x}dx. \end{aligned}$$

Next,

$$\lim_{R\to\infty} I_4(R)=0$$

by Jordan's Lemma, because $\lim_{R\to\infty}\sup_{0\le\theta\le\pi}|\frac{1}{Re^{i\theta}}|=\lim_{R\to\infty}\frac{1}{R}=0$. Letting $R\to\infty$, we have

$$2i\int_\epsilon^\infty \frac{\sin x}{x}dx+I_2(\epsilon)=0.$$

Accordingly the next stage is to find $\lim_{\epsilon\downarrow 0} I_2(\epsilon)$. There are important general theorems to handle such problems; they are difficult; I recommend the following approach. The point is that we have a Laurent series for $\frac{e^{iz}}{z}$:

$$\begin{aligned} \frac{e^{iz}}{z} &= \frac{1}{z}(1+iz-\frac{z^2}{2!}-i\frac{z^3}{3!}+\dots) \\ &= \frac{1}{z}+(i-\frac{1}{2}z-\frac{i}{6}z^2+\dots) \\ &= \frac{1}{z}+g(z) \text{ say,} \end{aligned}$$

where g is an analytic function defined at and near 0 (in fact, everywhere). Now because $0\in\operatorname{dom} g$, we know that $\lim_{z\to 0}|g(z)|=|\lim_{z\to 0}g(z)|=|g(0)|$ exists, so for all z small enough we shall have $|g(z)|\le |g(0)|+1=M$ say. Accordingly, for all ϵ small enough,

$$|\int_{\Delta_\epsilon} g(z)dz|\le M\times\text{length of }\Delta_\epsilon=M\pi\epsilon,$$

and

$$\lim_{\epsilon\downarrow 0}\int_{\Delta_\epsilon} g(z)dz=0.$$

Thus

$$\lim_{\epsilon\downarrow 0}\int_{\Delta_\epsilon}\frac{e^{iz}}{z}dz=\lim_{\epsilon\downarrow 0}\int_{\Delta_\epsilon}\frac{1}{z}dz.$$

But this last expression is easy to calculate; parametrizing Δ_ϵ with $z=\epsilon e^{-i\theta}$ for $-\pi\le\theta\le 0$, $dz=-i\epsilon e^{-i\theta}d\theta$, we have

$$\int_{\Delta_\epsilon}\frac{1}{z}dz=\int_{-\pi}^0\frac{1}{\epsilon e^{-i\theta}}(-i\epsilon e^{-i\theta})d\theta=\int_{-\pi}^0(-i)d\theta=-i\pi.$$

Accordingly

$$\lim_{\epsilon\downarrow 0} I_2(\epsilon)=\lim_{\epsilon\downarrow 0}(-i\pi)=-i\pi.$$

Putting this together with the results above,

$$\lim_{\epsilon\downarrow 0} 2i\int_{\epsilon}^{\infty}\frac{\sin x}{x}dx - i\pi = 0,$$

and

$$\int_0^{\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}.$$

7F Notes (a) This is a basic formula for the theory of Fourier transforms.

(b) We could have worked through the whole thing with $\int_0^\infty \frac{\sin\lambda x}{x}dx$, where $\lambda > 0$, and I suppose it would be good for you to do this. However the substitution $y = \lambda x$ reduces this at once to the case $\lambda = 1$. (Question: what happens if $\lambda < 0$?)

(c) The method is good, in principle, for other singularities on the real axis. But note that the reason for the convergence of $I_2(\epsilon)$ was the fact that 0 was a simple pole of $\frac{e^{iz}}{z}$, so that we could reduce $\lim\int_{\Delta_\epsilon}\frac{e^{iz}}{z}dz$ to $\lim\int_{\Delta_\epsilon}\frac{d_1}{z}dz = -i\pi d_1$. If it had been a pole of order 2, or worse, we should have had to worry about terms of the form

$$\begin{aligned}\int_{\Delta_\epsilon}\frac{1}{z^2}dz &= \int_{-\pi}^{0}\frac{1}{\epsilon^2 e^{-2i\theta}}(-i\epsilon e^{-i\theta})d\theta\\ &= -\frac{i}{\epsilon}\int_{-\pi}^{0} e^{i\theta}d\theta\\ &= -\frac{2}{\epsilon},\end{aligned}$$

which has no finite limit.

(d) There is also the point that what we find ourselves calculating, at best, is

$$\lim_{\epsilon\downarrow 0}\lim_{R\to\infty}\Big(\int_{-R}^{-\epsilon} f(x)e^{i\lambda x}dx + \int_{\epsilon}^{R} f(x)e^{i\lambda x}dx\Big).$$

There are cases in which this limit can exist even though neither of the limits

$$\lim_{\epsilon\downarrow 0}\lim_{R\to\infty}\int_{-R}^{-\epsilon} f(x)e^{i\lambda x}dx, \quad \lim_{\epsilon\downarrow 0}\lim_{R\to\infty}\int_{\epsilon}^{R} f(x)e^{i\lambda x}dx$$

exists, and in such cases we should not like to speak of $\int_0^\infty f(x)e^{i\lambda x}dx$.

(d) To summarise: we can hope to use this singularity-dodging technique to find $\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}\sin\lambda x$ or $\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}\cos\lambda x$ in cases where p and q are real polynomials, the degree of q is greater than that of p, $\lambda > 0$, any singularities of $\frac{p(x)}{q(x)}$ on the real axis are either removable or simple poles, and the integrand $\frac{p(x)}{q(x)}\sin\lambda x$ or $\frac{p(x)}{q(x)}\cos\lambda x$ has a finite limit at each of these singularities.

Version of 14.1.99

8. Trick integrals III: Logarithms and Fractional Powers

Further versions of big-semicircle methods give us a variety of integrals of the form $\int_0^\infty f(x)\ln x\,dx$, $\int_0^\infty x^\alpha f(x)dx$.

8A Complex Logarithms Recall the following properties of the complex function Ln:

dom(Ln) $= \{z : z \text{ is not real and } \le 0\}$;

$\mathrm{Ln}(z) = \ln(|z|) + i\arg z$;

$\exp\mathrm{Ln}\,z = z$ for every $z \in \mathrm{dom}(\mathrm{Ln})$;

$\mathrm{Ln} \exp z = x$ iff $|\mathcal{I}\mathrm{m}(z)| < \pi$.

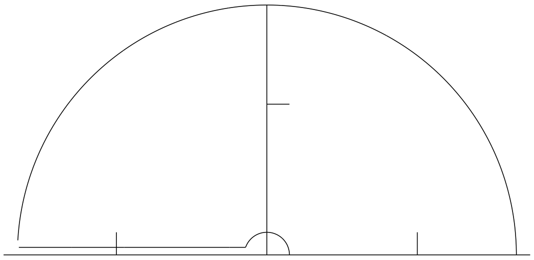
Note also that there is something arbitrary about this. We choose to cut the plane along the negative real axis from $-\infty$ to 0, and a very wide variety of alternative cuts are possible. Indeed, if θ is any real number, we can define a function Ln^* by setting

$$\mathrm{Ln}^*(z) = \mathrm{Ln}(e^{-i\theta}z) + i\theta$$

whenever this is defined; that is, $\mathrm{Ln}^*(z) = \ln(|z|) + i\arg(e^{-i\theta}z) + i\theta$ except when $e^{-i\theta}z$ is real and negative; and Ln^* will have many of the properties of Ln – for instance, $\mathrm{Ln}^*(x) = \ln x$ for real $x > 0$ (at least when $-\pi < \theta < \pi$), $\exp \mathrm{Ln}^* z = z$ for every $z \in \mathrm{dom}(\mathrm{Ln}^*)$, and $\mathrm{Ln}^{*\prime}(z) = \frac{1}{z}$. But it will be undefined along a different cut – this time, the 'bad' line will be $\{z : e^{-i\theta z}$ is real $\le 0\} = \{z : \arg z = \theta w \pm \pi\}$ if $-\pi < \theta < \pi$ – and this may be more convenient. A typical choice might be $\theta = \frac{\pi}{2}$; then Ln^* is defined except on the negative imaginary axis.

8B Example To find $\int_0^\infty \frac{\ln x}{1+x^2+x^4}dx$.

(a) The idea is to integrate $\frac{\mathrm{Ln}\, z}{1+z^2+z^4}$ along a curve consisting of a large semicircle with a small pimple at 0, as in 7E; that is, a straight-line segment $[\epsilon, R]$, where $0 < \epsilon \ll 1 \ll R$, a semicircle Γ_R from R to $-R$ in the upper half-plane, a straight-line segment $[-R, -\epsilon]$ and a semicircle Δ_ϵ, again in the upper half-plane, from $-\epsilon$ to ϵ. Of course there is an immediate difficulty: the function Ln is not defined on the segment $[-R, -\epsilon]$. There are two possible methods of dealing with this: (i) to replace Ln by Ln^*, where $\mathrm{Ln}^* z = \mathrm{Ln}(z/i) + \mathrm{Ln}(i) = \mathrm{Ln}(-iz) + \frac{i\pi}{2}$, so that Ln^* agrees with Ln on the upper half-plane and in addition sets $\mathrm{Ln}^* z = \ln(|z|) + i\pi$ if z is real and negative; (ii) to regard the interval $[-R, -\epsilon]$ as replaced by one lying infinitesimally above the axis, so that again we can treat $\mathrm{Ln}\, z$ as being $\ln(|z|) + i\pi$. Of these, (i) seems easier to justify in terms of the methods I have used in the course so far. On the other hand, (ii) is easier to use in practice, and is a technique which will be useful later in this chapter. Thus I suggest that you regard the left-hand straight section as running from $-\sqrt{R^2 - \eta^2} + \eta i$ to $-\sqrt{\epsilon^2 - \eta^2} + \eta i$, where $0 < \eta \ll \epsilon$, so that as a rule we take the limit as $\eta \downarrow 0$ in formulae before even writing them down.



Keeping clear of trouble

(b) Now for the value of the integral, calculated by using residues. The singularities of the integrand $\frac{\mathrm{Ln}\, z}{1+z^2+z^4}$ are at the solutions of $1 + z^2 + z^4 = 0$, that is, when

$$z^2 = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} = e^{\pm 2\pi i/3},$$

so that we have four simple poles at

$$e^{\pi i/3}, \quad -e^{\pi i/3} = e^{-2\pi i/3}, \quad e^{-\pi i/3}, \quad e^{2\pi i/3}.$$

Of these, the first and last lie within the contour, so these are the residues we need to calculate. Now if a is any zero of $1 + z^2 + z^4$,

$$\operatorname*{Res}_{z=a} \frac{\operatorname{Ln} z}{1+z^2+z^4} = \lim_{z\to a}(z-a)\frac{\operatorname{Ln} z}{1+z^2+z^4} = \operatorname{Ln} a \lim_{z\to a}\frac{1}{4z^3+2z} = \frac{\operatorname{Ln} a}{4a^3+2a}.$$

Taking $a = z_1 = e^{i\pi/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, we get

$$\begin{aligned}
\operatorname*{Res}_{z=z_1} \frac{\operatorname{Ln} z}{1+z^2+z^4} &= \frac{\operatorname{Ln} z_1}{4z_1^3+2z_1} \\
&= \frac{\frac{i\pi}{3}}{-4+1+i\sqrt{3}} \\
&= \frac{i\pi(-3-i\sqrt{3})}{3.12} \\
&= \frac{\pi(\sqrt{3}-3i)}{36}.
\end{aligned}$$

Similarly, taking $a = z_2 = e^{2i\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, we get

$$\begin{aligned}
\operatorname*{Res}_{z=z_2} \frac{\operatorname{Ln} z}{1+z^2+z^4} &= \frac{\operatorname{Ln} z_2}{4z_2^3+2z_2} \\
&= \frac{\frac{2i\pi}{3}}{4-1+i\sqrt{3}} \\
&= \frac{2i\pi(3-i\sqrt{3})}{3.12} \\
&= \frac{2\pi(\sqrt{3}+3i)}{36}.
\end{aligned}$$

Adding, the sum of the residues is

$$\frac{\pi(\sqrt{3}-3i+2\sqrt{3}+6i)}{36} = \frac{\pi(3\sqrt{3}+3i)}{36} = \frac{\pi}{12}(i+\sqrt{3});$$

multiplying by $2\pi i$, the integral round the whole contour is

$$\frac{\pi^2}{6}(i\sqrt{3}-1).$$

(c) Now let us consider the four sections of our contour:

(i) ($[\epsilon, R]$) Parametrize with $z = x$ for $\epsilon \le x \le R$, $dz = dx$ so

$$I_1(\epsilon, R) = \int_\epsilon^R \frac{\ln x}{1+x^2+x^4}dx.$$

(ii) (Γ_R) We shall not need to parametrize this part of the curve; write

$$I_2(R) = \int_{\Gamma_R} \frac{\operatorname{Ln} z}{1+z^2+z^4}dz.$$

(iii) ('$[-R, -\epsilon]$') Strictly speaking, parametrize with $z = x + i\eta$ for $-\sqrt{R^2-\eta^2} \le x \le -\sqrt{\epsilon^2-\eta^2}$; but it is safe to move directly to the limit, so that we have $z = x$ for $-R \le x \le -\epsilon$, $dz = dx$ and

$$\begin{aligned}
I_3(\epsilon, R) &= \int_{-R}^{-\epsilon} \frac{\ln(|x|)+i\pi}{1+x^2+x^4}dx \\
&= \int_R^\epsilon \frac{\ln y + i\pi}{1+y+y^2}(-dy)
\end{aligned}$$

(substituting $x = -y$ for $\epsilon \le y \le x$)

$$= \int_\epsilon^R \frac{\ln x + i\pi}{1+x^2+x^4} dx.$$

(iv) (Δ_ϵ) Again, there is no virtue in writing down a parametrization, so set

$$I_4(\epsilon) = \int_{\Delta_\epsilon} \frac{\operatorname{Ln} z}{1+z^2+z^4} dz.$$

We have

$$I_1(\epsilon,R) + I_2(R) + I_3(\epsilon,R) + I_4(\epsilon) = \frac{\pi^2}{6}(i\sqrt{3}-1).$$

(d) From the expressions above,

$$\begin{aligned} I_1(\epsilon,R) + I_3(\epsilon,R) &= \int_\epsilon^R \frac{\ln x}{1+x^2+x^4} dx + \int_\epsilon^R \frac{\ln x + i\pi}{1+x^2+x^4} dx \\ &= 2\int_\epsilon^R \frac{\ln x}{1+x^2+x^4} dx + i\pi \int_\epsilon^R \frac{1}{1+x^2+x^4} dx. \end{aligned}$$

Now consider the two curved sections.

(i) To estimate $I_2(R)$, observe that the length of Γ_R is πR (strictly speaking, slightly less), while the greatest value of $\frac{\operatorname{Ln} z}{1+z^2+z^4}$ on Γ_R is at most

$$M_R = \frac{\ln R + \pi}{R^4 - R^2 - 1};$$

this is because any z on Γ_R is of the form $Re^{i\theta}$ for some $\theta \in [0,\pi[$, so that

$$|\operatorname{Ln} z| = |\ln(|z|) + i\arg z| = |\ln R + i\arg z| \leq \ln R + |\arg z| \leq \ln R + \pi,$$

while $|z^4+z^2+1| \geq |z^4| - |z^2+1| \geq |z^4| - |z^2| - 1 = R^4 - R^2 - 1$. Accordingly

$$\begin{aligned} |I_2(R)| &= |\int_{\Gamma_R} \frac{\operatorname{Ln} z}{1+z^2+z^4} dz| \\ &\leq \pi R M_R \\ &= \frac{\pi R(\ln R + \pi)}{R^4 - R^2 - 1} \\ &\to 0 \text{ as } R \to \infty. \end{aligned}$$

(ii) To estimate $I_4(\epsilon)$, observe that the length of Δ_ϵ is $\pi\epsilon$ (strictly speaking, slightly less), while the greatest value of $\frac{\operatorname{Ln} z}{1+z^2+z^4}$ on Δ_ϵ is at most

$$M_\epsilon = \frac{|\ln\epsilon| + \pi}{1-\epsilon^2-\epsilon^4};$$

this is because any z on Δ_ϵ is of the form $\epsilon e^{i\theta}$ for some $\theta \in [0,\pi[$, so that

$$|\operatorname{Ln} z| = |\ln(|z|) + i\arg z| = |\ln\epsilon + i\arg z| \leq |\ln\epsilon| + |\arg z| \leq |\ln\epsilon| + \pi,$$

while $|z^4+z^2+1| \geq 1 - |z^4+z^2| \geq 1 - |z^2| - |z^4| = 1 - \epsilon^2 - \epsilon^4$. Accordingly

$$\begin{aligned} |I_2(\epsilon)| &= |\int_{\Delta_\epsilon} \frac{\operatorname{Ln} z}{1+z^2+z^4} dz| \\ &\leq \pi\epsilon M_\epsilon \\ &= \frac{\pi\epsilon(|\ln\epsilon| + \pi)}{1-\epsilon^2-\epsilon^4} \\ &\to 0 \text{ as } \epsilon \downarrow 0 \end{aligned}$$

because $\lim_{\epsilon\downarrow 0}\epsilon\ln\epsilon = 0$.

This means that, taking $R\to\infty$,

$$2\int_\epsilon^\infty \frac{\ln x}{1+x^2+x^4}dx + i\pi\int_\epsilon^\infty \frac{1}{1+x^2+x^4}dx + I_4(\epsilon) = \frac{\pi^2}{6}(i\sqrt{3}-1),$$

and taking $\epsilon\downarrow 0$

$$2\int_0^\infty \frac{\ln x}{1+x^2+x^4}dx + i\pi\int_0^\infty \frac{1}{1+x^2+x^4}dx = \frac{\pi^2}{6}(i\sqrt{3}-1).$$

Taking the real parts, we have

$$2\int_0^\infty \frac{\ln x}{1+x^2+x^4}dx = -\frac{\pi^2}{6},$$

that is,

$$\int_0^\infty \frac{\ln x}{1+x^2+x^4}dx = -\frac{\pi^2}{12} \doteqdot 0.82.$$

Taking imaginary parts, we also get

$$\pi\int_0^\infty \frac{1}{1+x^2+x^4}dx = \frac{\pi^2}{6}\sqrt{3},$$

that is

$$\int_0^\infty \frac{1}{1+x^2+x^4}dx = \frac{\pi\sqrt{3}}{6} \doteqdot 0.91.$$

8C Notes (a) This is not of course an efficient method of calculating $\int_0^\infty \frac{1}{1+x^2+x^4}dx$; but it is valuable as a check.

(b) The method will work, in principle, for integrals of the form $\int_0^\infty \frac{p(x)}{q(x)}\ln x\,dx$ where p and q are *even* real polynomials, the degree of p is less than that of q, and q has no real zeroes. We need to have even functions because the integral I_3 depends on $\frac{p(x)}{q(x)}$ for negative x and we need to combine it with the integral I_1. We need to have the degree of p less than that of q in order to make $\lim_{R\to\infty} I_2(R) = 0$; when we estimate

$$|I_2(R)| \le \pi R M_R \le \pi R(\ln R + \pi)\sup_{|z|=R}\frac{|p(z)|}{|q(z)|},$$

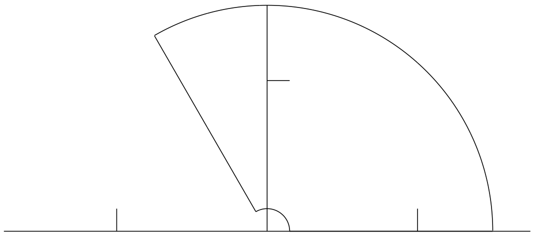
we need an extra R^2 on the bottom to control the $R\ln R$ on the top, and because p and q are both supposed to be even this will work whenever the degree of p is less than that of q.

(c) As in 7E-7F, we can cope with a zero of q on the real axis if it is compensated for by a zero of ln; that is, if it is a simple zero at 1. So, for instance, the method would give an expression for $\int_0^\infty \frac{\ln x}{x^2-1}dx$.

(d) In fact we can cope with integrals of the form $\int_0^\infty \frac{p(x)}{q(x)}\ln x\,dx$ in further cases, in which p and q are not even, by using different contours.

***8D Example** To find $\int_0^\infty \frac{\ln x}{1+x^3}dx$.

(a) This time we take the integral of $\frac{\mathrm{Ln}\,z}{1+z^3}$ around a contour consisting of a straight-line segment $[\epsilon, R]$, one third of a circle (centre 0) from R to $Re^{2\pi i/3}$, a straight line from $Re^{2\pi i/3}$ to $\epsilon e^{2\pi i/3}$, and one third of a circle (again with centre 0) back from $\epsilon e^{2\pi i/3}$ to ϵ.



A third of a cake

The reason why this will work is that the two straight-line segments again match up, and this is so because our multiplier $\frac{1}{1+z^3}$ is a function of z^3.

(b) The singularities of the integrand $\frac{\operatorname{Ln} z}{1+z^3}$ are at the cube roots of -1, that is, at -1 and $\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$. Of these, only $z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2} i = e^{\pi i/3}$ lies inside the contour, so that is the residue we need. Now

$$\begin{aligned}
\operatorname*{Res}_{z=z_1} \frac{\operatorname{Ln} z}{1+z^3} &= \lim_{z \to z_1} (z - z_1) \frac{\operatorname{Ln} z}{1+z^3} \\
&= \operatorname{Ln} z_1 \lim_{z \to z_1} \frac{1}{3z^2} \\
&= \frac{\operatorname{Ln} z_1}{3z_1^2} \\
&= \frac{i\pi}{3} \frac{1}{3} e^{-i2\pi/3} \\
&= \frac{\pi}{9} e^{-i\pi/6}.
\end{aligned}$$

Multiplying by $2\pi i$, the integral is

$$\frac{2\pi^2}{9} e^{\pi i/3} = \frac{\pi^2}{9}(1 + i\sqrt{3}).$$

(c) Working through the parametrizations, we have

$$\begin{aligned}
I_1(\epsilon, R) &= \int_\epsilon^R \frac{\ln x}{1+x^3} dx; \\
I_2(R) &= \int_{\Gamma_R} \frac{\operatorname{Ln} z}{1+z^3} dz
\end{aligned}$$

(this time writing Γ_R for one third of the circle, rather than one half);

$$I_3(\epsilon, R) = \int_{-R}^{-\epsilon} \frac{\ln|x| + \frac{2\pi i}{3}}{1-x^3} (-e^{2\pi i/3}) dx$$

(parametrizing the second straight line with $z = -e^{2\pi i/3} x$ for $-R \le x \le -\epsilon$, $dz = -e^{2\pi i/3} dx$)

$$= -e^{2\pi i/3} \int_R^\epsilon \frac{\ln y + \frac{2\pi i}{3}}{1+y^3} (-dy)$$

(substituting $x = -y$)

$$\begin{aligned}
&= \frac{1}{2}(1 - i\sqrt{3}) \int_\epsilon^R \frac{\ln x}{1+x^3} dx + \frac{\pi}{3}(\sqrt{3} + i) \int_\epsilon^R \frac{1}{1+x^3} dx; \\
I_4(\epsilon) &= \int_{\Delta_\epsilon} \frac{\operatorname{Ln} z}{1+z^3} dz,
\end{aligned}$$

writing Δ_ϵ for the final one-third circle.

From (b), we have

$$I_1(\epsilon, R) + I_2(R) + I_3(\epsilon, R) + I_4(\epsilon) = \frac{\pi^2}{9}(1 + i\sqrt{3}).$$

(d) Using the expressions above

$$I_1(\epsilon, R) + I_3(\epsilon, R) = (\frac{1}{2}(3 - i\sqrt{3}) \int_\epsilon^R \frac{\ln x}{1 + x^3} dx + \frac{\pi}{3}(\sqrt{3} + i) \int_\epsilon^R \frac{1}{1 + x^3} dx.$$

(i) To estimate $I_2(R)$, observe that the length of Γ_R is $2\pi R/3$, while the greatest value of $|\frac{\mathrm{Ln}\, z}{1+z^3}|$ on Γ_R is at most

$$M_R = \frac{\ln R + \frac{2\pi}{3}}{R^3 - 1};$$

so that

$$|I_2(R)| \leq \frac{2\pi R}{3} \frac{\ln R + \frac{2\pi}{3}}{R^3 - 1} \to 0$$

as $R \to \infty$.

(ii) To estimate $I_4(\epsilon)$, observe that the length of Δ_ϵ is $2\pi\epsilon/3$, while the greatest value of $|\frac{\mathrm{Ln}\, z}{1+z^3}|$ on Δ_ϵ is at most

$$M_\epsilon = \frac{|\ln \epsilon| + \frac{2\pi}{3}}{1 - \epsilon^3};$$

so that

$$|I_4(\epsilon)| \leq \frac{2\pi\epsilon}{3} \frac{|\ln \epsilon| + \frac{2\pi}{3}}{1 - \epsilon^3} \to 0$$

as $\epsilon \downarrow 0$.

Taking the limit as $R \to \infty$,

$$\frac{1}{2}(3 - i\sqrt{3}) \int_\epsilon^\infty \frac{\ln x}{1 + x^3} dx + \frac{\pi}{3}(\sqrt{3} + i) \int_\epsilon^\infty \frac{1}{1 + x^3} dx + I_4(\epsilon) = \frac{\pi^2}{9}(1 + i\sqrt{3});$$

taking the limit as $\epsilon \downarrow 0$,

$$\frac{1}{2}(3 - i\sqrt{3}) \int_0^\infty \frac{\ln x}{1 + x^3} dx + \frac{\pi}{3}(\sqrt{3} + i) \int_0^\infty \frac{1}{1 + x^3} dx = \frac{\pi^2}{9}(1 + i\sqrt{3}).$$

Setting $J = \int_0^\infty \frac{\ln x}{1+x^3} dx$, $K = \int_0^\infty \frac{1}{1+x^3} dx$, and taking real and imaginary parts, we have

$$\begin{aligned} \frac{3}{2} J + \frac{\pi\sqrt{3}}{3} K &= \frac{\pi^2}{9}, \\ -\frac{\sqrt{3}}{2} J + \frac{\pi}{3} K &= \frac{\pi^2\sqrt{3}}{9}. \end{aligned}$$

Solving these, we get

$$\begin{aligned} \int_0^\infty \frac{\ln x}{1 + x^3} dx &= J = -\frac{2\pi^2}{27} \simeq -0.73, \\ \int_0^\infty \frac{1}{1 + x^3} dx &= K = \frac{2\pi\sqrt{3}}{9} \simeq 1.21. \end{aligned}$$

*__8E Note__ The point of the method is that the emphasis on 'even' functions in 8B-8C is slightly misplaced; really what we need is a function $\frac{p(z^n)}{q(z^n)}$ where $n \geq 2$, p and q are real polynomials, the degree of q is greater than that of p, and q has no zeroes on $[0, \infty[$ except perhaps a simple zero at 1. We can then use a contour around $\frac{1}{n}$th of a cake, returning along the line $\{z : \arg z = \frac{2\pi i}{n}\}$.

8F Fractional powers Recall that for any $z \in \operatorname{dom} \operatorname{Ln}$ and any $a \in \mathbb{C}$ we can define

$$z^a = \exp(a \operatorname{Ln} z).$$

For real α, this becomes

$$z^\alpha = \exp(\alpha(\ln|z| + i \arg z)) = e^{\alpha \ln|z|} e^{i\alpha \arg z} = |z|^\alpha e^{i\alpha \arg z}.$$

This is defined except along a slit consisting of the negative real axis. If z is close to this slit, then

$$\begin{aligned} z^\alpha &\simeq |z|^\alpha e^{i\alpha\pi} \text{ if } \mathcal{I}\mathrm{m}\, z > 0, \\ &\simeq |z|^\alpha e^{-i\alpha\pi} \text{ if } \mathcal{I}\mathrm{m}\, z < 0. \end{aligned}$$

If α is an integer, then $e^{i\alpha\pi} = e^{-i\alpha\pi} = (-1)^\alpha$, so these two expressions match up to give $z^\alpha = (-1)^\alpha |z|^\alpha$ along the negative real axis (there is still a problem at 0 if $\alpha < 0$); but if α is not an integer, they differ, and z^α jumps by a factor of $e^{2i\alpha\pi}$ as z crosses the slit.

As with Ln, it is essentially arbitrary where we set the slit. For any real number θ, we could examine

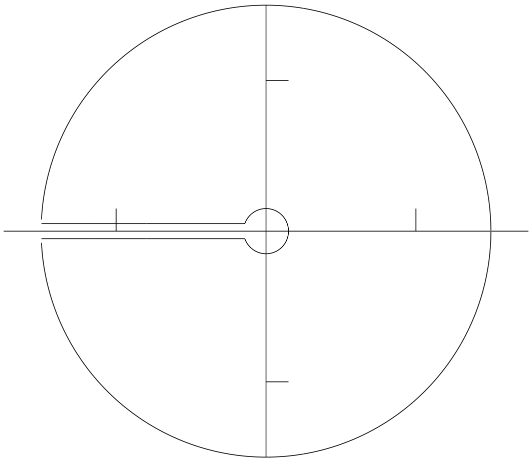
$$e^{i\theta a}(e^{-i\theta} z)^a$$

and get a version of z^a undefined along a different slit.

The fact that (for non-integer α) z^α jumps across the negative real axis gives us a chance to calculate integrals of the form $\int_0^\infty x^\alpha f(x) dx$ for certain functions f.

8G Example To find $\int_0^\infty \frac{x^\alpha}{1+x} dx$.

(a) The idea is to integrate $\frac{z^\alpha}{1-z}$ around a contour which proceeds from (nearly) $-R$ to (nearly) $-\epsilon$ in the upper half-plane, circumnavigates the origin along the circle of radius ϵ, returns to (nearly) $-R$ in the lower half-plane, and then goes round a big circle to get back to $-R$ above the axis:



Going both ways

We think of the straight-line sections as being so close to the axis that we can treat them as if they actually lay on it, but interpreting z^α as $|z|^\alpha e^{\alpha\pi i}$ for the upper section and as $|z|^\alpha e^{-\alpha\pi i}$ for the lower section.

(b) The only singularity of the integrand $\frac{z^\alpha}{1-z}$ is at 1, where it has a simple pole with residue $-1^\alpha = -1$; so the integral is $-2\pi i$.

(c) Now for the four parts of the contour:

(i) ('upper $[-R, -\epsilon]$') Parametrizing as $z = x$ for $-R \le x \le -\epsilon$, we have

$$\begin{aligned} I_1(\epsilon, R) &= \int_{-R}^{-\epsilon} \frac{|x|^\alpha e^{\alpha\pi i}}{1-x} dx \\ &= e^{\alpha\pi i} \int_R^\epsilon \frac{y^\alpha}{1+y}(-dy) \\ &= e^{\alpha\pi i} \int_\epsilon^R \frac{y^\alpha}{1+y} dy \\ &= e^{\alpha\pi i} \int_\epsilon^R \frac{x^\alpha}{1+x} dx. \end{aligned}$$

(ii) (inner circle) Write this part of the contour as Δ_ϵ, and

$$I_2(\epsilon) = \int_{\Delta_\epsilon} \frac{z^\alpha}{1-z} dz.$$

(iii) ('lower$[-R, -\epsilon]$') Parametrize as $z = -x$ for $\epsilon \le x \le R$, $dz = -dx$, to get

$$\begin{aligned} I_3(\epsilon, R) &= \int_\epsilon^R \frac{x^\alpha e^{-\alpha\pi i}}{1+x}(-dx) \\ &= -e^{-\alpha\pi i} \int_\epsilon^R \frac{x^\alpha}{1+x} dx. \end{aligned}$$

(iv) (outer circle) Write this part of the contour as Γ_R, and

$$I_4(R) = \int_{\Gamma_R} \frac{z^\alpha}{1-z} dz.$$

From (b), we have

$$I_1(\epsilon, R) + I_2(\epsilon) + I_3(\epsilon, R) + I_4(R) = -2\pi i.$$

(d)(i) Putting the expressions for I_1 and I_3 together, we have

$$\begin{aligned} I_1(\epsilon, R) + I_3(\epsilon, R) &= (e^{\alpha\pi i} - e^{-\alpha\pi i}) \int_\epsilon^R \frac{x^\alpha}{1+x} dx \\ &= 2i \sin \alpha\pi \int_\epsilon^R \frac{x^\alpha}{1+x} dx. \end{aligned}$$

(ii) To estimate $I_2(\epsilon)$, we observe that the length of Δ_ϵ is slightly less than $2\pi\epsilon$, while the greatest value of $|\frac{z^\alpha}{1-z}|$ on Δ_ϵ is at most $M_\epsilon = \frac{\epsilon^\alpha}{1-\epsilon}$. So

$$|I_2(\epsilon)| \le 2\pi\epsilon M_\epsilon = \frac{2\pi\epsilon^{\alpha+1}}{1-\epsilon} \to 0$$

as $\epsilon \downarrow 0$ *provided* that $\alpha + 1 > 0$, that is, that $\alpha > -1$.

(iii) To estimate $I_4(R)$, we observe that the length of Γ_R is slightly less than $2\pi R$, while the greatest value of $|\frac{z^\alpha}{1-z}|$ on Γ_R is at most $M_R = \frac{R^\alpha}{R-1}$. So

$$|I_2(R)| \le 2\pi R M_R = \frac{2\pi R^{\alpha+1}}{R-1} \to 0$$

as $R \to \infty$ *provided* that $\alpha + 1 < 1$, that is, that $\alpha < 0$.

(iv) Supposing that $-1 < \alpha < 0$, therefore, we may take the limit as $R \to \infty$ to get

$$2i \sin \alpha\pi \int_\epsilon^\infty \frac{x^\alpha}{1+x} dx + I_2(\epsilon) = -2\pi i,$$

and now the limit as $\epsilon \downarrow 0$ to get

$$2i \sin \alpha\pi \int_0^\infty \frac{x^\alpha}{1+x} dx = -2\pi i,$$

so that

$$\int_0^\infty \frac{x^\alpha}{1+x} dx = -\frac{\pi}{\sin \alpha\pi},$$

provided that $-1 < \alpha < 0$. You may think this looks prettier if we write it as

$$\int_0^\infty \frac{x^{-\gamma}}{1+x} dx = \frac{\pi}{\sin \gamma\pi} \text{ if } 0 < \gamma < 1.$$

8H Notes (a) In this case, the requirement $-1 < \alpha < 0$ imposed by the proof corresponds to a real restriction on the admissible values of α. For if $\alpha \ge 0$ then $\int_1^\infty \frac{x^\alpha}{1+x} dx = \infty$ (compare with $\int_1^\infty \frac{1}{2x} dx = \infty$), while if $\alpha \le -1$ then $\int_0^1 \frac{x^\alpha}{1+x} dx = \infty$ (compare with $\int_0^1 \frac{1}{2} x^\alpha dx = \infty$).

(b) The method is applicable to any integral of the form $\int_0^\infty \frac{p(x)}{q(x)} x^\alpha dx$ where p and q are real polymomials, q has no zeroes in $[0, \infty[$,

$$-1 < \alpha < \text{degree}(q) - \text{degree}(p) - 1,$$

and α is not an integer. We need $\alpha > -1$ to show that $\lim_{\epsilon\downarrow 0} I_2(\epsilon) = 0$, and $\alpha < \text{degree}(q) - \text{degree}(p) - 1$ to show that $\lim_{R\to\infty} I_4(R) = 0$; moreover, just as in 8G, these are exactly the limits needed to ensure that $\int_0^\infty \frac{p(x)}{q(x)} x^\alpha dx$ is finite. We need $\alpha \notin \mathbb{Z}$ to ensure that the two integrals $I_1(\epsilon, R)$, $I_3(\epsilon, R)$ don't cancel out; in fact their sum will always be

$$2i \sin \alpha\pi \int_\epsilon^R \frac{p(x)}{q(x} x^\alpha dx,$$

so integral α, corresponding to zero $\sin \alpha\pi$, must involve a zero sum of the residues.

(c) Of course if the degree of q is two or more above the degree of p, there will be $\alpha \in \mathbb{Z}$ for which the integral is finite. The integral cannot be calculated directly by this method (and, as always, there will be better methods of calculating the integral of a rational function $\frac{p(x)}{q(x)} x^n$), but if $0 \le n < \text{degree}(q) - \text{degree}(p)$ we shall have

$$\int_0^\infty \frac{p(x)}{q(x)} x^n dx = \lim_{\alpha\to n} \int_0^\infty \frac{p(x)}{q(x)} x^\alpha dx,$$

so that we can find the exceptional values quickly if we need them.

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9. Infinite sums: the cotangent trick

I describe here a method of calculating certain infinite sums. It depends on special properties of the cotangent function.

9A Cotangents (a) Recall that

$$\cot z = \frac{\cos z}{\sin z} = \frac{e^{iz}+e^{-iz}}{2} \Big/ \frac{e^{iz}-e^{-iz}}{2i} = i\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}} = i\frac{1+e^{-2iz}}{1-e^{-2iz}}$$

is an analytic function defined except when $e^{-2iz} = 1$, that is, except when $-2iz$ is an integral multiple of $2\pi i$, that is, except when $z = n\pi$ for some $n \in \mathbb{Z}$.

Because exp is periodic with period $2\pi i$, cot is periodic with period π. At each singularity,

$$\operatorname*{Res}_{z=n\pi} \cot z = \lim_{z\to n\pi}(z-n\pi)\cot z = \lim_{z\to n\pi}\cos z\frac{z-n\pi}{\sin z} = \cos n\pi \lim_{z\to n\pi}\frac{1}{\cos z} = 1.$$

This shows also that cot has a simple pole at each $n \in \mathbb{Z}$. It is necessary sometimes to know the early terms of its Laurent series near 0. Because cot has a simple pole with residue 1 at 0, it is expressible in the form

$$\cot z = \frac{1}{z} + c_0 + c_1 z + c_2 z^2 + \dots$$

for $0 < |z| < \pi$. Because sin is odd and cos is even, cot is odd, so $c_0 = c_2 = \dots = 0$, and we have

$$\cot z = \frac{1}{z} + c_1 z + c_3 z^3 + \dots .$$

Multiplying this by the series for sin we get

$$(\frac{1}{z} + c_1 z + c_3 z^3 + \dots)(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots ;$$

equating coefficients, we get

(coefficient of z^2) $c_1 - \frac{1}{3!} = -\frac{1}{2!}$ so $c_1 = -\frac{1}{3}$;

(coefficient of z^4) $c_3 - \frac{1}{3!}c_1 + \frac{1}{5!} = \frac{1}{4!}$ so $c_3 = -\frac{1}{45}$;

(coefficient of z^6) $c_5 - \frac{1}{3!}c_3 + \frac{1}{5!}c_1 - \frac{1}{7!} = -\frac{1}{6!}$ so $c_5 = -\frac{2}{945}$.

(b) It will in fact be convenient to do calculations with $\cot \pi z$ rather than with z. This has a simple pole at $z = n$ for every $n \in \mathbb{Z}$; the residue is

$$\lim_{z\to n}(z-n)\cot \pi z = \lim_{z\to n}\cos \pi z\frac{z-n}{\sin \pi z} = \cos n\pi \lim_{z\to n}\frac{1}{\pi\cos \pi z} = \frac{1}{\pi}.$$

Generally, if $n \in \mathbb{Z}$ and f is any analytic function defined at n,

$$\begin{aligned}\operatorname*{Res}_{z=n} f(z)\cot \pi z = \lim_{z\to n}(z-n)f(z)\cot \pi z &= \lim_{z\to n} f(z)\cos \pi z\frac{z-n}{\sin \pi z} \\ &= f(n)\cos n\pi \lim_{z\to n}\frac{1}{\pi\cos \pi z} = \frac{1}{\pi}f(n).\end{aligned}$$

We can also compute the Laurent series of $\cot \pi z$ for $0 < |z| < 1$; this is

$$\frac{1}{\pi z} - \frac{\pi}{3}z - \frac{\pi^3}{45}z^3 - \frac{2\pi^5}{945}z^5 - \dots .$$

This gives us the possibility of computing the residue of $f(z)\cot \pi z$ at 0 when f has a pole at 0; thus

$$\operatorname*{Res}_{z=0}\frac{1}{z}\cot \pi z = \operatorname*{Res}_{z=0}\frac{1}{\pi z^2} - \frac{\pi}{3} - \frac{\pi^3}{45}z^2 - \frac{2\pi^5}{945}z^4 - \dots = 0;$$

$$\operatorname*{Res}_{z=0}\frac{1}{z^2}\cot \pi z = \operatorname*{Res}_{z=0}\frac{1}{\pi z^3} - \frac{\pi}{3z} - \frac{\pi^3}{45}z - \frac{2\pi^5}{945}z^3 - \dots = -\frac{\pi}{3};$$

$$\operatorname*{Res}_{z=0}\frac{1}{z^3}\cot \pi z = \operatorname*{Res}_{z=0}\frac{1}{\pi z^4} - \frac{\pi}{3z^2} - \frac{\pi^3}{45} - \frac{2\pi^5}{945}z^2 - \dots = 0;$$

$$\operatorname*{Res}_{z=0}\frac{1}{z^4}\cot\pi z=\operatorname*{Res}_{z=0}\frac{1}{\pi z^5}-\frac{\pi}{3z^3}-\frac{\pi^3}{45z}-\frac{2\pi^5}{945}z-\ldots=-\frac{\pi^3}{45};$$

$$\operatorname*{Res}_{z=0}\frac{1}{z^5}\cot\pi z=\operatorname*{Res}_{z=0}\frac{1}{\pi z^6}-\frac{\pi}{3z^4}-\frac{\pi^3}{45z^2}-\frac{2\pi^5}{945}-\ldots=0;$$

and so on. If f has a pole of order at most 5 at 0, so that it is expressible as $\frac{d_5}{z^5}+\ldots+\frac{d_1}{z}+c_0+c_1z+\ldots$ for z near to 0, then

$$\operatorname*{Res}_{z=0}f(z)\cot\pi z=-\frac{\pi^3}{45}d_4-\frac{\pi}{3}d_2+c_0.$$

Because cot is periodic with period 1, we can use the same ideas to calculate residues at n for non-zero integers n; if f has a pole of order at most 5 at n, so that it is expressible as $\frac{d_5}{(z-n)^5}+\ldots+\frac{d_1}{z-n}+c_0+c_1(z-n)+\ldots$ for z near to n, then

$$\operatorname*{Res}_{z=n}f(z)\cot\pi z=-\frac{\pi^3}{45}d_4-\frac{\pi}{3}d_2+c_0.$$

9B Lemma If *either* $|\mathcal{I}\mathrm{m}\,z|\ge 1$ *or* $\mathcal{R}\mathrm{e}\,z=n+\frac{1}{2}$ for some integer n, then

$$|\cot\pi z|\le\frac{1+e^{-2\pi}}{1-e^{-2\pi}}\le 2.$$

proof

$$|\cot\pi z|=\left|\frac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}}\right|=\left|\frac{1+e^{2\pi iz}}{1-e^{2\pi iz}}\right|=\left|\frac{1+e^{-2\pi iz}}{1-e^{-2\pi iz}}\right|.$$

Now if $z=x+iy$, then $e^{2\pi iz}=e^{-2\pi y}e^{2\pi ix}$, so $|e^{2\pi iz}|=e^{-2\pi y}$; similarly, $|e^{-2\pi iz}|=e^{2\pi y}$.

case 1 If $y=\mathcal{I}\mathrm{m}\,z\ge 1$, then

$$|\cot\pi z|=\left|\frac{1+e^{2\pi iz}}{1-e^{2\pi iz}}\right|\le\frac{1+|e^{2\pi iz}|}{1-|e^{2\pi iz}|}=\frac{1+e^{-2\pi y}}{1-e^{-2\pi y}}\le\frac{1+e^{-2\pi}}{1-e^{-2\pi}}.$$

case 2 If $y=\mathcal{I}\mathrm{m}\,z\le -1$, then

$$|\cot\pi z|=\left|\frac{1+e^{-2\pi iz}}{1-e^{-2\pi iz}}\right|\le\frac{1+|e^{-2\pi iz}|}{1-|e^{-2\pi iz}|}=\frac{1+e^{2\pi y}}{1-e^{2\pi y}}\le\frac{1+e^{-2\pi}}{1-e^{-2\pi}}.$$

case 3 If $x=n+\frac{1}{2}$, where $n\in\mathbb{Z}$, then $e^{2\pi iz}=e^{-2\pi y}e^{2\pi ix}=e^{-2\pi y}e^{2\pi in+\pi i}=-e^{-2\pi y}$, so

$$|\cot\pi z|=\left|\frac{1+e^{2\pi iz}}{1-e^{2\pi iz}}\right|=\left|\frac{1-e^{-2\pi y}}{1+e^{-2\pi y}}\right|<1.$$

Thus in all cases

$$|\cot\pi z|\le\frac{1+e^{-2\pi}}{1-e^{-2\pi}};$$

now $e\ge 1+1+\frac{1}{2}=\frac{5}{2}$ and $\pi\ge 3$, so $e^{2\pi}\ge e^6\ge(\frac{5}{2})^6\ge 6^3=216$, and

$$\frac{1+e^{-2\pi}}{1-e^{-2\pi}}\le\frac{1+\frac{1}{216}}{1-\frac{1}{216}}=\frac{217}{215}=1\frac{2}{215}\le 2.$$

9C Example To find $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

(a) The idea is to integrate $\frac{1}{z^2}\cot \pi z$ around a square with corners at $(\pm 1 \pm i)(m+\frac{1}{2})$, where $m \geq 1$ is a large integer. We shall find that for large m the integral is necessarily small, so that the sum of the residues inside the contour must also be small, and this will give the result.

(b) Let us calculate the residues. There is a singularity at 0, from both the $\frac{1}{z^2}$ and the $\cot \pi z$, and also at $\pm n$ for integers n between 1 and m; larger integers n lie outside the contour and don't contribute. The residue at 0, as computed in 9A above, is $-\frac{\pi}{3}$; at each other n with $-m \leq n \leq m$ it is $\frac{1}{\pi n^2}$. Accordingly the sum of the residues is

$$-\frac{\pi}{3} + 2\sum_{n=1}^{m} \frac{1}{\pi n^2},$$

and the integral will be

$$2\pi i(-\frac{\pi}{3} + 2\sum_{n=1}^{m} \frac{1}{\pi n^2}).$$

(c) To estimate the integral, we observe that the length of the contour is $8(m+\frac{1}{2})$, while the maximum value of $|\frac{1}{z^2}\cot \pi z|$ on the contour is at most $\frac{2}{m^2}$, because (by 9B) $|\cot \pi z| \leq 2$ for every z on the contour, while also $|z| \geq m + \frac{1}{2}$ so $|\frac{1}{z^2}| \leq \frac{1}{m^2}$. Accordingly the integral is at most

$$8(m+\frac{1}{2})\frac{2}{m^2} \to 0 \text{ as } m \to \infty.$$

(d) Accordingly

$$\lim_{m\to\infty} 2\pi i(-\frac{\pi}{3} + 2\sum_{n=1}^{m} \frac{1}{\pi n^2}) = 0$$

and

$$\lim_{m\to\infty} \sum_{n=1}^{m} \frac{1}{\pi n^2} = \frac{\pi}{3},$$

that is,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \simeq 1.64.$$

9D Note The method can be adapted to find

$$\sum_{n=k}^{\infty} \frac{p(-n)}{q(-n)} + \sum_{n=k}^{\infty} \frac{p(n)}{q(n)}$$

whenever p and q are (real or complex) polynomials, the degree of q is at least 2 more than the degree of p, and $q(n) \neq 0$ for any integer n with $|n| \geq k$, provided (as usual) that we can factorize q, so as to identify the residues of $\cot \pi z \frac{p(z)}{q(z)}$. The condition

$$\text{degree}(q) \geq 2 + \text{degree}(p)$$

is what is needed to ensure that the sums are summable, as well as to ensure that the integral around the square with corners $(m+\frac{1}{2})(\pm 1 \pm i)$ is small for large m. If $\frac{p(x)}{q(x)}$ is even, of course, we can get a sum $\sum_{n=k}^{\infty} \frac{p(n)}{q(n)}$, as in 9C. But this method does not (for instance) give us an expression for $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

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10. Conformal Mappings

I turn now to a quite different subject: the problem of understanding analytic functions as geometrical transformations.

10A Definition If $D \subseteq \mathbb{R}^2$ is an open set, a function $f : D \to \mathbb{R}^2$ is **conformal** if for small objects $A \subseteq D$ the image $f[A]$ of A is approximately the same shape as A; possibly rotated and/or expanded or contracted, but not reflected or significantly distorted.

10B Theorem If f is an analytic function and D is an open subset of $\{z : z \in \operatorname{dom} f,\ f'(z) \neq 0\}$, then f is conformal on D.

remark Here I identify $\mathbb{C}$ with the plane $\mathbb{R}^2$.

idea of proof Take any $z_0 \in D$. Then we have

$$f(z) = f(z_0) + (z-z_0)f'(z_0) + \frac{1}{2}(z-z_0)^2 f''(z_0) + \dots$$

for z near z_0. If z is very near z_0 then

$$f(z) \simeq f(z_0) + f'(z_0)(z-z_0) = g(z) \text{ say,}$$

with error small compared with $z - z_0$. So if $A \subseteq D$ is an object lying very close to z_0, the image $f[A]$ is closely approximated by $g[A]$, with errors small compared with the diameters of A and $f[A]$. But the map $g : \mathbb{C} \to \mathbb{C}$ is a composition of the maps

$$\begin{aligned} z &\mapsto z - z_0, \\ z &\mapsto f'(z_0)z, \\ z &\mapsto f(z_0) + z. \end{aligned}$$

Of these, the first and last are simple translations, so certainly leave shapes unaffected, while the middle one is an expansion or contraction (by a factor of $|f'(z_0)| \neq 0$) with a rotation (through $\arg f'(z_0)$), so also leaves shapes unchanged. Thus g is certainly conformal, and $f[A]$ will also be very nearly the same shape as A.

10C Remarks (a) Conversely, any conformal map on an open subset of $\mathbb{R}^2$ or $\mathbb{C}$ corresponds to an analytic function with non-zero derivative.

(b) Note that if $f : D \to \mathbb{C}$ is conformal, then near any $z_0 \in D$ the transformation is locally approximated by $z \mapsto f(z_0) + f'(z_0)(z - z_0)$. In particular, the scale is changed by a factor of $|f'(z_0)|$ and figures are rotated through an angle of $\arg f'(z_0)$.

(c) One of the reasons for taking conformal maps seriously is their use in solving partial differential equations, using the following theorem. I star this because we shall not be going further in this direction, but it is worth knowing about.

***10D Theorem** Suppose that D, E are open sets in $\mathbb{R}^2$ and that $f : D \to E$ is conformal, $\phi : E \to \mathbb{R}$ is twice continuously differentiable. Set $\psi = \phi \circ f$. Then

$$\frac{\partial^2 \psi}{\partial x^2}(a,b) + \frac{\partial^2 \psi}{\partial y^2}(a,b) = |f'(a+ib)|^2 \big(\frac{\partial^2 \phi}{\partial x^2}(f(a,b)) + \frac{\partial^2 \phi}{\partial y^2}(f(a,b))\big)$$

for every $(a,b) \in D$, where f' is the derivative of f when regarded as an analytic function.

In particular, if ϕ is **harmonic** (i.e., $\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}=0$), so is ψ.

10E Möbius transformations Consider the formula

$$f(z)=\frac{az+b}{cz+d}$$

where a, b, c and d are complex numbers and c, d are not both 0. This defines an analytic function, defined everywhere if $c=0$ (so $d\neq 0$), and everywhere except at $-d/c$ if $c\neq 0$. If $ad=bc$ then f is constant, with value $\frac{a}{c}=\frac{b}{d}$. Otherwise, it is invertible, with inverse given by

$$\begin{aligned} w&=\frac{az+b}{cz+d}\\ \iff\ & w(cz+d)=az+b\\ \iff\ & z(cw-a)=b-wd\\ \iff\ & z=\frac{b-wd}{cw-a}\end{aligned}$$

(we cannot have either $cz+d=az+b=0$ or $b-wd=cw-a=0$ because $ad\neq bc$). Observe that the inverse transformation

$$f^{-1}(w)=\frac{b-wd}{cw-a}$$

is another transformation of the same kind, this time defined except at a/c (unless $c=0$, in which case it is defined everywhere).

If we look at the derivative of f, we get

$$f'(z)=\frac{a(cz+d)-c(az+b)}{(cz+d)^2}=\frac{ad-bc}{(cz+d)^2},$$

which (still assuming that $ad\neq bc$) is never 0. So f defines a conformal transformation.

Transformations of this type are called **Möbius** or **linear fractional** transformations.

10F Structure of Möbius transformations Let $f(z)=\frac{az+b}{cz+d}$ be a Möbius transformation. (Remember that $ad\neq bc$.)

case 1 If $c=0$ then $a\neq 0$, $d\neq 0$ and $f(z)=\frac{a}{d}z+\frac{b}{d}$. We can regard this as the composition of the maps $z\mapsto\frac{a}{d}z$ and $z\mapsto z+\frac{b}{d}$. The first is a rotation together with a contraction or expansion; the second is a translation. Both preserve all shapes; in particular, circles are taken to circles and straight lines are taken to straight lines. The transformation is a bijection from $\mathbb{C}$ to itself.

case 2 Before considering the general case of $c\neq 0$, let us examine the special case $f(z)=\frac{1}{z}$ (that is, $a=c=1$, $b=d=0$). In this case, consider the general equation of a straight line in the plane

$$2Bx+2Cy+D=0,$$

where B, C, D are real and B, C are not both 0, and the general equation of a circle in the plane

$$(x+B)^2+(y+C)^2=r^2,$$

where B, C are real and $r>0$. These can both be put in the form

$$A(x^2+y^2)+2Bx+2Cy+D=0,$$

where A, B and C are not all 0, and if $A \neq 0$ then $(\frac{B}{A})^2 + (\frac{C}{A})^2 > \frac{D}{A}$, that is, $B^2 + C^2 > AD$, while if $A = 0$ then $B^2 + C^2 > 0$ – thus the condition $B^2 + C^2 > AD$ ensures a non-trivial figure in all cases, which will be a circle if $A \neq 0$ and a straight line if $A = 0$.

In terms of a complex variable z, the equation becomes

$$Az\overline{z} + 2B\frac{z+\overline{z}}{2} + 2C\frac{z-\overline{z}}{2i} + D = 0.$$

Setting $w = \frac{1}{z}$ this becomes

$$\frac{A}{w\overline{w}} + B\left(\frac{1}{w} + \frac{1}{\overline{w}}\right) + \frac{C}{i}\left(\frac{1}{w} - \frac{1}{\overline{w}}\right) + D = 0,$$

that is,

$$A + 2B\frac{w+\overline{w}}{2} - 2C\frac{w-\overline{w}}{2i} + Dw\overline{w} = 0,$$

which is an equation of the same sort as before (because $B^2 + (-C)^2 > DA$), so again corresponds to either a straight line or a circle. Moreover, the transformed figure is a straight line iff $D = 0$, that is, iff the original figure passed through 0; just as the original figure was a straight line iff $A = 0$, that is, iff the transformed figure passes through 0.

Summarising, we have

a straight line through 0 becomes a straight line through 0,
a straight line not through 0 becomes a circle through 0,
a circle through 0 becomes a straight line not through 0,
a circle not through 0 becomes a circle not through 0.

Both the transformation and its inverse are bijections from $\mathbb{C} \setminus \{0\}$ to itself.

case 3 Now let us turn to the general case $c \neq 0$. In this case

$$w = \frac{az+b}{cz+d} = \frac{\frac{a}{c}z + \frac{b}{c}}{z + \frac{d}{c}} = \frac{a}{c} + \frac{\frac{bc-ad}{c^2}}{z + \frac{d}{c}},$$

so we have the composition of the transformations

$z \mapsto z + \frac{d}{c}$ (translation)

$z \mapsto \frac{1}{z}$ (inversion, as considered in case 2)

$z \mapsto \frac{bc-ad}{c^2}z$ (rotation & magnification)

$z \mapsto \frac{a}{c} + z$ (translation).

Since each of these transform straight lines and circles into straight lines and circles, so does the composition.

The *pole* of the transformation $\frac{az+b}{cz+d}$ is the point $-\frac{d}{c}$ where it is undefined; straight lines through this point are transformed into straight lines through $\frac{a}{c}$, while circles through the pole are transformed into straight lines not through $\frac{a}{c}$, and other straight lines are transformed into circles through $\frac{a}{c}$. ($\frac{a}{c}$ is of course the pole of the inverse transformation $z = \frac{b-dw}{cw-a}$.) The transformation is a bijection from $\mathbb{C} \setminus \{-\frac{d}{c}\}$ to $\mathbb{C} \setminus \{-\frac{a}{c}\}$.

Remark It sometimes helps to think of a Möbius transformation $w = \frac{az+b}{cz+d}$ as a map from $\mathbb{C} \cup \{\infty\}$ to itself; $z = \infty$ corresponds to $w = \frac{a}{c}$ (or to $w = \infty$ if $c = 0$), $w = \infty$ corresponds to $z = -\frac{d}{c}$ (or to $z = \infty$ if $c = 0$). Under this convention, a straight line behaves like a 'circle through ∞', so the rule becomes just 'generalized circles are transformed into generalized circles', and the rest is a matter of checking what $z = \infty$, $w = \infty$ mean.

10G Techniques for finding particular Möbius transformations (a) If z_1, z_2, z_3 are three distinct points in $\mathbb{C}$ and w_1, w_2, w_3 are three distinct points in $\mathbb{C}$, then the formula

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_1-w_2)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_1-z_2)}$$

defines a Möbius transformation which moves z_1 to w_1, z_2 to w_2 and z_3 to w_3.

(b) In this formula, we can allow '∞' as a value for one of the w_j or one of the z_j (or both), subject to the convention that $\frac{\infty}{\infty} = 1$. To say that '$z_1$ moves to ∞' declares that z_1 is to be the pole of the transformation; to say that '∞ moves to w_1' declares that $\lim_{|z|\to\infty} w = w_1$; to say that '$\infty$ moves to ∞' declares that the transformation is to be of the form $w = az + b$, with no pole.

Generally, we can think of a Möbius transformation $w = \frac{az+b}{cz+d}$ as a bijection from $\mathbb{C} \cup \{\infty\}$ to itself; if $c = 0$ then ∞ is taken to itself; if $c \neq 0$ then $z = \infty$ corresponds to $w = \frac{b}{d}$, and $z = -\frac{d}{c}$ corresponds to $w = \infty$. A straight line is now a 'circle through ∞', so a straight line in the z-plane gets transformed into either a straight line or a circle through $-\frac{d}{c}$.

(c) To move a given line L to the unit circle, a point z_1 on L to 1, and a point z_0 not on L to 0, let z_∞ be the reflection of z_0 in L, and use the transformation

$$w = \frac{z-z_0}{z-z_\infty} \times \frac{z_1-z_\infty}{z_1-z_0};$$

note that this is just the formula in (a) with $w_0 = 0$, $w_1 = 1$ and $w_\infty = \infty$, so that

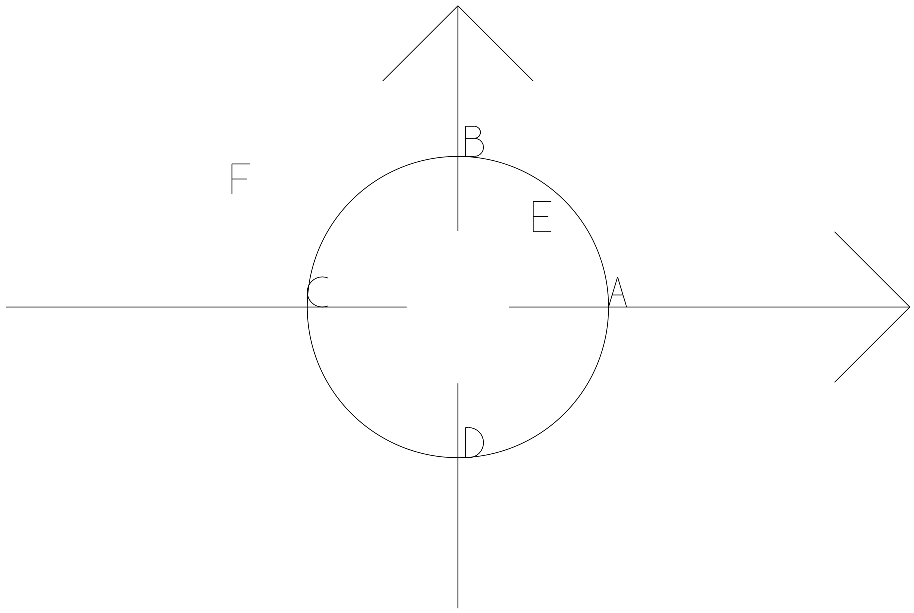
$$w = \frac{(w-0)(w_1-\infty)}{(1-\infty)(1-0)} = \frac{(w-w_0)(w_1-w_\infty)}{(w-w_\infty)(w_1-w_0)} = \frac{(z-z_0)(z_1-z_\infty)}{(z-z_\infty)(z_1-z_0)}.$$

This transformation can be run backwards to transform the unit circle into a given line.

(d) Now to transform lines into other circles, we can use the transformation in (c) and then a suitable rotation, magnification and translation.

10H Examples I will now present some diagrams which will I hope help to give a notion of the geometric nature of conformal transformations.

(a) I begin with $w = f(z) = 1/z$. This transforms



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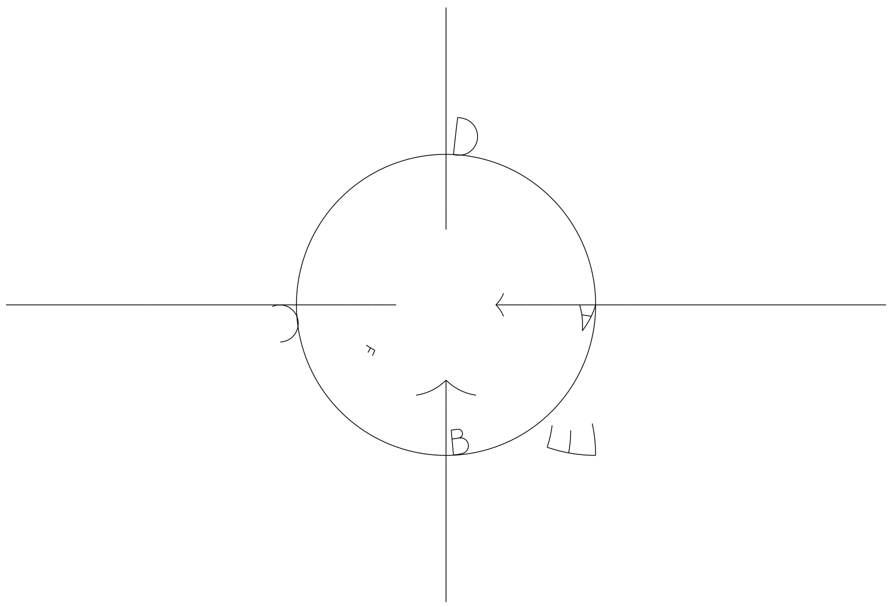
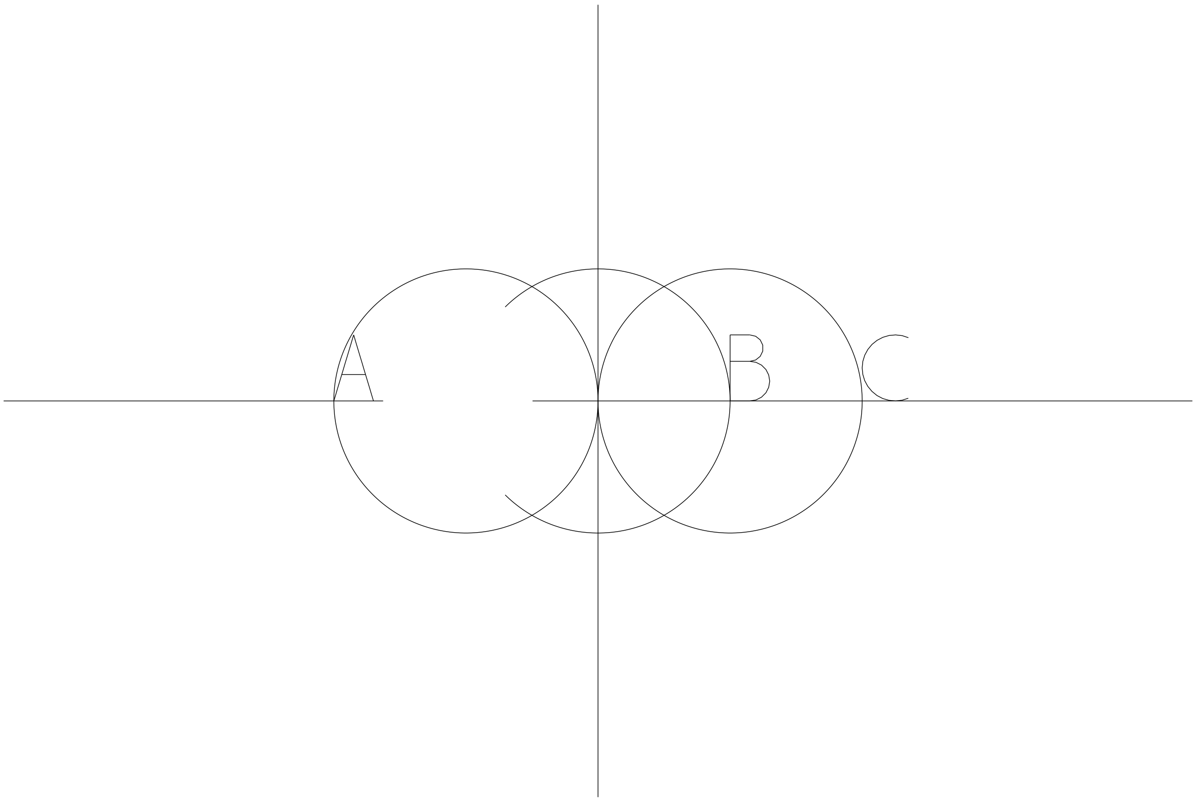


The letters `A`, `B`, `C` and `D`, all more or less on the unit circle in the z-diagram, are moved again to the unit circle in the w-diagram; `A` and `C` are inverted (because $f'(1) = f'(-1) = -1$), while `B` and `D` remain upright (because $f'(i) = f'(-i) = 1$). `C` and `E` start inside the circle and finish outside the circle, magnified as well as rotated, because $|f'| > 1$ inside the circle; while `A` and `F` start outside and finish inside, contracted because $|f'| < 1$ outside the unit circle. The axis lines look as if they haven't moved, but in fact they've been turned inside out, and the upper and lower sections of the imaginary axis have been swapped.

(b) Now consider $w = \frac{3z-2}{2z+2}$. This transforms



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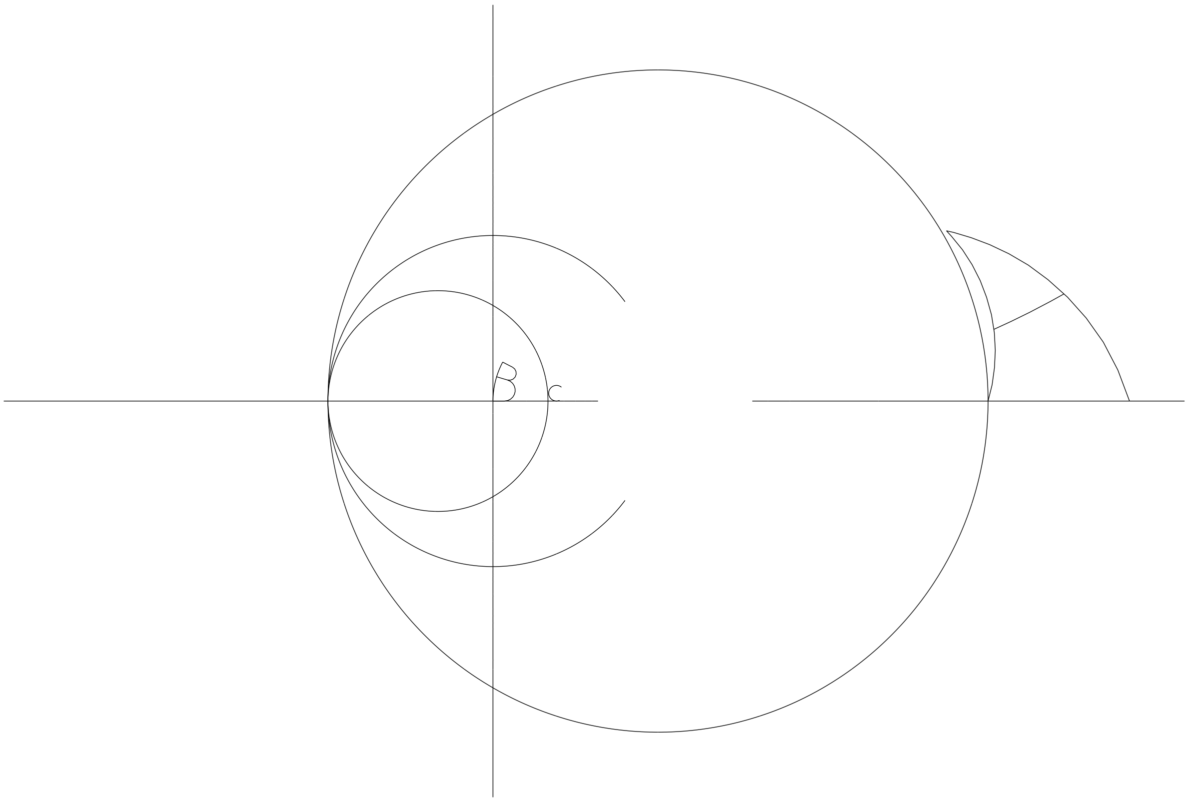
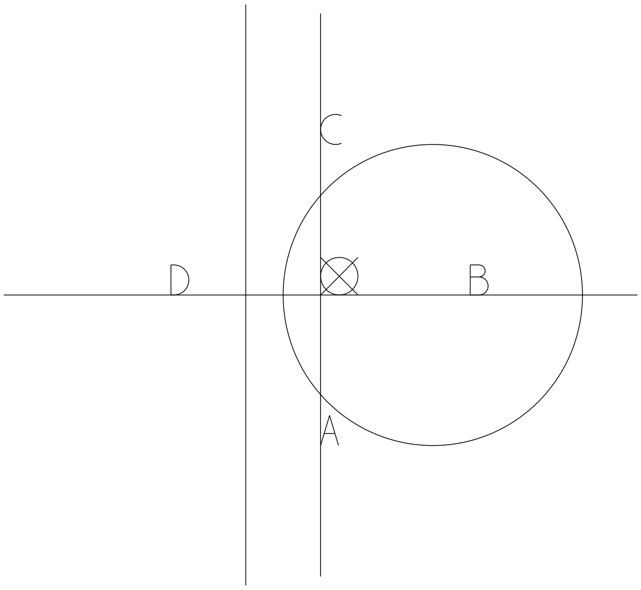


Note that $f'(z) = \frac{5}{2(z+1)^2}$. In particular, this is real and positive for any real $z \neq -1$, so that all points on the real axis are left right way up.

The real axis remains the real axis, but the two sections (above and below -1) are exchanged left-to-right. The imaginary axis, a line not through the pole, turns into a circle through the pole $w = \frac{3}{2}$ of the new diagram. Of the three circles in the z-diagram, the left-hand one, a circle centered on the pole $z = -1$, becomes a circle centered on the pole $w = \frac{3}{2}$; the middle one, passing through the pole $z = -1$, becomes a straight line; and the third, which doesn't enclose the pole $z = -1$, becomes a circle not enclosing the pole $w = \frac{3}{2}$.

(c) Now consider $w = z^2$. This transforms the grid



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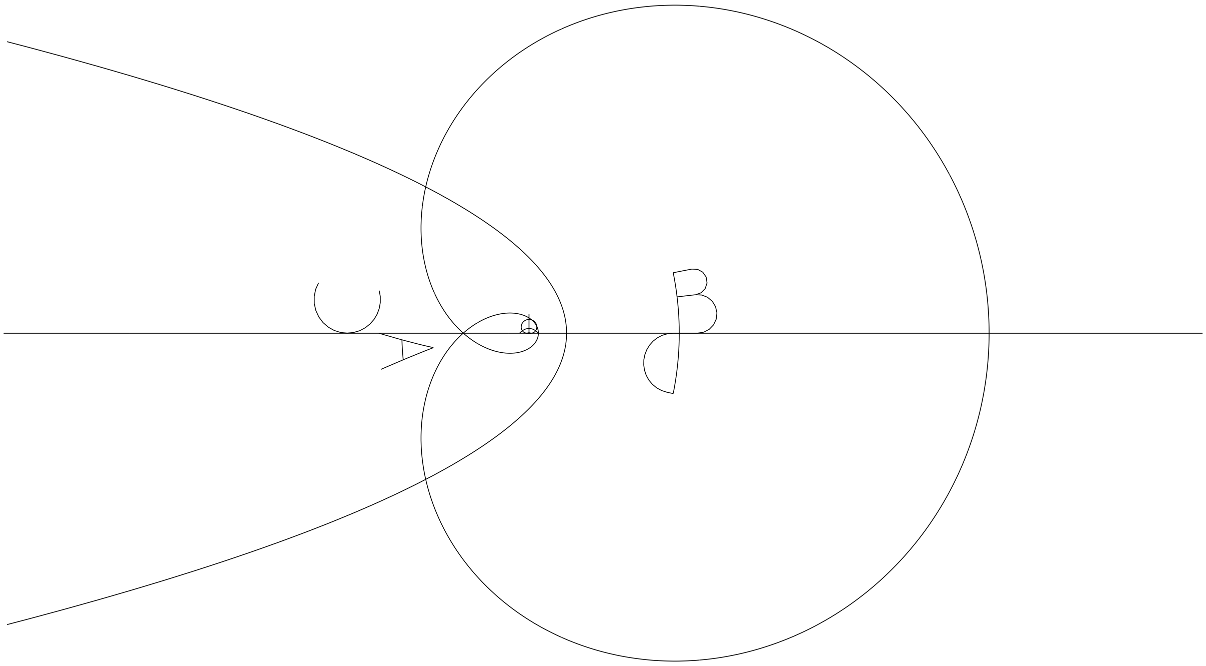
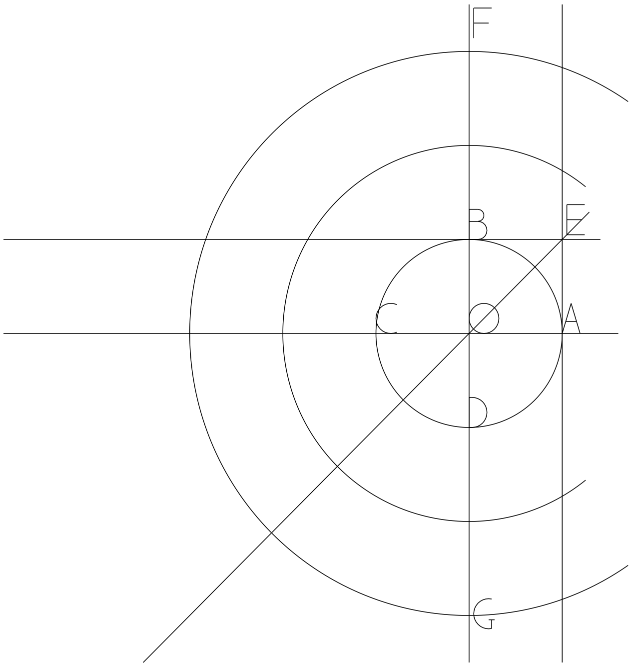


We have $f(1) = f(-1)$, so B and D are brought together; similarly, the letters A and C are brought together. $|f'(z)| = 2$ for $|z| = 1$, so all four are approximately doubled in size. $f'(1) = 2$, so B stays the right way up; $f'(-1) = -2$, so D is rotated through $180°$; $f'(i) = 2i$, so C is rotated anticlockwise; and $f'(-i) = -2i$, so A is rotated clockwise. The real axis is folded on itself and becomes the positive real axis; the imaginary axis is also folded to become the negative real axis. *The circle surrounding* 0 *now winds round* 0 *twice*; this is almost the most characteristic behaviour of the function $z \mapsto z^2$; and something of this sort will happen with any analytic function g near a point z_0 such that $g'(z_0) = 0$, $g''(z_0) \neq 0$, so that $g(z) \simeq a + b(z - z_0)^2$

for z near z_0. The vertical line at the left of the original diagram is bent into a parabola.

At 0, we have $f'(0) = 0$, so *the mapping is not conformal*; you can see that not only has an axis line apparently disappeared, but the `X` has been dramatically distorted. It has also, of course, been shrunk.

(d) Now consider $w = \exp z$. This transforms



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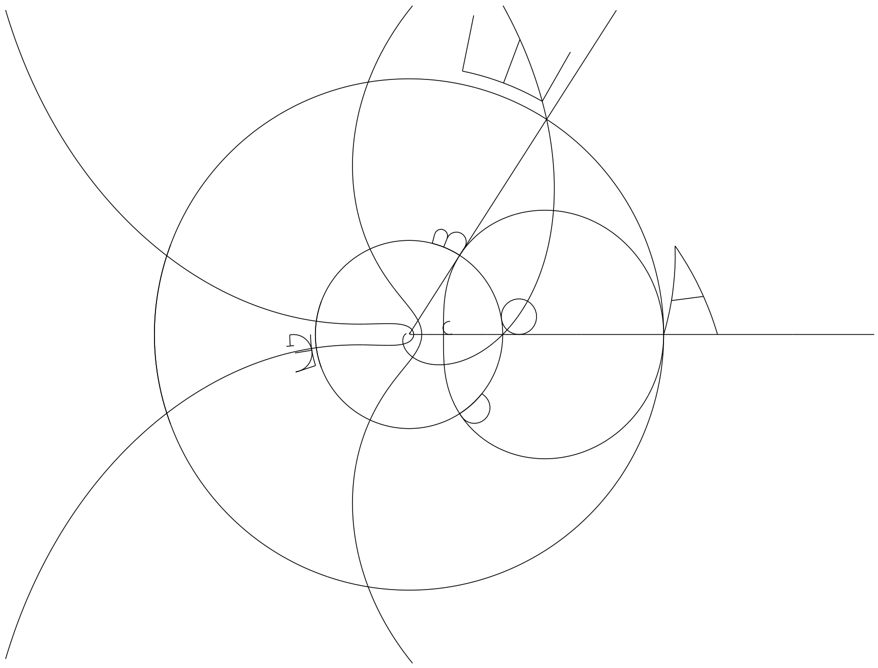
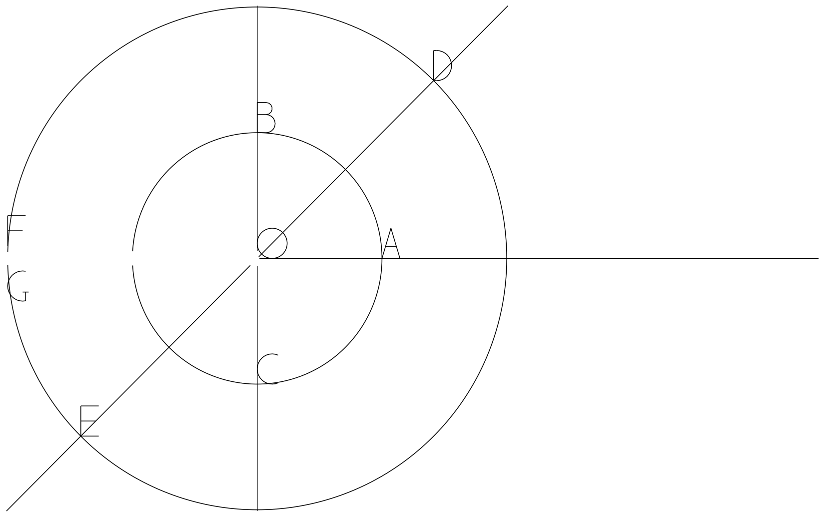


Observe that $z = 0$ goes to $w = 1$; $f'(0) = 1$ so the spokes are not rotated. $z = 1$ (the letter `A`) goes to $w = e$; no rotation, magnification by a factor of e. The line $\mathcal{I}\mathrm{m}\, z = 0$ (letters `C`, `O`, `A`) goes to the half-line $\{w : w \text{ is real } > 0\}$; the line $\mathcal{R}\mathrm{e}\, z = 0$ (letters `G`, `D`, `O`, `B`, `F`) becomes the circle $|w| = 1$, rotating around it, no magnification or contraction (because $|f'(z)| = 1$). The line $\mathcal{R}\mathrm{e}\, z = 1$ (past `A` and `E`) becomes the circle $|w| = e$; the line $\mathcal{I}\mathrm{m}\, z = 1$ (past `B` and `E`) becomes the line $\arg w = 1$. The circles $|z| = r$, for $0 < r < \pi$, become simple closed curves around $w = 1$, indented to pass to the right of $w = 0$; but the circles $|z| = 2$, $|z| = 3$ on the upper diagram are incomplete because the corresponding values of w are too large to fit onto the lower diagram. The line $\mathcal{R}\mathrm{e}\, z = \mathcal{I}\mathrm{m}\, z$, passing through `O` and `E`, corresponds to the spiral $|w| = e^{\arg w \pm 2k\pi}$; only fragments of these appears on the diagrams, since it is truncated at one end because w goes off the page and at the other end because z goes off the page (to the left, as $w \to 0$). The letters `F` and `G`, close to $z = \pm i$, are superimposed in the lower diagram near $w = -1$.

(e) Now consider $w = \operatorname{Ln} z$. This transforms



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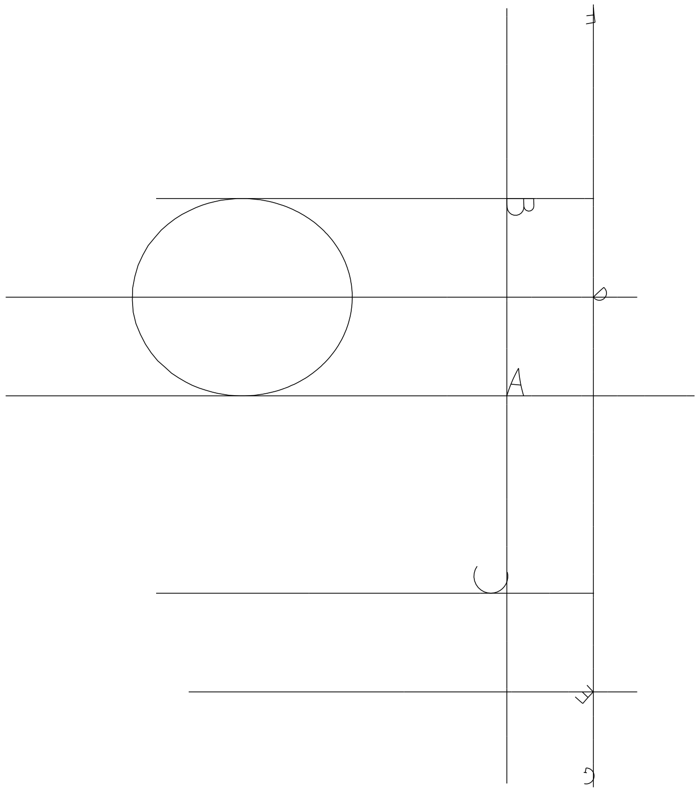
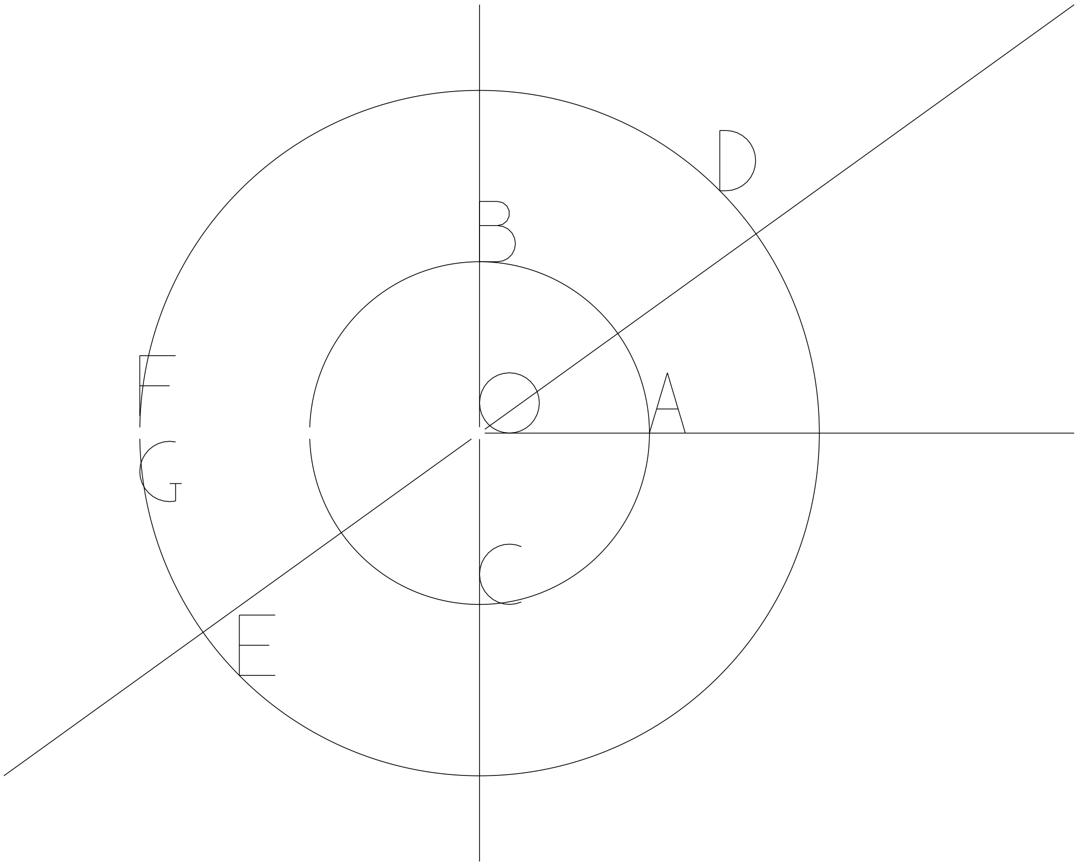


Here rays starting at 0 (on which $\arg z$ is constant) are transformed into horizontal lines, while circles centred at zero (on which $|z|$ is constant) are transformed into vertical lines. Since $|f'(z)| = 1$ when $|z| = 1$, `A`, `B` and `C` come out approximately their original size, while `D`, `E`, `F` and `G` are shrunk. The function opens the slit along the negative real axis so wide as to invert `F` and `G` on opposite sides of the diagram. As for the `O`, this is close to the singularity at 0, and is spread all over the left-hand side of the diagram (the apparently polygonal shape of the figure is due to the crudeness of the program generating it). But note that the singularity is only logarithmic, so that the `0` does appear in the lower picture, and is not actually pushed off the page.

$z = 1$ moves to $w = 0$; $z = i$ moves to $w = \frac{\pi}{2}i$, with a rotation of 90° clockwise; points near the negative real axis move to points near the lines $\mathcal{I}\mathrm{m}\, w = \pm\pi$, rotated through 180°. The circles $|z| =$ constant move to lines $\mathcal{R}\mathrm{e}\, w =$ constant; the spokes $\arg z =$ constant move to lines $\mathcal{I}\mathrm{m}\, w =$ constant.

(f) Now consider $w = z^{1/2}$. This transforms



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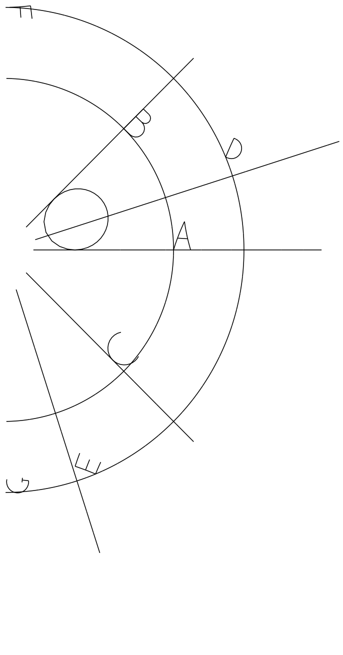


This time, the slit along the negative real axis is opened out to become the imaginary axis. Circles centered on the origin become semi-circles, and rays from 0 are still rays from 0, rotated half way to the positive real axis. `A`, `B` and `C` are halved in size, while the outlying letters are reduced still more; but `O` is enlarged.

10I Riemann Mapping Theorem Suppose that $D \subseteq \mathbb{C}$ is a simply connected open set, *not the whole of* $\mathbb{C}$. Then there is a bijective conformal mapping taking D onto the open unit disk $\{z : |z| < 1\}$.

Note Recall that an open set $D \subseteq \mathbb{C}$ is **simply connected** if whenever Γ is a simply closed curve lying within D then the interior of Γ also lies entirely within D.

D is simply connected iff $\int_\Gamma \frac{dz}{z-z_0} = 0$ whenever Γ is a closed curve of finite length lying within D and $z_0 \in \mathbb{C} \setminus D$.

10J Example If $D = \{z : |\mathcal{I}\mathrm{m}\, z| < \pi\}$, then use the following transformations:

exp to transform D into $D_1 = \{z : z \text{ is not real } \leq 0\}$,

$z \mapsto z^{1/2}$ to transform D_1 into $D_2 = \{z : \mathcal{R}\mathrm{e}\, z > 0\}$,

$z \mapsto z + 1$ to transform D_2 into $D_3 = \{z : \mathcal{R}\mathrm{e}\, z > 1\}$,

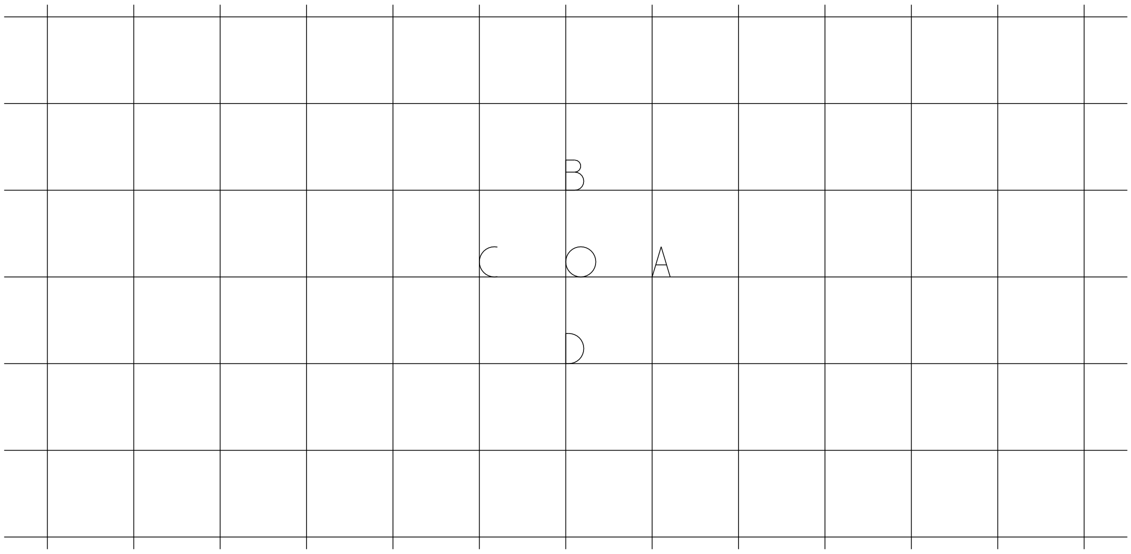
$z \mapsto \frac{1}{z}$ to transform D_3 into $D_4 = \{z : |z - \frac{1}{2}| < \frac{1}{2}\}$,

$z \mapsto 2z - 1$ to transform D_4 into the open unit disk.

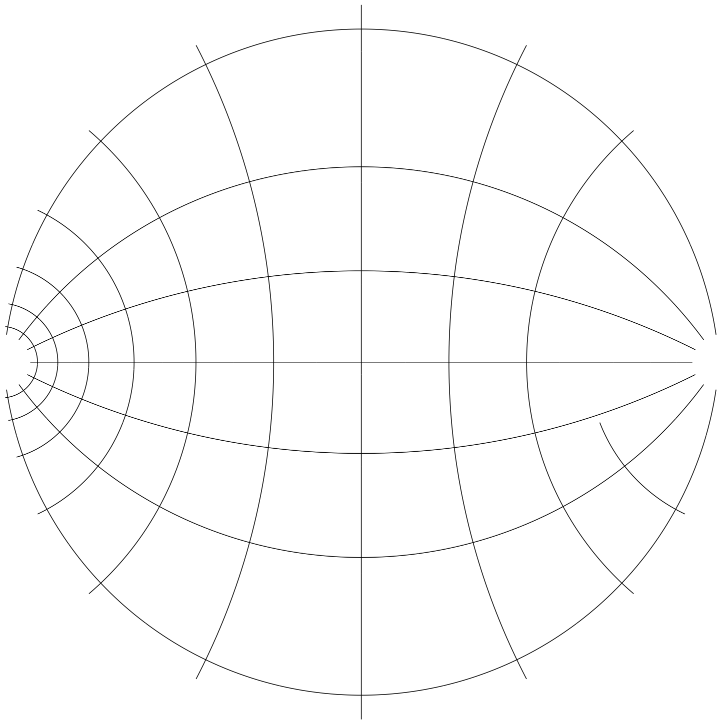
Putting these together,

$$f(z) = 2\frac{1}{1 + (\exp z)^{1/2}} - 1 = \frac{1 - \exp\frac{z}{2}}{1 + \exp\frac{z}{2}} = -\tanh\frac{z}{4}.$$

Evidently $z \mapsto f(z) = \tanh\frac{z}{4}$ will also map D bijectively onto the unit circle, and the picture will be a little easier to interpret, because $f'(0) = \frac{1}{4}$ is real and positive, so the transformed grid is right way up near 0:



is transformed to



Note that the domain $\{z : |\mathcal{I}\mathrm{m}\, z| < \pi\}$ chosen for the function is comfortably inside the natural domain of $\tanh \frac{z}{4}$, the nearest singularities being at $\pm 2\pi i$; so that the curves remain reasonably smooth right up to the boundary, except near $w = \pm 1$, corresponding to z far to the right or left. The lines $\mathcal{I}\mathrm{m}\, z = \pm\pi$ correspond respectively to the upper and lower parts of the circle $|w| = 1$.

10K It is natural to ask whether the condition '$D \neq \mathbb{C}$' in the Riemann Mapping Theorem (10I) is really necessary. Indeed it is, as the following theorem shows:

Liouville's Theorem If f is a bounded analytic function with domain $\mathbb{C}$, it is constant. In particular, there can be no analytic surjection $f : \mathbb{C} \to \{z : |z| < 1\}$.

proof Let $f : \mathbb{C} \to \mathbb{C}$ be a bounded analytic function; suppose that $|f(z)| \le M$ for every $z \in \mathbb{C}$. If $z_0 \in \mathbb{C}$, let Γ_R be the circle centre z_0 radius R, taken once anticlockwise. Then, by Cauchy's Integral Formula (4K),

$$\begin{aligned} |f'(z_0)| &= \Big|\frac{1}{2\pi i}\int_{\Gamma_R} \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta\Big| \\ &\le \frac{1}{2\pi} 2\pi R \frac{M}{R^2} = \frac{M}{R}. \end{aligned}$$

As R is arbitrary, $|f'(z_0)| = 0$. As z_0 is arbitrary, f is constant.

10L Example Consider the function $f(z) = \frac{1}{2}(z + \frac{1}{z})$ with domain restricted to the punctured disk $\{z : 0 < |z| < 1\}$.

(a) f is injective. **P** If $f(z_1) = f(z_2)$ then $z_1 - z_2 = \frac{1}{z_2} - \frac{1}{z_1} = (z_1 - z_2)/z_1 z_2$, so either $z_1 = z_2$ or $z_1 z_2 = 1$; but if z_1, $z_2 \in \mathrm{dom}(f)$ then $|z_1 z_2| = |z_1||z_2| < 1$, so the latter never happens, and we always have $z_1 = z_2$. **Q**

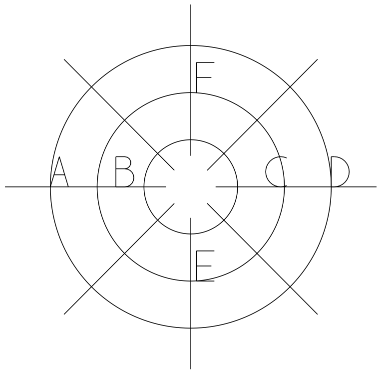
(b) The set of values of f is $\{w : \textit{either}\, w \text{ is not real } \textit{or}\, |w| > 1\}$. **P** If $z = r(\cos\theta + i\sin\theta)$, where $0 < r < 1$, then $\frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta)$ so $f(z) = \frac{1}{2}(\frac{1}{r} + r)\cos\theta - \frac{i}{2}(\frac{1}{r} - r)\sin\theta$ lies on the ellipse

$$\{\xi + i\eta : \frac{\xi^2}{\alpha^2} + \frac{\eta^2}{\beta^2} = 1\},$$

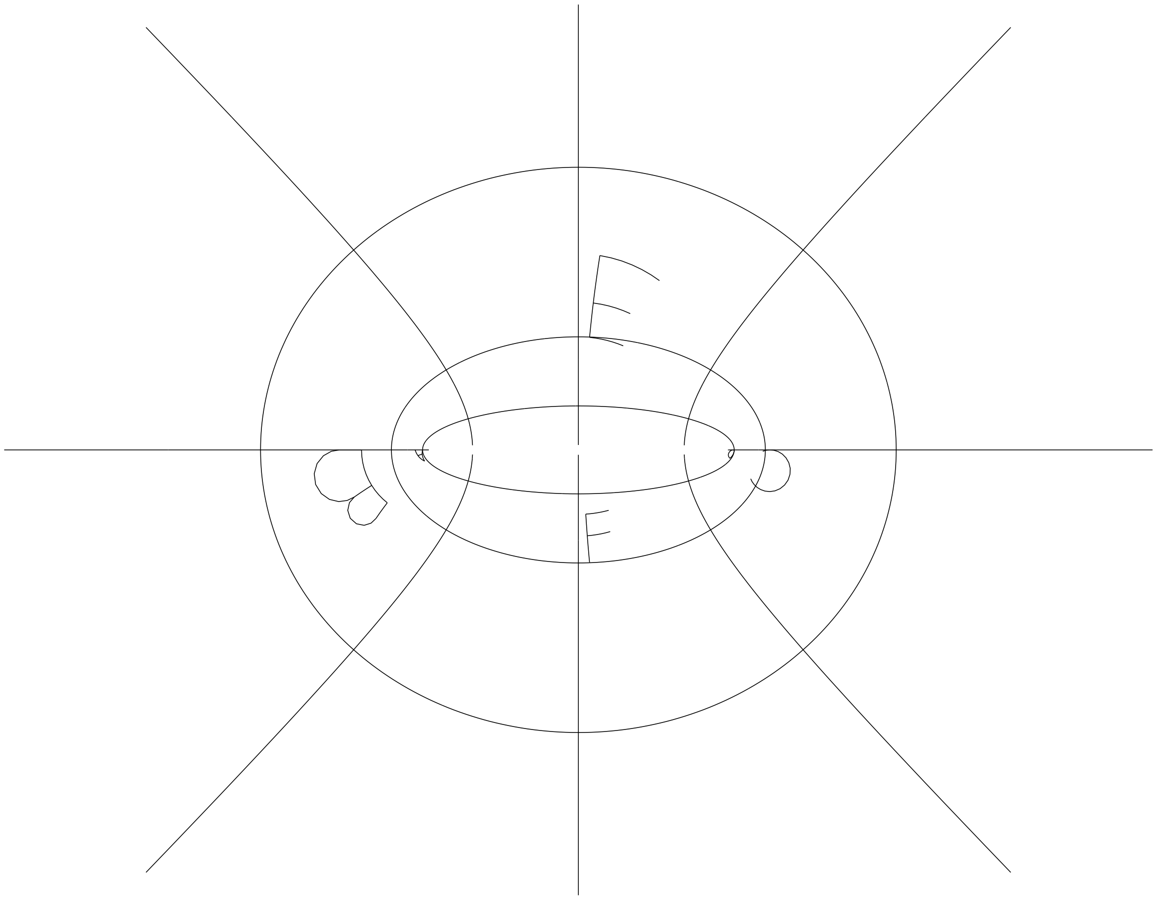
where $\alpha = \frac{1}{2}(\frac{1}{r} + r)$, $\beta = \frac{1}{2}(\frac{1}{r} - r)$. Observe that this ellipse has major axis running from $-\alpha$ to α and minor axis running from $-i\beta$ to $i\beta$. Now as r rises from 0 to 1 (exclusive), α falls from ∞ to 1 and β falls from ∞ to 0. So f takes all values except for the slit from -1 to 1 which lies inside all the ellipses; as claimed. **Q**

(c) $f^{-1}(w) = w(1 - (1 - \frac{1}{w^2})^{1/2})$ for all $w \in \mathrm{dom}(f^{-1})$, that is, unless w is a real number in $[-1, 1]$. **P** $w = \frac{1}{2}(z + \frac{1}{z}) \iff z^2 - 2wz + 1 = 0 \iff z = w \pm \sqrt{w^2 - 1}$. To decide which square root we must take, I argue as follows. I know that $f^{-1}(w)$ must be of the form $w + v$ where v is one of the square roots of $w^2 - 1$. Now $(w + v)(w - v) = w^2 - v^2 = w^2 - (w^2 - 1) = 1$, and also $|w + v| = |z| < 1$, so we must have $|w + v| < |w - v|$, that is, v is closer to $-w$ than to w. This means that $\frac{v}{w}$ must be closer to -1 than to 1 (recall that by (b) above we have $w \ne 0$); that is, $\mathcal{R}\mathrm{e}(\frac{v}{w}) < 0$. Now as $v^2 = w^2 - 1$, $(\frac{v}{w})^2 = 1 - \frac{1}{w^2}$; as $\mathcal{R}\mathrm{e}(\frac{v}{w}) < 0$, $\frac{v}{w} = -(1 - \frac{1}{w^2})^{1/2}$ (because $a^{1/2}$ is always the square root of a with positive real part). So $v = -w(1 - \frac{1}{w^2})^{1/2}$ and $z = w(1 - (1 - \frac{1}{w^2})^{1/2})$, as claimed. **Q**

The grid



becomes



As already remarked, circles $|z| = r$ move to ellipses enclosing the interval $[-1, 1]$. The transformation turns the grid inside out; the circle $|z| = 1$ corresponds to the line interval $[-1, 1]$ itself, while $|w| \to \infty$ as $z \to 0$. *Real* parts stay the same sign; *imaginary* parts reverse sign; but letters on the real axis are inverted (because $f'(z) = \frac{1}{2}(1 - \frac{1}{z^2})$ is negative for real z of modulus < 1) and letters on the imaginary axis stay the right way up (because $f'(z)$ is real and positive for purely imaginary z). Each spoke becomes a quarter-hyperbola, except for the spokes along the axes, which become half-lines. The letters `A` and `D` have not quite vanished, but they were too close to the edge $|z|=1$, and have been shrunk nearly to invisibility near $w=\mp 1$.

Version of 16.2.93

11. Z-transforms

In the last two chapters of the course, I give a necessarily very brief introduction to the theory of 'transforms'. These chapters should be read in parallel, because while the details of the formulae and arguments differ, the underlying ideas are really very similar.

11A Definition Let $\langle a_n\rangle_{n\ge 0}$ be a sequence of complex numbers. Its **Z-transform** is the function A defined by setting

$$A(z)=\sum_{n=0}^{\infty}\frac{a_n}{z^n}$$

for all those complex numbers z for which the sum is defined.

11B Remarks (a) This is a special kind of Laurent series; but really it behaves more like a Taylor series in $\frac{1}{z}$.

It would be materially more convenient to use the function $\sum_{n=0}^{\infty}a_nz^n$. I do not know why this is not customary.

(b) For any sequence $\langle a_n\rangle_{n\ge 0}$ there will be an $R\in[0,\infty]$ such that $A(z)$ is defined for $|z|>R$ and not for $|z|<R$. ($\frac{1}{R}$ is the radius of convergence of the series $\sum_{n=0}^{\infty}a_nz^n$.) On the open set $\{z:|z|>R\}$, A will be an analytic function; its derivative will be

$$A'(z)=-\sum_{n=0}^{\infty}\frac{na_n}{z^{n+1}}=-\sum_{n=1}^{\infty}\frac{(n-1)a_{n-1}}{z^n}=-\sum_{n=2}^{\infty}\frac{(n-1)a_{n-1}}{z^n}.$$

(c) Note that for any Z-transform with finite R, we must have $a_0=\lim_{|z|\to\infty}A(z)$, $a_1=\lim_{z\to\infty}z^2A'(z)=\lim_{|z|\to\infty}z(A(z)-a_0)$; these formulae give useful checks on our calculations.

11C Basic rules for calculating Z-transforms (a) If $\langle a_n\rangle_{n\ge 0}$, $\langle b_n\rangle_{n\ge 0}$ have Z-transforms A, B defined on $\{z:|z|>R_1\}$, $\{z:|z|>R_2\}$ then $\langle a_n+b_n\rangle_{n\ge 0}$ has Z-transform $A+B$ defined (at least) on $\{z:|z|>\max(R_1,R_2)\}$.

(b) If $\langle a_n\rangle_{n\ge 0}$ has Z-transform A defined on $\{z:|z|>R\}$, and $c\in\mathbb{C}$, then $\langle ca_n\rangle_{n\ge 0}$ has Z-transform cA defined (at least) on $\{z:|z|>R\}$.

(c) If $\langle a_n\rangle_{n\ge 0}$ has Z-transform $A(z)$ defined on $\{s:\mathcal{R}\mathrm{e}\,s>R\}$, and $c\in\mathbb{C}\setminus\{0\}$, then $\langle c^na_n\rangle_{n\ge 0}$ has Z-transform $A(\frac{z}{c})$ defined on $\{z:|z|>R|c|\}$. **P** $\sum_{n=0}^{\infty}\frac{a_nc^n}{z^n}=\sum_{n=0}^{\infty}\frac{a_n}{(z/c)^n}$. **Q**

(d) If $\langle a_n\rangle_{n\ge 0}$ has Z-transform $A(z)$ defined on $\{z:|z|>R\}$, then $\langle a_{n+1}\rangle_{n\ge 0}$ has Z-transform $z(A(z)-a_0)$ defined on $\{z:|z|>R\}$.

$$\textbf{P}\ \sum_{n=0}^{\infty}\frac{a_{n+1}}{z_n}=z\sum_{n=0}^{\infty}\frac{a_{n+1}}{z^{n+1}}=z\sum_{n=1}^{\infty}\frac{a_n}{z^n}=z(A(z)-a_0).\ \textbf{Q}$$

(e) If $\langle a_n\rangle_{n\ge 0}$ has Z-transform $A(z)$ defined for $|z|>R$, then $\langle na_n\rangle_{n\ge 0}$ has Z-transform $-zA'(z)$ defined for $|z|>R$. **P** Using 11Bb,

$$-zA'(z)=\sum_{n=1}^{\infty}\frac{(n-1)a_{n-1}}{z^{n-1}}=\sum_{n=0}^{\infty}\frac{na_n}{z^n}.\ \textbf{Q}$$

11D Basic transforms: three to memorise (a) The Z-transform of the constant sequence 1 is

$$\sum_{n=0}^{\infty}\frac{1}{z^n}=\frac{1}{1-1/z}=\frac{z}{z-1}.$$

whenever $|z|>1$.

(b) For real ω, the Z-transform of $\langle\cos\omega n\rangle_{n\ge0}$ is $\frac{z^2-z\cos\omega}{z^2-2z\cos\omega+1}$; the Z-transform of $\langle\sin\omega n\rangle_{n\ge0}$ is $\frac{z\sin\omega}{z^2-2z\cos\omega+1}$, both defined for $|z|>1$.

P Consider the Z-transform of $\langle e^{i\omega n}\rangle_{n\ge0}$. By 11Cc, with $c=e^{i\omega}$ and $a_n=1$, $A(z)=\frac{z}{z-1}$ this is $A(z/c)=A(e^{-i\omega}z)=\frac{e^{-i\omega}z}{e^{-i\omega}z-1}=\frac{z}{z-e^{i\omega}}$ for $|z|>1$. Similarly, the Z-transform of $\langle e^{-i\omega n}\rangle_{n\ge0}$ is $\frac{z}{z-e^{-i\omega}}$. By 11Ca-b we get the Z-transform of $\langle\cos\omega n\rangle_{n\ge0}=\langle\frac{1}{2}(e^{i\omega n}+e^{-i\omega n})\rangle_{n\ge0}$ to be

$$\begin{aligned}\frac{1}{2}\Big(\frac{z}{z-e^{i\omega}}+\frac{z}{z-e^{-i\omega}}\Big)&=\frac{z(z-e^{-i\omega})+z(z-e^{i\omega})}{2(z-e^{i\omega})(z-e^{-i\omega})}\\&=\frac{z^2-z\cos\omega}{z^2-2z\cos\omega+1},\end{aligned}$$

while the Z-transform of $\langle\sin\omega n\rangle_{n\ge0}=\langle\frac{1}{2i}(e^{i\omega n}-e^{-i\omega n})\rangle_{n\ge0}$ is

$$\frac{1}{2i}\Big(\frac{z}{z-e^{i\omega}}-\frac{z}{z-e^{-i\omega}}\Big)=\frac{z\sin\omega}{z^2-2z\cos\omega+1}.\ \mathbf{Q}$$

11E Derived transforms (a) Putting 11Da and 11Cc together, the Z-transform of the sequence $\langle c^n\rangle_{n\ge0}$ is $\frac{\frac{z}{c}}{\frac{z}{c}-1}=\frac{z}{z-c}$, defined for $|z|>|c|$, whenever $c\ne0$.

(b) Combining this with 11Ce, the Z-transform of $\langle nc^n\rangle_{n\ge0}$ is

$$-z\frac{d}{dz}\frac{z}{z-c}=\frac{cz}{(z-c)^2};$$

now the Z-transform of $\langle n^2c^n\rangle_{n\ge0}$ is

$$-z\frac{d}{dz}\frac{cz}{(z-c)^2}=\frac{cz(z+c)}{(z-c)^3}.$$

In particular, the Z-transforms of $\langle n\rangle_{n\ge0}$ and $\langle n^2\rangle_{n\ge0}$ are $\frac{z}{(z-1)^2}$, $\frac{z(z+1)}{(z-1)^3}$.

(c) Putting 11Db and 11Cc together, the Z-transforms of $\langle c^n\cos\omega n\rangle_{n\ge0}$, $\langle c^n\sin\omega n\rangle_{n\in\mathbb{N}}$ are $\frac{z^2-cz\cos\omega}{z^2-2cz\cos\omega+c^2}$, $\frac{2cz\sin\omega}{z^2-2cz\cos\omega+c^2}$. Adding 11Ce, we get the Z-transforms of $\langle nc^n\cos\omega n\rangle_{n\ge0}$ and $\langle nc^n\sin\omega n\rangle_{n\ge0}$ to be

$$-z\frac{d}{dz}\frac{z^2-cz\cos\omega}{z^2-2cz\cos\omega+c^2}=\frac{cz(z^2\cos\omega-2cz+c^2\cos\omega)}{(z^2-2cz\cos\omega+c^2)^2},$$

$$-z\frac{d}{dz}\frac{cz\sin\omega}{z^2-2cz\cos\omega+c^2}=\frac{cz\sin\omega(z^2-c^2)}{(z^2-2cz\cos\omega+c^2)^2};$$

always taking $c\ne0$ and $|z|>|c|$.

(d) Using 11Cd twice, we see that if $\langle a_n\rangle_{n\ge0}$ is a sequence with Z-transform $A(z)$ defined on $\{z:|z|>R\}$, then $\langle a_{n+2}\rangle_{n\ge0}$ has Z-transform $z^2A(z)-a_0z^2-a_1z$, defined on $\{z:|z|>R\}$.

11F Uniqueness Theorem Suppose $\langle a_n\rangle_{n\ge0}$, $\langle b_n\rangle_{n\ge0}$ are two sequences with Z-transforms $A(z)$ and $B(z)$, and suppose that $A(z)=B(z)$ at least whenever $|z|$ is large enough. Then in fact $A(z)=B(z)$ wherever either is defined, and $a_n=b_n$ for every n.

11G Solution of difference equations One of the uses of the Z-transform is in the solution of difference equations. Consider, for instance, the equation

$$a_{n+2}+4a_n=1,\ a_0=0,\ a_1=1.$$

We proceed by *assuming* that there is a solution in which $\langle a_n\rangle_{n\ge 0}$ has a Z-transform defined for $|z| > R$ for some finite R. In this case, if the Z-transform of $\langle a_n\rangle_{n\ge 0}$ is A, the Z-transform of $\langle a_{n+2}\rangle_{n\ge 0}$ will be $z^2A - z^2a_0 - za_1 = z^2A - z$; on the other side, the Z-transform of the constant sequence 1 is $\frac{z}{z-1}$, so (using 11Ca-b to find the Z-transform of $\langle a_{n+2} + 4a_n\rangle_{n\ge 0}$) we get

$$z^2A - z + 4A = \frac{z}{z-1}.$$

Now this is easy to solve for A; we have

$$A = \frac{z^2}{(z-1)(z^2+4)}.$$

We seek a sequence $\langle a_n\rangle_{n\ge 0}$ with this Z-transform. There are various approaches to this: the one I favour is to say that we know some relevant ones, being

$$\begin{array}{rcl} 1 & : & \dfrac{z}{z-1} \\ \cos\dfrac{\pi}{2}n & : & \dfrac{z^2}{z^2+1} \\ \sin\dfrac{\pi}{2}n & : & \dfrac{z}{z^2+1}. \end{array}$$

These aren't quite enough, because our denominator has a factor z^2+4, not z^2+1; but try

$$\begin{array}{rcl} 2^n\cos\dfrac{\pi}{2}n & : & \dfrac{(z/2)^2}{(z/2)^2+1} = \dfrac{z^2}{z^2+4}, \\ 2^n\sin\dfrac{\pi}{2}n & : & \dfrac{z/2}{(z/2)^2+1} = \dfrac{2z}{z^2+4}. \end{array}$$

This gives us a target to aim at: if we can express $A(z)$ as $\alpha\frac{z}{z-1} + \beta\frac{z^2}{z^2+4} + \gamma\frac{2z}{z^2+4}$, then we shall have $a_n = \alpha + 2^n\beta\cos\frac{\pi}{2}n + 2^n\gamma\sin\frac{\pi}{2}n$. Now we want

$$\alpha\frac{z}{z-1} + \beta\frac{z^2}{z^2+4} + \gamma\frac{2z}{z^2+4} = \frac{z^2}{(z-1)(z^2+4)},$$

that is,

$$\alpha z(z^2+4) + \beta z^2(z-1) + 2\gamma z(z-1) = z^2.$$

Equating coefficients, this becomes

$$\begin{array}{ll} (\text{coefficient of } z^3)1 & \alpha + \beta 2{=}0{,}3 \\ (\text{coefficient of } z^2)1 & -\beta + 2\gamma 2{=}1{,}3 \\ (\text{coefficient of } z)1 & 4\alpha - 2\gamma 2{=}0{,}3 \end{array}$$

giving $\alpha = \frac{1}{5}$, $\beta = -\frac{1}{5}$ and $\gamma = \frac{2}{5}$, so that

$$a_n = \frac{1}{5} - \frac{2^n}{5}\cos\frac{\pi}{2}n + \frac{2^{n+1}}{5}\sin\frac{\pi}{2}n.$$

11H Notes (a) We can use this method on a wide variety of linear difference equations with constant coefficients. In principle, it will work on equations of all orders, because we can find the Z-transform of all the sequences $\langle a_{n+k}\rangle_{n\ge 0}$ in terms of the Z-transform A of $\langle a_n\rangle_{n\ge 0}$ – for instance, the Z-transform of $\langle a_{n+3}\rangle_{n\ge 0}$ is $z^3A - z^3a_0 - z^2a_1 - za_2$. We do need to be able to find the Z-transform of the sequence on the right-hand side of the equation, and we do need to be able to calculate, or guess, the sequence whose Z-transform is the A we finally extract from the equation.

(b) An equation with constant coefficients, of the form

$$c_k a_{n+k} + \ldots + c_1 a_{n+1} + c_0 a_n = d_n,$$

will always turn into an equation of the form

$$(c_k z^k + \ldots + c_1 z + c_0)A(z) + p(z) = D(z),$$

where p is a polynomial depending on the initial values given for $a_0, \ldots, a_{k-1}$, and D is the Z-transform of $\langle d_n \rangle_{n \geq 0}$. The polynomial $c_k z^k + \ldots + c_1 z + c_0$ is the characteristic polynomial of the equation, and the method depends on factorising it, at least into quadratic factors. It may well happen, of course, that the transform $D(z)$ introduces further factors into the denominator of the final expression for $A(z)$. Given this factorisation, a suitable combination of transforms of the form

$$\frac{z}{z-c}, \frac{cz}{(z-c)^2}, \frac{cz(z+c)}{(z-c)^3}, \ldots,$$

$$\frac{(z^2 - cz\cos\omega)}{z^2 - 2cz\cos\omega + c^2}, \frac{cz\sin\omega}{z^2 - 2cz\sin\omega + c^2}, \frac{cz(z^2\cos\omega - 2cz + c^2\cos\omega)}{(z^2 - 2cz\cos\omega + c^2)^2} \ldots$$

corresponding to the sequences

$$c^n, nc^n, n^2c^n, \ldots,$$

$$c^n \cos\omega n, c^n \sin\omega n, nc^n \cos\omega n, \ldots$$

ought to work, at least when $D(z)$ is a rational function of z.

(c) In the example above, we can identify a 'particular solution' as the constant sequence $\frac{1}{5}$, while the terms in $2^n \cos\frac{\pi}{2}n$ and $2^n \sin\frac{\pi}{2}n$ are solutions of the homogeneous equation $a_{n+2} + 4a_n = 0$. (Note that of course these can be rearranged, if we allow ourselves complex coefficients, as combinations of $(2i)^n = 2^n \exp(i\frac{\pi}{2}n)$ and $(-2i)^n = 2^n \exp(-i\frac{\pi}{2}n)$; the point being that $\pm 2i$ are the two complex roots of the characteristic polynomial $z^2 + 4$.)

(d) Note the use of the Uniqueness Theorem here. We started by assuming that there was a solution, and that it would have a Z-transform. We worked out what that Z-transform would have to be, and found a sequence which had that transform. I then cheerfully asserted that that sequence did indeed solve the equation. The point is that I – if not you – could have proved, from the general theory of difference equations, that there was a solution, and that it would have a Z-transform defined at least for z with large modulus. So the first part of the argument was in fact sound, even if it relied on deeper results than I wish to go into here. Now, having found a sequence with the right Z-transform, the uniqueness theorem tells me that it must be the right sequence.

Of course any prudent person does not rely entirely on such abstract considerations, but substitutes the supposed solution into the original equation (and especially the initial conditions) as a check.

(e) Of course in this form the method cannot solve any equation not soluble by the basically more elementary method of trying suitable combinations of the sequences on the list just given, choosing c and ω among those parameters occuring in the sequence $\langle d_n \rangle_{n \geq 0}$, or from roots of the characteristic polynomial in the form $c \pm i\omega$.

For the real power of the method, we need some more theorems.

11I Delay switch Suppose a sequence $\langle a_n \rangle_{n \geq 0}$ has Z-transform A defined for $|z| > R$. Suppose that $k \geq 0$ and that we define a new sequence $\langle b_n \rangle_{n \geq 0}$ by setting

$$b_n = 0 \text{ for } 0 \leq n < k, \; b_n = a_{n-k} \text{ if } n \geq k.$$

Then the Z-transform of $\langle b_n \rangle_{n \geq 0}$ is $A(z)/z^k$, defined for $|z| > R$.

proof

$$\sum_{n=0}^{\infty}\frac{b_n}{z^n}=\sum_{n=k}^{\infty}\frac{b_n}{z^n}=\sum_{n=k}^{\infty}\frac{a_{n-k}}{z^n}=\sum_{n=0}^{\infty}\frac{a_n}{z^{n+k}}=\frac{A(z)}{z^k}$$

whenever $A(z)$ is defined.

11J Example Consider the equation $a_{n+1}+a_n=d_n$, where $d_n=2^n$ for $0\le n<k$, 0 for $n\ge k$, with $a_0=0$.

Express d_n as 2^n-g_n, where $g_n=0$ for $0\le x<k$, 2^n for $x\ge k$. The Z-transform of 2^n is $\frac{z}{z-2}$. Now we can express g_n as $g_n=0$ for $x<n$, $2^k d_{n-k}$ for $x\ge k$; so that the Z-transform of $\langle g_n\rangle_{n\ge 0}$ is $2^k z^{-k}\frac{z}{z-2}$, by 11I, and the Z-transform of $\langle d_n\rangle_{n\ge0}$ is $(1-2^k z^{-k})\frac{z}{z-2}$.

If the Z-transform of $\langle a_n\rangle_{n\ge0}$ is A, then the Z-transform of $\langle a_{n+1}+a_n\rangle_{n\ge0}$ is $zA+za_0+A=(z+1)A$, so we have

$$(z+1)A=(1-2^k z^{-k})\frac{z}{z-2}$$

and

$$A=(1-2^k z^{-k})\frac{z}{(z+1)(z-2)}.$$

As before, we have to assemble this out of a table of appropriate Z-transforms:

$$\begin{aligned}
2^n &: \frac{z}{z-2}\\
(-1)^n &: \frac{z}{z+1}\\
2^n \text{ delayed by } k &: z^{-k}\frac{z}{z-2}\\
(-1)^n \text{ delayed by } k &: z^{-k}\frac{z}{z+1}.
\end{aligned}$$

Thus we need to find constants α, β, γ, δ such that

$$\frac{\alpha z}{z-2}+\frac{\beta z}{z+1}+\gamma z^{-k}\frac{z}{z-2}+\delta z^{-k}\frac{z}{z+1}(1-2^k z^{-k})\frac{z}{(z+1)(z-2)}.$$

Multiplying up, and dropping a constant factor z, this becomes

$$\alpha(z+1)+\beta(z-2)+\gamma z^{-k}(z+1)+\delta z^{-k}(z-2)=1-2^k z^{-k}.$$

Equating coefficients of 1, z, z^{-k+1} and z^{-k} we get

$$\begin{aligned}
\alpha-2\beta&=1\\
\alpha+\beta&=0\\
\gamma+\delta&=0\\
\gamma-2\delta&=-2^k,
\end{aligned}$$

so $\alpha=\frac{1}{3}$, $\beta=-\frac{1}{3}$, $\gamma=-\frac{1}{3}2^k$, $\delta=\frac{1}{3}2^k$. Putting these together, our solution is

$$a_n=\frac{1}{3}2^n-\frac{1}{3}(-1)^n-\frac{1}{3}2^k l_n+\frac{1}{3}2^k m_n,$$

where l_n, m_n are the sequences 2^n, $(-1)^n$ 'delayed to k', that is,

$$l_n=0 \text{ if } n<k,\ 2^{n-k} \text{ if } n\ge k,$$

$$m_n = 0 \text{ if } n < k,\ (-1)^{n-k} \text{ if } n \geq k.$$

Accordingly

$$\begin{aligned} a_n &= \frac{1}{3}2^n - \frac{1}{3}(-1)^n \text{ if } n < k, \\ &= \frac{1}{3}2^n - \frac{1}{3}(-1)^n - \frac{1}{3}2^k 2^{n-k} + \frac{1}{3}2^k(-1)^{n-k} \\ &= \frac{1}{3}((-2)^k - 1)(-1)^n \text{ if } n \geq k. \end{aligned}$$

11K Notes (a) Observe that on each section the answer is in the expected form. For $0 \leq n < k$, we have $a_n = \frac{1}{3}2^n - \frac{1}{3}(-1)^n$; the term $\frac{1}{3}2^n$ is a particular solution to the equation $a_{n+1} + a_n = 2^n$, while the term $-\frac{1}{3}(-1)^n$ is a solution to the homogeneous equation $a_{n+1} + a_n$; the two together give $a_0 = 0$. For $n \geq k$ we have just a multiple of $(-1)^n$, because in that region we have $a_{n+1} + a_n = 0$. The constant $\frac{1}{3}((-2)^k - 1)$ is just that required to make the value of $a_k = \frac{1}{3}(2^k - (-1)^k)$ match up from both sides.

(b) In this particular case it would I think be easier to solve the equations in the two halves independently, doing $0 \leq n \leq k$ first, calculating a_k from the solution there, and then doing $n > k$. In other cases (and in particular with higher-order equations and sequences d_n subject to more than one delayed switch) the method here is quicker and more reliable.

11L Convolutions The next theorem which gives further scope to the Z-transform method is the **convolution theorem**, as follows:

Theorem Let $\langle a_n \rangle_{n \geq 0}$, $\langle b_n \rangle_{n \geq 0}$ be two sequences with Z-transforms A, B, defined respectively for $|z| > R$ and $|z| > S$. Set

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

for $n \geq 0$. Then $\langle c_n \rangle_{n \geq 0}$ has Z-transform $A(z)B(z)$, at least for $|z| > \max(R, S)$.

11M Example Consider the recurrence relation

$$a_0 = 1, \quad a_{n+1} = \sum_{j=0}^{n} a_j a_{n-j}.$$

Suppose, optimistically, that the solution to this has a Z-transform $A(z)$ defined for large $|z|$. Then the Z-transform of $\langle a_{n+1} \rangle_{n \geq 0}$ is $z(A(z) - 1)$, while the right-hand side is the convolution of $\langle a_n \rangle_{n \geq 0}$ with itself, so has Z-transform $A(z)^2$. Thus we have

$$z(A - 1) = A^2,$$

yielding

$$A = \frac{z \pm \sqrt{z^2 - 4z}}{2}.$$

Which square root should we take? We want $\lim_{|z| \to \infty} A(z) = a_0 = 1$. If we express A as

$$\frac{z}{2}(1 \pm \sqrt{1 - \frac{4}{z}}),$$

then plainly we must take

$$A = \frac{z}{2}(1-(1-\frac{4}{z})^{\frac{1}{2}})$$

to have a chance. Now we know that

$$(1+w)^{\frac{1}{2}} = 1+\frac{1}{2}w+\frac{1}{2!}(\frac{1}{2})(-\frac{1}{2})w^2+\frac{1}{3!}(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})w^3 \\ +\frac{1}{4!}(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})w^4+\dots,$$

so

$$\begin{aligned}(1-\frac{4}{z})^{\frac{1}{2}} &= 1-\frac{2}{z}-\frac{1}{2!}\frac{4}{z^2}-\frac{1}{3!}.3.\frac{8}{z^3}-\frac{1}{4!}.3.5.\frac{16}{z^4}-\frac{1}{5!}.3.5.7.\frac{32}{z^5}-\dots \\ &= 1-\frac{2}{z}-\frac{1}{2!}\frac{1!}{1}\frac{2^2}{z^2}-\frac{1}{3!}\frac{3!}{2}\frac{2^3}{z^3}-\frac{1}{4!}\frac{5!}{2.4}\frac{2^4}{z^4}-\frac{1}{5!}\frac{7!}{2.4.6}\frac{2^5}{z^5}-\dots \\ &= 1-\frac{2}{z}-\frac{1}{2!}\frac{1!}{1}\frac{2^2}{z^2}-\frac{1}{3!}\frac{3!}{2}\frac{2^3}{z^3}-\frac{1}{4!}\frac{5!}{2^2 2!}\frac{2^4}{z^4}-\frac{1}{5!}\frac{7!}{2^3 3!}\frac{2^5}{z^5}-\dots \\ &= 1-\frac{2}{z}-\frac{1}{2!}\frac{1!}{0!}\frac{4}{z^2}-\frac{1}{3!}\frac{3!}{1!}\frac{4}{z^3}-\frac{1}{4!}\frac{5!}{2!}\frac{4}{z^4}-\frac{1}{5!}\frac{7!}{3!}\frac{4}{z^5}-\dots \\ &= 1-\frac{2}{z}-\sum_{n=2}^{\infty}\frac{4(2n-3)!}{n!(n-2)!z^n}\end{aligned}$$

and

$$\begin{aligned}\frac{z}{2}(1-(1-\frac{4}{z})^{\frac{1}{2}}) &= 1+\sum_{n=2}^{\infty}\frac{2(2n-3)!}{n!(n-2)!z^{n-1}} \\ &= 1+\sum_{n=1}^{\infty}\frac{2(2n-1)!}{(n+1)!(n-1)!z^n} \\ &= 1+\sum_{n=1}^{\infty}\frac{(2n)!}{(n+1)!n!z^n} \\ &= \sum_{n=0}^{\infty}\frac{(2n)!}{(n+1)!n!z^n}.\end{aligned}$$

This means that

$$a_n = \frac{(2n)!}{(n+1)!n!}$$

for every $n \geq 0$.

(To check this, run the argument backwards. Observe that $\langle\frac{(2n)!}{(n+1)!n!}\rangle_{n\geq 0}$ does have a Z-transform $A(z)$ defined for $|z| > 4$ (use the ratio test), that $A(z) = \frac{z}{2}(1-(1-\frac{4}{z})^{1/2})$ so that $z(A-1) = A^2$, and that therefore $a_{n+1} = \sum_{j=0}^{n} a_j a_{n-j}$ for every n.)

11N Hermite polynomials (a) Consider the real polynomials He_n defined by starting with

$$\mathrm{He}_0(x) = 1, \quad \mathrm{He}_1(x) = x,$$

and proceeding with the recurrence relation

$$\mathrm{He}_{n+1}(x) = x\,\mathrm{He}_n(x) - n\,\mathrm{He}_{n-1}(x),$$

so that $\mathrm{He}_2(x) = x^2-1$, $\mathrm{He}_3(x) = x^3-3x$, $\mathrm{He}_4(x) = x^4-6x^2+3$, and so on.

(b) Fix x for a while, and set $a_n = \frac{1}{n!}\mathrm{He}_n(x)$, so that

$$(n+1)!a_{n+1} = xn!a_n - n(n-1)!a_{n-1},$$

and

$$(n+1)a_{n+1} = xa_n - a_{n-1}, \quad (n+2)a_{n+2} = xa_{n+1} - a_n.$$

Recall our table:

$$\begin{array}{rcl} a_n & : & A \\ na_n & : & -zA' \\ a_{n+1} & : & z(A - a_0) \\ (n+1)a_{n+1} & : & z(-zA' - 0) = -z^2A' \\ (n+2)a_{n+2} & : & z(-z^2A' - a_1) = -z^3A' - za_1. \end{array}$$

In the present case we have $a_0 = 1$, $a_1 = x$, so if A is the Z-transform of $\langle a_n \rangle_{n \ge 0}$ (assuming, as usual, that there is one defined for sufficiently large $|z|$), we get

$$-z^3A' - zx = xz(A-1) - A,$$

that is,

$$z^3A' + A(xz - 1) = 0.$$

Solving this differential equation by separation of variables, we have

$$\frac{A'}{A} + \frac{x}{z^2} - \frac{1}{z^3} = 0,$$

so that

$$\mathrm{Ln}\, A - \frac{x}{z} + \frac{1}{2z^2} = \text{constant}, \quad A = c\exp(\frac{x}{z} - \frac{1}{2z^2});$$

without looking too deeply into what $\mathrm{Ln}\, A$ means in this context, we can check that this solves the differential equation $z^3A' + A(xz - 1) = 0$. Now, letting $|z| \to \infty$, we find (if this is going to work at all) that $c = \lim_{|z|\to\infty} A(z) = a_0 = 1$, so that $A = \exp(\frac{x}{z} - \frac{1}{2z^2})$, and

$$\sum_{n=0}^{\infty} \frac{\mathrm{He}_n(x)}{n!z^n} = \exp(\frac{x}{z} - \frac{1}{z^2}).$$

(c) Given this formula for A, we can now justify all the preceding steps, because $A(z)$ surely has a Laurent series $\sum_{n=0}^{\infty} \frac{a_n}{z^n}$ valid for all non-zero z, and the differential equation which A satisfies translates readily into the correct recurrence relation for the a_n.

(d) Now let x vary, and write

$$A(z,x) = \exp(\frac{x}{z} - \frac{1}{2z^2}) = \sum_{n=0}^{\infty} \frac{\mathrm{He}_n(x)}{n!z^n}.$$

We can compute

$$\begin{aligned}\int_{-\infty}^{\infty} A(z,x)A(w,x)e^{-x^2/2}dx &= \int_{-\infty}^{\infty} \exp(\frac{x}{z}-\frac{1}{2z^2})\exp(\frac{x}{w}-\frac{1}{2w^2})e^{-x^2/2}dx \\ &= \int_{-\infty}^{\infty} \exp(\frac{x}{z}-\frac{1}{2z^2}+\frac{x}{w}-\frac{1}{2w^2}-\frac{x^2}{2})dx \\ &= \int_{-\infty}^{\infty} \exp(-\frac{1}{2}(x-\frac{1}{z}-\frac{1}{w})^2+\frac{1}{wz})dx \\ &= \exp(\frac{1}{wz})\int_{\infty}^{\infty} \exp(-\frac{1}{2}y^2)dy,\end{aligned}$$

at least if w, z are real and non-zero, using the substitution $y = x - \frac{1}{w} - \frac{1}{z}$; that is,

$$\begin{aligned}\int_{-\infty}^{\infty} A(z,x)A(w,x)e^{-x^2/2}dx &= \sqrt{2\pi}\exp(\frac{1}{wz}) \\ &= \sqrt{2\pi}\sum_{n=0}^{\infty}\frac{1}{n!w^n z^n}.\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}\int_{-\infty}^{\infty} A(z,x)A(w,x)e^{-x^2/2}dx &= \int_{\infty}^{\infty}\sum_{m=0}^{\infty}\frac{\mathrm{He}_m(x)}{m!w^m}\sum_{n=0}^{\infty}\frac{\mathrm{He}_n(x)}{n!z^n}e^{-x^2/2}dx \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_{\infty}^{\infty}\frac{\mathrm{He}_m(x)}{m!w^m}\frac{\mathrm{He}_n(x)}{n!z^n}e^{-x^2/2}dx \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!n!w^m z^n}\int_{\infty}^{\infty}\mathrm{He}_m(x)\,\mathrm{He}_n(x)e^{-x^2/2}dx.\end{aligned}$$

Equating coefficients of $\frac{1}{w^m z^n}$, we get

$$\begin{aligned}\int_{\infty}^{\infty}\mathrm{He}_m(x)\,\mathrm{He}_n(x)e^{-x^2/2}dx &= n!\sqrt{2\pi} \text{ if } m = n, \\ &= 0 \text{ if } m \neq n.\end{aligned}$$

Version of 19.2.93

12. Laplace Transforms

12A Definition Let y be a complex-valued function. Its **Laplace transform** is the function Y defined by setting

$$Y(s) = \int_0^{\infty} e^{-sx}y(x)dx$$

for all those complex numbers s for which the integral is defined.

12B Remarks (a) The question immediately arises – I mean rather: you ought to ask at once – which functions y have Laplace transforms? The answer depends on which theory of integration you use. For our purposes here, Riemann integration will be adequate, taking $\int_0^\infty f(x)dx$ to be $\int_0^\infty \mathcal{R}\mathrm{e}(f(x))dx + i\int_0^\infty \mathcal{I}\mathrm{m}(f(x))dx$ if both of these are defined as integrals of the real functions $\mathcal{R}\mathrm{e}\, f$, $\mathcal{I}\mathrm{m}\, f$, and $\int_0^\infty f(x)dx$ to be $\lim_{a\to\infty}\int_0^a f(x)dx$ if the limit is defined.

It will sometimes happen that we shall want a Laplace transform of a function which is not defined on the whole of $[0,\infty[$. In particular, if for every $a \geq 0$ there are only finitely many points in $[0,a]$ at which y is undefined, we can list them in increasing order as $a_1, a_2, \ldots, a_n$, and then say that

$$\int_0^a e^{-sx}y(x)dx = \int_0^{a_1} e^{-sx}y(x)dx + \int_{a_1}^{a_2} e^{-sx}y(x)dx + \ldots + \int_{a_n}^{a} e^{-sx}y(x)dx,$$

ignoring the points $a_1, \ldots, a_n$ altogether.

(b) Under these rules, we find that there is always an $R \in [0, \infty]$ such that $Y(s)$ is defined if $\mathcal{R}\mathrm{e}\, s > R$ and not if $\mathcal{R}\mathrm{e}\, s < R$; the domain of Y will always be either $\mathbb{C}$ or $\emptyset$ or a half-space, with various problems possible on the line $\{s : \mathcal{R}\mathrm{e}\, s = R\}$. On the open set $\{s : \mathcal{R}\mathrm{e}\, s > R\}$, moreover, Y will be an analytic function; its derivative will be

$$Y'(s) = -\int_0^\infty xe^{-sx}y(x)dx.$$

(c) In fact it may help if we restrict our attention to certain functions which are adequate for our needs and are relatively easy to visualize. I will say that a complex-valued function y on a subset of $[0, \infty[$ is **locally piecewise continuous** if for every $a \in [0, \infty[$ there are finitely many points $a_0, \ldots, a_n$ such that $0 = a_0 < a_1 < \ldots < a_n = a$ and

y is defined and continuous on (at least) each of the open intervals $]a_{k-1}, a_k[$,
$\lim_{x \downarrow a_k} y(x)$ is defined in $\mathbb{C}$ whenever $0 \le k \le n-1$,
$\lim_{x \uparrow a_k} y(x)$ is defined in $\mathbb{C}$ whenever $1 \le k \le n$.

Examples (i) If $y(x) = |x-1|$, then $y : [0, \infty[\to \mathbb{R}$ is continuous. Its derivative y' is defined except at 1, and is locally piecewise continuous.

(ii) If $y(x) = \frac{\sin x}{|\sin x|}$ then y takes only the values ± 1; it is defined except at multiples of π, and is locally piecewise continuous.

(d) It is possible for the Laplace transform $Y(s)$ to be undefined solely because y is too irregular a function. However, a more interesting problem arises if, for instance, $y(x) = e^{x^2}$; then $\int_0^\infty e^{-sx}y(x)dx = \infty$ for every real s, so that $Y(s)$ is undefinable for any (real or complex) s.

(e) When the Laplace transform is defined anywherev2n11.tex (in the strict sense used here – there are interesting, and valuable, extensions of the idea with different properties) then we must have $\lim_{\mathcal{R}\mathrm{e}\, s \to \infty} Y(s) = \lim_{\mathcal{R}\mathrm{e}\, s \to \infty} Y'(s) = \lim_{\mathcal{R}\mathrm{e}\, s \to \infty} Y''(s) = \ldots = 0$; this provides a useful check on some of our formulae.

12C Basic rules for calculating Laplace transforms (a) If y_1, y_2 have Laplace transforms Y_1, Y_2 defined on $\{s : \mathcal{R}\mathrm{e}(s) > R_1\}$, $\{s : \mathcal{R}\mathrm{e}\, s > R_2\}$ then $y_1 + y_2$ has Laplace transform $Y_1 + Y_2$ defined (at least) on $\{s : \mathcal{R}\mathrm{e}\, s > \max(R_1, R_2)\}$.

(b) If y has Laplace transform Y defined on $\{s : \mathcal{R}\mathrm{e}(s) > R\}$, and $c \in \mathbb{C}$, then cy has Laplace transform cY defined (at least) on $\{s : \mathcal{R}\mathrm{e}\, s > R\}$.

(c) If $y(x)$ has Laplace transform $Y(s)$ defined on $\{s : \mathcal{R}\mathrm{e}\, s > R\}$, and $a \in \mathbb{C}$, then $e^{ax}y(x)$ has Laplace transform $Y(s-a)$ defined on $\{s : \mathcal{R}\mathrm{e}\, s > R + \mathcal{R}\mathrm{e}\, a\}$. **P** $\int_0^\infty e^{-sx}e^{ax}y(x)dx = \int_0^\infty e^{-(s-a)x}y(x)dx$. **Q**

(d) If a function y is such that
(i) y is defined and continuous on $[0, \infty[$
(ii) y has Laplace transform $Y(s)$ defined on $\{s : \mathcal{R}\mathrm{e}\, s > R\}$
(iii) its derivative y' is locally piecewise continuous on $[0, \infty[$
(iv) $\alpha \in \mathbb{R}$ is such that $\lim_{x \to \infty} e^{-\alpha x}y(x) = 0$,
then the Laplace transform of $y'(x)$ is $sY(s) - y(0)$, defined (at least) on $\{s : \mathcal{R}\mathrm{e}\, s > \max(R, \alpha)\}$.

idea of proof Integrating by parts,

$$\begin{aligned}\int_0^\infty e^{-sx}y'(x)dx &= \left[e^{-sx}y(x)\right]_0^\infty - \int_0^\infty (-se^{-sx})y(x)dx \\ &= \lim_{x \to \infty} e^{-sx}y(x) - y(0) + sY(s) \\ &= sY(s) - y(0),\end{aligned}$$

because $\lim_{x\to\infty}|e^{-sx}y(x)| \le \lim_{x\to\infty} e^{-\alpha x}|y(x)| = 0$ if $\mathcal{R}\mathrm{e}\, s > \alpha$.

(e) If $y(x)$ has Laplace transform $Y(s)$ defined for $\mathcal{R}\mathrm{e}\, s > R$, then $xy(x)$ has Laplace transform $-Y'(s)$ defined for $\mathcal{R}\mathrm{e}\, s > R$. (See 12Bb.)

12D Basic transforms: three to memorise (a) The Laplace transform of the constant function 1 is

$$Y(s) = \int_0^\infty e^{-sx}dx = \left[-\frac{1}{s}e^{-sx}\right]_0^\infty = \frac{1}{s}(1 - \lim_{x\to\infty} e^{-sx}) = \frac{1}{s}$$

whenever $\mathcal{R}\mathrm{e}\, s > 0$ (so that $\lim_{x\to\infty}|e^{-sx}| = \lim_{x\to\infty} e^{-x\,\mathcal{R}\mathrm{e}\, s} = 0$).

(b) For real ω, the Laplace transform of $\cos\omega x$ is $\frac{s}{s^2+\omega^2}$; the Laplace transform of $\sin\omega x$ is $\frac{\omega}{s^2+\omega^2}$, both defined for $\mathcal{R}\mathrm{e}\, s > 0$.

P Consider the Laplace transform of $e^{i\omega x}$. By 12Cc, with $a = i\omega$ and $y(x) = 1$, $Y(s) = \frac{1}{s}$ this is $Y(s-a) = \frac{1}{s-i\omega}$ for $\mathcal{R}\mathrm{e}\, s > 0 + \mathcal{R}\mathrm{e}(i\omega) = 0$. Similarly, the Laplace transform of $e^{-i\omega x}$ is $\frac{1}{s+i\omega}$. By 12Ca-b we get the Laplace transform of $\cos\omega x = \frac{1}{2}(e^{i\omega x} + e^{-i\omega x})$ to be

$$\frac{1}{2}\left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right) = \frac{s}{s^2+\omega^2},$$

while the Laplace transform of $\sin\omega x = \frac{1}{2i}(e^{i\omega x} - e^{-i\omega x})$ is

$$\frac{1}{2i}\left(\frac{1}{s-i\omega} - \frac{1}{s+i\omega}\right) = \frac{\omega}{s^2+\omega^2}.$$ **Q**

12E Derived transforms (a) Putting 12Da and 12Cc together, the Laplace transform of e^{ax} is $\frac{1}{s-a}$, defined for $\mathcal{R}\mathrm{e}\, s > \mathcal{R}\mathrm{e}\, a$.

(b) Combining this with 12De, the Laplace transform of xe^{ax} is

$$-\frac{d}{ds}\frac{1}{s-a} = \frac{1}{(s-a)^2},$$

defined for $\mathcal{R}\mathrm{e}\, s > \mathcal{R}\mathrm{e}\, a$; now the Laplace transform of x^2e^{ax} is

$$-\frac{d}{ds}\frac{1}{(s-a)^2} = \frac{2}{(s-a)^3}.$$

In particular, the Laplace transforms of x and x^2 are $\frac{1}{s^2}$ and $\frac{2}{s^3}$. Generally, the Laplace transform of x^k is $\frac{k!}{s^{k+1}}$, defined for $\mathcal{R}\mathrm{e}\, s > 0$.

(c) Putting 12Db and 12Cc together, the Laplace transforms of $e^{ax}\cos\omega x$ and $e^{ax}\sin\omega x$ are $\frac{s-a}{(s-a)^2+\omega^2}$, $\frac{\omega}{(s-a)^2+\omega^2}$. Adding 12Ce, the Laplace transform of $xe^{ax}\cos\omega x$ is

$$-\frac{d}{ds}\frac{(s-a)}{(s-a)^2+\omega^2} = \frac{\omega^2-(s-a)^2}{((s-a)^2+\omega^2)^2};$$

while the Laplace transform of $xe^{ax}\sin\omega x$ is

$$-\frac{d}{ds}\frac{\omega}{(s-a)^2+\omega^2} = \frac{2\omega(s-a)}{((s-a)^2+\omega^2)^2}.$$

(d) Using 12Cd twice, we see that if y is a function such that

(i) y and y' are defined and continuous on $[0,\infty[$
(ii) y has Laplace transform $Y(s)$ defined on $\{s : \mathcal{R}\mathrm{e}\, s > R\}$
(iii) the second derivative y'' is locally piecewise continuous on $[0,\infty[$
(iv) $\alpha \in \mathbb{R}$ is such that $\lim_{x\to\infty} e^{-\alpha x}y(x) = \lim_{x\to\infty} e^{\alpha x}y'(x) = 0$,
then the Laplace transform of $y'(x)$ is $s^2Y(s) - sy(0) - y'(0)$, defined (at least) on $\{s : \mathcal{R}\mathrm{e}\, s > \max(R,\alpha)\}$.

12F Uniqueness Theorem Suppose y_1, y_2 are two functions with Laplace transforms Y_1 and Y_2, and suppose that $Y_1(s) = Y_2(s)$ at least for all large real s. Then in fact $Y_1(s) = Y_2(s)$ wherever either is defined, and $y_1(x) = y_2(x)$ wherever both are continuous.

Remark Note that of course we cannot say that y_1, y_2 are exactly the same function, because it might be (for instance) that they differ at only finitely many points, and then they will still have the same Laplace transform.

12G Solution of differential equations One of the uses of the Laplace transform is in the solution of differential equations. Consider, for instance, the equation

$$y'' + 4y = 1,\ y(0) = 0,\ y'(0) = 1.$$

We proceed by *assuming* that there is a solution in which y, y' are continuous, y'' is piecewise continuous, and all have Laplace transforms defined for $\mathcal{R}\mathrm{e}\, s > R$ for some finite R. In this case, if the Laplace transform of y is Y, the Laplace transform of y'' will be $s^2Y - sy(0) - y'(0) = s^2Y - 1$; on the other side, the Laplace transform of the constant function 1 is $\frac{1}{s}$, so (using 12Ca-b to find the Laplace transform of $y'' + 4y$) we get

$$s^2Y - 1 + 4Y = \frac{1}{s}.$$

Now this is easy to solve for Y; we have

$$Y = \frac{s+1}{s(s^2+4)}.$$

We seek a function y with this Laplace transform. There are various approaches to this: the one I favour is to say that we know some relevant ones, being

$$\begin{array}{rcl} 1 & : & \dfrac{1}{s} \\ \cos 2x & : & \dfrac{s}{s^2+4} \\ \sin 2x & : & \dfrac{2}{s^2+4}. \end{array}$$

An appropriate combination of these has a good chance of working. Specifically, the Laplace transform of $\alpha + \beta\cos 2x + \gamma \sin 2x$ will have Laplace transform $\frac{\alpha}{s} + \frac{\beta s}{s^2+4} + \frac{2\gamma}{s^2+4}$.
So we seek constants α, β, γ such that

$$\frac{\alpha}{s} + \frac{\beta s}{s^2+4} + \frac{2\gamma}{s^2+4} = \frac{s+1}{s(s^2+4)}.$$

This is almost a problem in partial fractions, and indeed can be done that way if you wish; but I think it is quicker and safer just to multiply up and say

$$\alpha(s^2+4) + \beta s^2 + 2\gamma s = s + 1.$$

Equating coefficients, this becomes

(coefficient of s^2)1 $\alpha + \beta$2=0,3
(coefficient of s)1 2γ2=1,3
(coefficient of 1)1 4α2=1,3

giving $\alpha = \frac{1}{4}$, $\beta = -\frac{1}{4}$ and $\gamma = \frac{1}{2}$, so that

$$y = \frac{1}{4} - \frac{1}{4}\cos 2x + \frac{1}{2}\sin 2x.$$

12H Notes (a) We can use this method on a wide variety of linear ordinary differential equations with constant coefficients. In principle, it will work on equations of all orders, because we can find the Laplace transform of all the repeated derivatives y', y'', y''', ... in terms of the Laplace transform Y of y – for instance, the Laplace transform of y''' is $s^3Y - s^2y(0) - sy'(0) - y''(0)$. We do need to be able to find the Laplace transform of the function on the right-hand side of the equation, and we do need to be able to calculate, or guess, the function whose Laplace transform is the Y we finally extract from the equation.

(b) An equation with constant coefficients, of the form

$$a_k y^{(k)} + \ldots + a_1 y' + a_0 y = f,$$

will always turn into an equation of the form

$$(a_k s^k + \ldots + a_1 s + a_0)Y(s) + p(s) = F(s),$$

where p is a polynomial depending on the initial conditions $y(0) = c_0$, $y'(0) = c_1, \ldots, y^{(n-1)}(0) = c_{n-1}$ imposed, and F is the Laplace transform of f. You will I hope recognise the polynomial $a_k s^k + \ldots + a_1 s + a_0$ as the characteristic polynomial of the equation, and the method depends on factorising it, at least into quadratic factors. It may well happen, of course, that the transform $F(s)$ introduces further factors into the denominator of the final expression for $Y(s)$. Given this factorisation, a suitable combination of transforms of the form

$$\frac{1}{s+a}, \frac{1}{(s+a)^2}, \frac{2}{(s+a)^3}, \ldots,$$

$$\frac{(s+a)}{(s+a)^2+\omega^2}, \frac{\omega}{(s+a)^2+\omega^2}, \frac{\omega^2-(s+a)^2}{((s+a)^2+\omega^2)^2}, \frac{2\omega(s+a)}{(s+a)^2+\omega^2)^2}, \ldots$$

corresponding to the functions

$$e^{-ax}, xe^{-ax}, x^2e^{-ax}, \ldots,$$

$$e^{-ax}\cos\omega x, e^{-ax}\sin\omega x, xe^{-ax}\cos\omega x, xe^{-ax}\sin\omega x, \ldots$$

ought to work, at least when $F(s)$ is a rational function of s.

(c) In the example above, we can identify a 'particular solution' as the constant function $\frac{1}{4}$, while the terms in $\cos 2x$ and $\sin 2x$ are solutions of the homogeneous equation $y'' + 4y = 0$.

(d) Note the use of the Uniqueness Theorem here. We started by assuming that there was a solution, and that it would have a Laplace transform. We worked out what that Laplace transform would have to be, and found a function which had that transform. I then cheerfully asserted that that function did indeed solve the equation. The point is that I – if not you – could have proved, from the general theory of ordinary differential equations, that there was a continuous solution, and that it would have a Laplace transform defined at least for s with large real part. So the first part of the argument was in fact sound, even if it relied on deeper results than I wish to go into here. Now, having found a function with the right Laplace transform, the uniqueness theorem tells me that it must be the right function, because they agree at least at all points where they are both continuous, and my function is continuous everywhere, like the true solution.

Of course any prudent person does not rely entirely on such abstract considerations, but substitutes the supposed solution into the original equation (and especially the given initial values) as a check.

(e) Of course in this form the method cannot solve any equation not soluble by the basically more elementary method of trying suitable combinations of the functions on the list just given, choosing a and ω

among those parameters occuring in the function f, or from roots of the characteristic equation in the form $a \pm i\omega$.

For the real power of the method, we need some more theorems.

12I Delay switch Suppose a function y has Laplace transform Y defined for $\mathcal{R}\mathrm{e}\, s > R$. Suppose that $\alpha \geq 0$ and that we define a new function y_1 by setting

$$y_1(x) = 0 \text{ for } 0 \leq x < \alpha,\ y_1(x) = y(x-\alpha) \text{ if } x \geq \alpha \text{ (and } x - \alpha \in \operatorname{dom} y).$$

Then the Laplace transform of y_1 is $e^{-as}Y(s)$, defined for $\mathcal{R}\mathrm{e}\, s > R$.

proof

$$\begin{aligned}
\int_0^\infty e^{-sx} y_1(x)dx &= \int_\alpha^\infty e^{-sx} y(x-\alpha)dx \\
&= \int_0^\infty e^{-s(x+\alpha)} y(x)dx \\
&= e^{-\alpha s} Y(s)
\end{aligned}$$

whenever $Y(s)$ is defined.

12J Example Consider the equation $y' + y = f$, where $f(x) = e^x$ for $0 \leq x < 2$, 0 for $x \geq 2$, with $y(0) = 0$.

Express $f(x)$ as $e^x - g(x)$, where $g(x) = 0$ for $0 \leq x < 2$, e^x for $x \geq 2$. The Laplace transform of e^x is $\frac{1}{s-1}$. Now we can express g as $g(x) = 0$ for $x < 2$, $e^2 e^{x-2}$ for $x \geq 2$; so that the Laplace transform of g is $e^2 e^{-2s} \frac{1}{s-1}$, by 12I, and the Laplace transform of f is $\frac{1-e^2e^{-2s}}{s-1}$.

If the Laplace transform of y is Y, then the Laplace transform of $y' + y$ is $sY + y(0) + Y = sY + Y$, so we have

$$sY + Y = \frac{1 - e^2 e^{-2s}}{s-1}$$

and

$$Y = \frac{1 - e^2 e^{-2s}}{(s+1)(s-1)}.$$

As before, we have to assemble this out of a table of appropriate Laplace transforms:

$$\begin{aligned}
e^x &: \quad \frac{1}{s-1} \\
e^{-x} &: \quad \frac{1}{s+1} \\
e^x \text{ delayed by } 2 &: \quad \frac{e^{-2s}}{s-1} \\
e^{-x} \text{ delayed by } 2 &: \quad \frac{e^{-2s}}{s+1}.
\end{aligned}$$

Thus we need to find constants α, β, γ, δ such that

$$\frac{\alpha}{s-1} + \frac{\beta}{s+1} + \frac{\gamma e^{-2s}}{s-1} + \frac{\delta e^{-2s}}{s+1} = \frac{1 - e^2 e^{-2s}}{(s+1)(s-1)}.$$

Multiplying up, this becomes

$$\alpha(s+1) + \beta(s-1) + \gamma e^{-2s}(s+1) + \delta e^{-2s}(s-1) = 1 - e^2 e^{-2s}.$$

Equating coefficients of s, 1, $e^{-2s}s$ and e^{-2s} we get

$$\begin{aligned}\alpha+\beta&=0\\ \alpha-\beta&=1\\ \gamma+\delta&=0\\ \gamma-\delta&=-e^2,\end{aligned}$$

so $\alpha=\frac{1}{2}$, $\beta=-\frac{1}{2}$, $\gamma=-\frac{1}{2}e^2$, $\delta=\frac{1}{2}e^2$. Putting these together, our solution is

$$y(x)=\frac{1}{2}e^x-\frac{1}{2}e^{-x}-\frac{e^2}{2}h_1(x)+\frac{e^2}{2}h_2(x),$$

where h_1, h_2 are the functions e^x, e^{-x} 'delayed to 2', that is,

$$h_1(x)=0 \text{ if } x<2,\ e^{x-2}=e^{-2}e^x \text{ if } x\geq 2,$$
$$h_2(x)=0 \text{ if } x<2,\ e^{-(x-2)}=e^2e^{-x} \text{ if } x\geq 2.$$

Accordingly

$$\begin{aligned}y(x)&=\frac{1}{2}e^x-\frac{1}{2}e^{-x} \text{ if } x<2,\\ &=\frac{1}{2}e^x-\frac{1}{2}e^{-x}-\frac{e^2}{2}e^{-2}e^x+\frac{e^2}{2}e^2e^{-x}=\frac{1}{2}(e^4-1)e^{-x} \text{ if } x\geq 2.\end{aligned}$$

12K Notes (a) Observe that on each section the answer is in the expected form. For $0<x<2$, we have $y=\frac{1}{2}e^x-\frac{1}{2}e^{-x}$; the term $\frac{1}{2}e^x$ is a particular solution to the equation $y'+y=e^x$, while the term $\frac{1}{2}e^{-x}$ is a solution to the homogeneous equation $y'+y=0$; the two together give $y(0)=0$. For $x>2$ we have just a multiple of e^{-x}, because in that region we have $y'+y=0$. The constant $\frac{1}{2}(e^4-1)$ is just that required to make the value of $y(2)=\frac{1}{2}(e^2-e^{-2})=\sinh 2$ match up from both sides.

(b) In this particular case it would I think be easier to solve the equations in the two halves independently, doing $[0,2]$ first, calculating $y(2)$ from the solution there, and then doing $[2,\infty[$. In other cases (and in particular with higher-order equations) the method here is quicker and more reliable.

(c) Note the way in which the Laplace transform method deals with the jump in the function f. If we are to have $y'+y=f$, at least one of y, y' must jump at 2. The Laplace transform method automatically puts the jump into y'. This is because the formula $sY(s)-y(0)$ for the Laplace transform of y' assumes that y is continuous, but allows y' to jump. In the same way, if we have a higher-order equation of the same type – such as $y''+y'+y=f$ – the jump will be put into y'', and y and y' will be kept continuous. Happily, this is the appropriate thing to do in most applications.

12L Convolutions The next theorem which gives further scope to the Laplace transform method is the **convolution theorem**, as follows:

Theorem Let f, g be two functions with Laplace transforms F, G, defined respectively for $\mathcal{R}\mathrm{e}\, s>R$ and $\mathcal{R}\mathrm{e}\, s>S$. Set

$$h(x)=\int_0^x f(t)g(x-t)dt$$

for $x\geq 0$. Then h has Laplace transform $H(s)=F(s)G(s)$, at least for $\mathcal{R}\mathrm{e}\, s>\max(R,S)$.

proof We have

$$\begin{aligned}H(s)&=\int_0^\infty e^{-sx}\Big(\int_0^x f(t)g(x-t)dt\Big)dx\\ &=\int_0^\infty f(t)\Big(\int_t^\infty e^{-sx}g(x-t)dx\Big)dt\end{aligned}$$

reversing the order of integration over the domain $\{(x,t):0\leq t\leq x\}$

$$= \int_0^\infty f(t)\Big(\int_0^\infty e^{-s(u+t)} g(u)du\Big)dt$$

substituting $u = x - t$

$$= \int_0^\infty f(t)e^{-st}\Big(\int_0^\infty e^{-su} g(u)du\Big)dt$$
$$= \int_0^\infty f(t)e^{-st}G(s)dt$$
$$= F(s)G(s),$$

as claimed.

12M Example Let us return to the equation $y' + y = f$, $y(0) = 0$, but this time without specifying the function f, except to declare that it does have a Laplace transform $F(s)$ defined at least when $\mathcal{R}\mathrm{e}\, s$ is large enough.

In this case, taking the Laplace transform of y to be Y as usual, we get $sY + Y = F$ and $Y(s) = F(s)\frac{1}{s+1}$. Now this is a product of Laplace transforms, so will be the Laplace transform of a convolution: $F(s)$ is the transform of $f(x)$ and $\frac{1}{s+1}$ is the transform of e^{-x}, so $\frac{F(s)}{s+1}$ will be the transform of $y(x)$, where

$$y(x) = \int_0^x f(t)e^{-(x-t)}dt = e^{-x}\int_0^x e^t f(t)dt.$$

We can run this through the previous analysis, with $f(t) = e^t$ for $t < 2$, 0 thereafter: if $x \le 2$, then

$$y(x) = e^{-x}\int_0^x e^{2t}dt = \frac{1}{2}e^{-x}(e^{2x} - 1) = \frac{1}{2}(e^x - e^{-x}),$$

while if $x \ge 2$ then

$$y(x) = e^{-x}\int_0^2 e^{2t}dt = \frac{1}{2}e^{-x}(e^4 - 1),$$

just as before.

12N Notes Of course the real value of this approach is when we don't know the function f, or can't calculate its Laplace transform. Of course, if we can't find a formula for the Laplace transform of f, it's unlikely that we shall be able to get a formula for such an expression as $\int_0^x e^t f(t)dt$. But it may be practicable to estimate values of this integral by numerical methods, and thereby get a table of values for y, even if we don't have it in 'closed form'.

12O Example Consider the equation

$$y(x) = \frac{1}{x^2}\int_0^x y(t)y(x-t)dt.$$

Let us, optimistically, assume that this equation has a solution y defined on $[0, \infty[$ which has a Laplace transform $Y(s)$ defined for at least some s. Then we have

$$x^2 y(x) = \int_0^x y(t)y(x-t)dt,$$

so that

$$Y'' = Y^2.$$

There is a standard trick for solving differential equations of the form $Y'' = f(Y, Y')$; set $V = Y'$, so that

$$Y'' = \frac{d^2Y}{d^2s} = \frac{dv}{ds} = \frac{dV}{dY}\frac{dY}{ds} = V\frac{dV}{dY},$$

and

$$V\frac{dV}{dY} = Y^2, \quad V\,dV = Y^2\,dY, \quad \frac{1}{2}V^2 = \frac{1}{3}Y^3 + \text{constant}.$$

To determine the value of the constant here, note that (if this is going to work at all) $\lim_{\mathcal{R}\mathrm{e}\, s\to\infty} V(s) = \lim_{\mathcal{R}\mathrm{e}\, s\to\infty} Y(s) = 0$ (see 12Be), so we must have

$$\Bigl(\frac{dY}{ds}\Bigr)^2 = V^2 = \frac{2}{3}Y^3.$$

Without fretting too much about whether $Y(s)^{3/2}$ is really definable on a strict interpretation, write

$$\frac{dY}{ds} = \sqrt{\frac{2}{3}}Y^{3/2}, \quad \frac{dY}{Y^{3/2}} = \sqrt{\frac{2}{3}}ds, \quad -\frac{2}{Y^{1/2}} = \sqrt{\frac{2}{3}}(s+c), \quad Y = \frac{6}{(s+c)^2}.$$

This gives us

$$y = 6e^{-cx}x.$$

All the arguments above are shaky enough so that we must immediately check that this works: we have

$$\begin{aligned}
\int_0^x y(t)y(x-t)dt &= \int_0^x 6e^{-ct}t.6e^{-c(x-t)}(x-t)dt \\
&= 36e^{-cx}\int_0^x t(x-t)dt \\
&= 36e^{-cx}\Bigl[\frac{1}{2}xt^2 - \frac{1}{3}t^3\Bigr]_0^x \\
&= 6x^3e^{-cx} \\
&= x^2y(x),
\end{aligned}$$

as required. Of course this check is no guarantee that there are not other, equally good solutions; for instance, it misses the solution $y = 0$ (corresponding to $c \to \infty$). But at least the method picks up one non-trivial family of solutions.

Version of 14.1.99

13. Fourier Transforms

For the last five lectures of the course, I seek to give a taste of Fourier analysis. This will necessarily be unnaturally curtailed, but I hope I can indicate both something of what it is and something of what it's for.

13A The formulae If f is a *complex-valued* function of a *real* variable, write

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iyx}f(x)dx,$$

$$\check{f}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iyx}f(x)dx = \hat{f}(-y).$$

$\hat{f}$ is the **Fourier transform** of f, and $\check{f}$ is the **inverse Fourier transform**.

13B Notes (a) There are two particular things to watch in these formulae; the constant $\frac{1}{\sqrt{2\pi}}$ and the sign $-$ in e^{-ixy} in the first. Several choices are being made here, and different authors do different things. The six commonest variations are, I think,

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mp iyx}f(x)dx,$$

$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{\mp iyx}f(x)dx,$$

$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{\mp 2\pi i y x} f(x) dx,$$

corresponding to inverse transforms

$$\check{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\pm i y x} f(x) dx,$$

$$\check{f}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\pm i y x} f(x) dx,$$

$$\check{f}(y) = \int_{-\infty}^{\infty} e^{\pm 2\pi i y x} f(x) dx.$$

The mathematics of the theory is identical whichever of these you use, but the formulae are different, in sometimes surprising ways. *I therefore strongly recommend* that you pick one and stick to it (as I shall do). If, at any point in your future career, you find yourself using one of the other systems, your first task will be to rewrite all your notes, from beginning to end, in the new system, before locking the original versions away where they can't confuse you.

In particular, I see that Kreyszig uses $\pm$ where I use $\mp$. Since he hardly touches on the theory of (complex) Fourier transforms I do not think this will affect you.

(b) I am deliberately not saying what I mean by $\int_{-\infty}^{\infty}$ here. The 'real' theory of Fourier analysis is astonishingly robust under changing interpretations of its formulae. Any pedantic insistence on saying exactly what one means is liable to exclude some important idea. Of course when one comes to *proofs*, it is necessary to be precise; and there are many traps into which blind faith can lead. This is not a course about proofs anyway, and I think that the theory as a whole is best introduced not as a logical structure but as a pattern of ideas; ideas which for a century and three-quarters have been among the most fertile in mathematics.

Fourier's thesis itself was criticized, and sent back for revision, on the grounds that the proofs were inadequate. And this was not just that the thesis committee was being fussy. If you take a serious pure mathematician's approach to Fourier analysis (e.g., T.W.Körner, *Fourier Analysis,* Cambridge U.P., 1988), and look at the theorems there, you will find that hardly one was proved by, or even known to, Fourier himself. The very language in which they are stated ('continuous function', 'differentiable function', 'function of bounded variation', 'square-integrable function') did not exist in his time. It was sixty years after Fourier started the subject before anybody had any solid idea of when the formulae worked and when they didn't. But Fourier's *idea*, that one could study functions (whatever they might be – even that was not clear then!) in terms of their Fourier series and transforms, has created as much mathematics as any since the invention of calculus.

13C Manipulating Fourier transforms I think it may help if I write down a list of the basic formulae of Fourier analysis, leaving explanations to one side for the moment. The fundamental ideas are encapsulated in the principles

Linearity $\widehat{f+g} = \hat{f} + \hat{g}, \quad \widehat{cf} = c\hat{f}$.

Shift If $g(x) = f(x+a)$ then $\hat{g}(y) = e^{iya}\hat{f}(y)$.

Scale If $g(x) = f(ax)$, where $a > 0$, then $\hat{g}(y) = \frac{1}{a}\hat{f}(\frac{y}{a})$.

Symmetry If $g(x) = f(-x)$ then $\hat{g}(y) = \hat{f}(-y)$.

Complex Conjugate If $g(x) = \overline{f(x)}$ then $\hat{g}(y) = \overline{\hat{f}(-y)}$.

Differentiation If $g(x) = f'(x)$ then $\hat{g}(y) = iy\hat{f}(y)$.

Duality $\int_{-\infty}^{\infty} f(x)\hat{g}(x)dx = \int_{-\infty}^{\infty} \hat{f}(y)g(y)dy$.

Convolution If we write $(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt$, then

$$(f * g)^\wedge(y) = \sqrt{2\pi}\hat{f}(y)\hat{g}(y), \quad (f \times g)^\wedge(y) = \frac{1}{\sqrt{2\pi}}(\hat{f} * \hat{g})(y).$$

Inversion $\overset{\vee}{\hat{f}} = \overset{\wedge}{\check{f}} = f$.

Reversal $\check{f}(y) = \hat{f}(-y)$.

Parseval $\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} \hat{f}(y)\overline{\hat{g}(y)}dy$.

13D Where the rules come from (I) I say again, we are not here in the business of 'proving' anything. But I think that they will be easier to remember, and make better sense, if we take a moment to look at ways in which some, at least, of the rules above grow out of the defining formulae.

(a) Linearity All this says is that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy}(f+g)(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy}f(x)dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy}g(x)dx,$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy}(cf)(x)dx = c\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy}f(x)dx;$$

that is, that $\int$ is a linear operator.

(b) Shift This is a simple substitution $\xi = x + c$:

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy}f(x+c)dx &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i(\xi-c)y}f(\xi)d\xi \\ &= e^{icy}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi y}f(\xi)d\xi.\end{aligned}$$

(c) Scale This is the substitution $\xi = cx$:

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy}f(cx)dx &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi y/c}f(\xi)\frac{d\xi}{c} \\ &= \frac{1}{c}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi y/c}f(\xi)d\xi.\end{aligned}$$

(d) Symmetry This is the substitution $\xi = -x$:

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy}f(-x)dx &= \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{i\xi y}f(\xi)(-d\xi) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi(-y)}f(\xi)d\xi.\end{aligned}$$

(e) Complex Conjugate This is because $\overline{e^{-ixy}} = e^{ixy}$ and $\int\overline{f} = \overline{\int f}$:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy}\overline{f(x)}dx = \overline{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixy}f(x)dx}.$$

13E Where the rules come from (II) The formulae in 13D rely only on the most fundamental properties of integration on the real line, and are accordingly universal and robust and can be trusted in almost any context in which they make sense at all. The next group use more sophisticated ideas and are correspondingly both less reliable and more important.

(a) Differentiation This is integration by parts:

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy} f'(x)dx \\
&\quad = \frac{1}{\sqrt{2\pi}}\Big[e^{-ixy}f(x)\Big]_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(-iy)e^{-ixy}f(x)dx \\
&\quad = \frac{1}{\sqrt{2\pi}}\big(\lim_{x\to\infty} e^{-ixy}f(x) - \lim_{x\to-\infty} e^{-ixy}f(x)\big) + iy\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy}f(x)dx \\
&\quad = iy\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy}f(x)dx
\end{aligned}$$

at least when $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$.

(b) Duality This involves a change in the order of integration:

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\Big(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy}g(y)dy\Big)dx \\
&\quad = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(y)\Big(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy}f(x)dx\Big)dy
\end{aligned}$$

(we hope).

(c) Convolution Another change in the order of integration:

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy}\Big(\int_{-\infty}^{\infty} f(t)g(x-t)dt\Big)dx \\
&\quad = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\Big(\int_{-\infty}^{\infty} e^{-ixy}g(x-t)dx\Big)dt
\end{aligned}$$

(we hope)

$$\quad = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\Big(\int_{-\infty}^{\infty} e^{-i(\xi+t)y}g(\xi)d\xi\Big)dt$$

(substituting $\xi = x - t$)

$$\begin{aligned}
&\quad = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ity}f(t)\sqrt{2\pi}\hat{g}(y)dt \\
&\quad = \sqrt{2\pi}\hat{f}(y)\hat{g}(y).
\end{aligned}$$

The other ‘convolution’ formula comes from putting this together with the ‘inversion’ formula.

13F The hard one: inversion Here we do not really have the apparatus to make sense of the general phenomenon. What I will do is to give a handful of examples in which we can do the calculations and see at least (i) a sort of explanation of the factor $\sqrt{2\pi}$ we use (ii) an indication of why the inverse Fourier transform is not identical to the Fourier transform.

(a) Example Set $f(x) = 1$ for $-1 \le x \le 1$, 0 otherwise. Then

$$\begin{aligned}
\hat{f}(y) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy}f(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-ixy}dx \\
&= \frac{1}{\sqrt{2\pi}}\Big[e^{-ixy}\Big]_{-1}^{1} = \frac{1}{\sqrt{2\pi}}\frac{1}{-iy}(e^{-iy} - e^{iy}) \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{iy}(e^{iy} - e^{-iy}) = \frac{1}{\sqrt{2\pi}}\frac{2\sin y}{y}.
\end{aligned}$$

So

$$\begin{aligned}\overset{\wedge\vee}{f}(0) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i0y}\hat{f}(y)dy\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\frac{2\sin y}{y}dy\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin y}{y}dy\\ &= \frac{2}{\pi}\int_{0}^{\infty}\frac{\sin y}{y}dy = 1 = f(0)\end{aligned}$$

by the calculation in 7E above.

Of course this is extraordinarily specialized; we cannot even find $\overset{\wedge\vee}{f}(x)$ for any non-zero x by this method, let alone for other functions f.

(b) Consider the function $f(x) = e^{-x}$ for $x \geq 0$, 0 for $x < 0$. Then

$$\begin{aligned}\hat{f}(y) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy}f(x)dx = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-ixy}e^{-x}dx\\ &= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{-1-iy}e^{-ixy}e^{-x}\right]_0^{\infty} = \frac{1}{\sqrt{2\pi}}\frac{1}{1+iy}.\end{aligned}$$

So

$$\begin{aligned}\overset{\wedge\vee}{f}(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixy}\hat{f}(y)dy\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\frac{e^{ixy}}{1+iy}dy\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ixy}}{1+iy}dy.\end{aligned}$$

Now, at least for $x > 0$, this is made for one of our standard techniques from Chapter 7. Integrate $\frac{e^{ixz}}{1+iz}$ around a large semicircle based on $[-R, R]$. By Jordan's Lemma, the integral around the curved section will converge to 0 (because $x > 0$ and $\lim_{|z|\to\infty}|\frac{1}{1+iz}| = 0$), while the integral around the whole contour will be

$$2\pi i \operatorname{Res}_i = 2\pi i\frac{e^{ixi}}{i} = 2\pi e^{-x}.$$

So

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ixy}}{1+iy}dy = \frac{1}{2\pi}\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{ixy}}{1+iy}dy = \frac{1}{2\pi}2\pi e^{-x} = e^{-x} = f(x).$$

For $x < 0$ we can use the same method, provided we first change the integral to

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i(-x)(-y)}}{1+iy}dy = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i(-x)y}}{1-iy}dy.$$

Again using the big-semicircle method, we find that (because $-x > 0$) we just have to integrate $\frac{e^{-ixz}}{1-iz}$ around large semicircles above the real axis; but this is always zero, because the singularity is now at $-i$, outside the curve. So $\overset{\wedge\vee}{f}(x) = 0$ for $x < 0$.

Finally, *provided* we interpret it as

$$\lim_{R\to\infty}\frac{1}{2\pi}\int_{-R}^{R}\frac{1}{1+iy}dy,$$

we can compute $\overset{\wedge\vee}{f}(0)$, as follows. $\frac{1}{1+iy} = \frac{1}{i}\frac{d}{dy}\operatorname{Ln}(1+iy)$ for all real y ($1 + iy \in \operatorname{dom}\operatorname{Ln}$ for all real y, and Ln is analytic). So the integral is

$$
\begin{aligned}
\lim_{R\to\infty}\frac{1}{2\pi}\int_{-R}^{R}\frac{1}{1+iy}dy &= \frac{1}{2\pi}\lim_{R\to\infty}\left[\frac{1}{i}\,\mathrm{Ln}(1+iy)\right]_{-R}^{R}\\
&= \frac{1}{2\pi i}\lim_{R\to\infty}\mathrm{Ln}(1+iR)-\mathrm{Ln}(1-iR)\\
&= \frac{1}{2\pi i}\lim_{R\to\infty}\ln(|1+iR|)+i\arg(1+iR)\\
&\qquad\qquad -\ln(|1-iR|)-i\arg(1-iR)\\
&= \frac{1}{2\pi}\lim_{R\to\infty}\arg(1+iR)-\arg(1-iR)\\
&= \frac{1}{2\pi}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)=\frac{1}{2}.
\end{aligned}
$$

I include this last step because it is characteristic of the theory of Fourier transforms, as in Fourier series, that the inverse Fourier transform, when interpreted as $\frac{1}{\sqrt{2\pi}}\lim_{R\to\infty}\int_{-R}^{R}e^{ixy}\hat{f}(y)dy$, converges at jumps to $\frac{1}{2}(\lim_{t\downarrow x}f(t)+\lim_{t\uparrow x}f(t))$; here we have

$$\frac{1}{2}(\lim_{t\downarrow 0}f(t)+\lim_{t\uparrow 0}f(t))=\frac{1}{2}(\lim_{t\downarrow 0}e^{-t})+\lim_{t\uparrow 0}0)=\frac{1}{2}.$$

(c) For a third example, I calculate the Fourier transform of the function $f(x)=e^{-x^2/2}$. Here we have

$$\hat{f}(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ixy}e^{-x^2/2}dx.$$

Once again, we have a contour-integration trick for this. Integrate $e^{-z^2/2}$ around a large rectangle with vertices at $\pm R$, $\pm R-iy$. $e^{-z^2/2}$ is an entire function, defined everywhere in $\mathbb{C}$, so the whole integral will be zero, by Cauchy's theorem. The four sides of the rectangle give contributions

(along base) parametrize as $z=x$ for $-R\le x\le R$, $dz=dx$,

$$I_1=\int_{-R}^{R}e^{-x^2/2}dx\to\int_{-\infty}^{\infty}e^{-x^2/2}dx=\sqrt{2\pi}$$

(I hope you knew this from MA102 or MA257).

(along right edge) Length of edge is $|y|$, modulus of function is at most

$$
\begin{aligned}
\sup_{|t|\le|y|}|\exp(-\tfrac{1}{2}(R+it)^2)| &= \sup_{|t|\le|y|}|\exp\frac{1}{2}(-R^2+t^2-2iRt)|\\
&= \sup_{|t|\le|y|}|\exp\frac{1}{2}(-R^2+t^2)\exp(-iRt)|\\
&= \sup_{|t|\le|y|}\exp\frac{1}{2}(-R^2+t^2)=e^{-R^2/2}e^{y^2/2}.
\end{aligned}
$$

So

$$|I_2|\le|y|e^{-R^2/2}e^{y^2/2}\to 0\text{ as }R\to\infty.$$

(other horizontal edge) parametrize as $z=-x-iy$ for $-R\le x\le R$, $dz=-dx$, to get

$$
\begin{aligned}
I_3 &= \int_{-R}^{R}\exp(-(-x-iy)^2/2)(-dx)\\
&= -\int_{-R}^{R}\exp(\tfrac{1}{2}(-x^2+y^2-2ixy))dx\\
&= -e^{y^2/2}\int_{-R}^{R}e^{-ixy}e^{-x^2/2}dx\\
&\to -e^{y^2/2}\int_{-\infty}^{\infty}e^{-ixy}e^{-x^2/2}dx
\end{aligned}
$$

as $R \to \infty$.

(left-hand edge) Again, the length of the curve is $|y|$ and the greatest value of the modulus of the integrand is

$$\begin{aligned}\sup_{|t|\le|y|} |\exp(-\tfrac{1}{2}(-R+it)^2)| &= \sup_{|t|\le|y|} |\exp\tfrac{1}{2}(-R^2+t^2+2iRt)| \\ &= \sup_{|t|\le|y|} |\exp\tfrac{1}{2}(-R^2+t^2)\exp iRt| \\ &= \sup_{|t|\le|y|} \exp\tfrac{1}{2}(-R^2+t^2) \\ &= e^{-R^2/2}e^{y^2/2}.\end{aligned}$$

So once again we have

$$|I_4| \le |y|e^{-R^2/2}e^{y^2/2} \to 0 \text{ as } R \to \infty.$$

Putting these together, we have

$$\begin{aligned}0 &= I_1 + I_2 + I_3 + I_4 \\ &\to \int_{-\infty}^{\infty} e^{-x^2/2}dx + 0 - e^{y^2/2}\int_{-\infty}^{\infty} e^{-ixy}e^{-x^2/2}dx + 0\end{aligned}$$

as $R \to \infty$. So we must have

$$\int_{-\infty}^{\infty} e^{-ixy}e^{-x^2/2}dx = e^{-y^2/2}\int_{-\infty}^{\infty} e^{-x^2/2}dx = \sqrt{2\pi}e^{-y^2/2},$$

and

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy}e^{-x^2}dx = e^{-y^2/2}.$$

(Note that since I have not decided what the sign of y is, I don't know whether I_1 or I_3 is the 'bottom' edge, so I don't know whether I am going round the rectangle in a positive or negative orientation. But as we are applying Cauchy's theorem, rather than the Residue Theorem, it doesn't matter.)

Remarks It is one of the delights of mathematics that a function like $e^{-x^2/2}$, of enormous importance in probability theory, can reappear in a pivotal role in a quite different subject. The result in (c) above is itself of great importance in probability theory, where we need to be able to calculate the 'characteristic functions' of distributions.

Note that in each of the three examples, the factor $\frac{1}{2\pi}$ or $\frac{1}{\sqrt{2\pi}}$ introduced by the Fourier transforms cancels out nicely against formulae ($\int_{-\infty}^{\infty} \frac{\sin x}{x}dx = \pi$, $\int_\Gamma f(z)dz = 2\pi i \sum \mathrm{Res}$, $\int_{-\infty}^{\infty} \exp(-x^2/2)dx = \sqrt{2\pi}$) which had π's in them for apparently unrelated reasons.

13G Parseval is now easy. Observe that

$$\begin{aligned}\overline{\hat{g}(y)} &= \overline{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy}g(x)dx} \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixy}\overline{g(x)}dx \\ &= \check{\bar{g}}(y).\end{aligned}$$

So

$$\begin{aligned}
&\int_{-\infty}^{\infty} \hat{f}(y)\overline{\hat{g}(y)}dy \\
&\quad = \int_{-\infty}^{\infty} \hat{f}(y)\check{\bar{g}}(y)dy \\
&\quad = \int_{-\infty}^{\infty} f(y)\check{\bar{g}}^{\wedge}(y)dy \\
&\quad = \int_{-\infty}^{\infty} f(y)\overline{g(y)}dy
\end{aligned}$$

because $\check{\bar{g}}^{\wedge} = \bar{g}$.

(I repeat: this is just to show that the formulae hang together. The question of just which functions f and g behave in such a way that the integrals here make sense is quite another matter.)

13H The heat equation One of Fourier's own applications of the ideas here (and one of the ones people criticized as having unsatisfactory foundations) was to a partial differential equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}.$$

This is one of the most important equations of applied mathematics. The characteristic feature of partial differential equations is that (because we have a function of two or more variables) the specification of boundary conditions is enormously subtle. Here I want to look at the problem in which we are given $f(x,0) = f_0(x)$ and seek to find $f(x,t)$ for $t \geq 0$. Now, let us take the Fourier transform of the equation. If we look at

$$\widehat{\frac{\partial f}{\partial t}}(y) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} e^{-ixy}\frac{\partial f}{\partial t}(x,t)dx$$

then, provided the operators $\int dx$ and $\frac{\partial}{\partial t}$ commute (which does happen, for well-enough-behaved functions – never mind how well-behaved they have to be) we can hope that, setting $g(x,t) = \dfrac{\partial f}{\partial t}(x,t)$, we shall get

$$\hat{g}(y,t) = \frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial t} \int_{\infty}^{\infty} e^{-ixy} f(x,t)dx = \frac{\partial}{\partial t}\hat{f}(y,t).$$

On the other hand, using the standard formula

$$(f')^{\wedge}(y) = iy\hat{f}(y), \quad (f'')^{\wedge}(y) = -y^2\hat{f}(y)$$

we get

$$\hat{g}(y,t) = -\frac{1}{2}y^2\hat{f}(y,t)$$

because $g = \dfrac{1}{2}\dfrac{\partial^2 f}{\partial x^2}$. So we have the equation

$$\frac{\partial}{\partial t}\hat{f} = -\frac{1}{2}y^2\hat{f}.$$

Now this we can solve; the solution is

$$\hat{f}(y,t) = e^{-ty^2/2}\hat{f}(y,0) = e^{-ty^2/2}\hat{f}_0(y).$$

Now we need to know the inverse Fourier transform of $e^{-ty^2/2}$. But we know that if $h(x) = e^{-x^2/2}$ then $\hat{h}(y) = e^{-y^2/2}$ (13Fc); so if we set $h_\gamma(x) = h(\gamma x) = e^{-\gamma^2x^2/2}$, then

$$\hat{h}_\gamma(y) = \frac{1}{\gamma}\hat{h}(\frac{y}{\gamma}) = \frac{1}{\gamma}e^{-y^2/2\gamma^2},$$

at least for $\gamma > 0$, by the 'scaling' rule. Setting $\gamma = 1/\sqrt{t}$, therefore, we see that the Fourier transform of $k_t(x) = \frac{1}{\sqrt{t}}e^{-x^2/2t}$ is $e^{-ty^2/2}$. Accordingly the Fourier transform of $k_t * f_0$ is

$$(k_t * f_0)^\wedge = \sqrt{2\pi}\hat{k}_t \times \hat{f}_0 = \sqrt{2\pi}\hat{f}$$

by the ‘convolution’ rule. So we get back to f by the formula

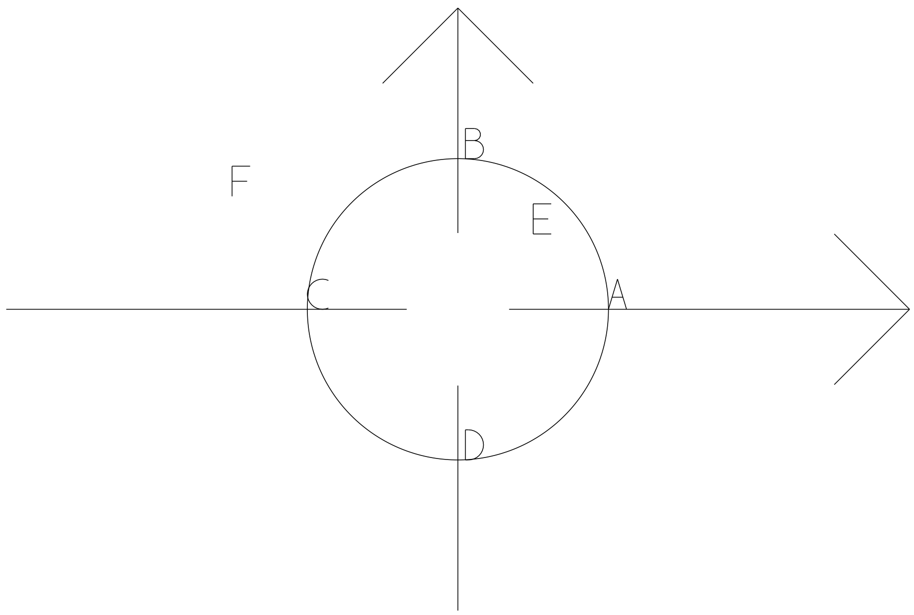
$$f_t = \frac{1}{\sqrt{2\pi}} k_t * f_0,$$

that is,

$$f(x,t) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-u^2/2t} f_0(x-u)du.$$

(Let me repeat, as a pure mathematician, that Fourier’s critics were absolutely right to doubt the validity of the method. Any ‘proof’ along the lines here must rely on some exceedingly deep ideas. But once we have the formula, derived by whatever method, we can then set out to check that it makes sense and gives valid solutions to the equations starting from whatever functions f_0 we are interested in. For instance, the final formula allows such a function as $f_0(x) = |x|$, for which the Fourier transform is a very doubtful construction.)

$w = f(z) = 1/z$ transforms



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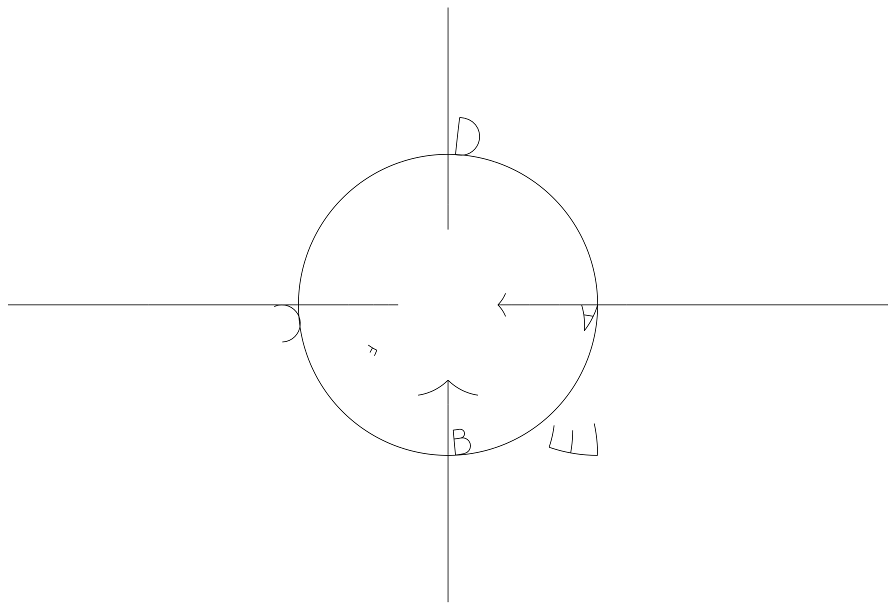
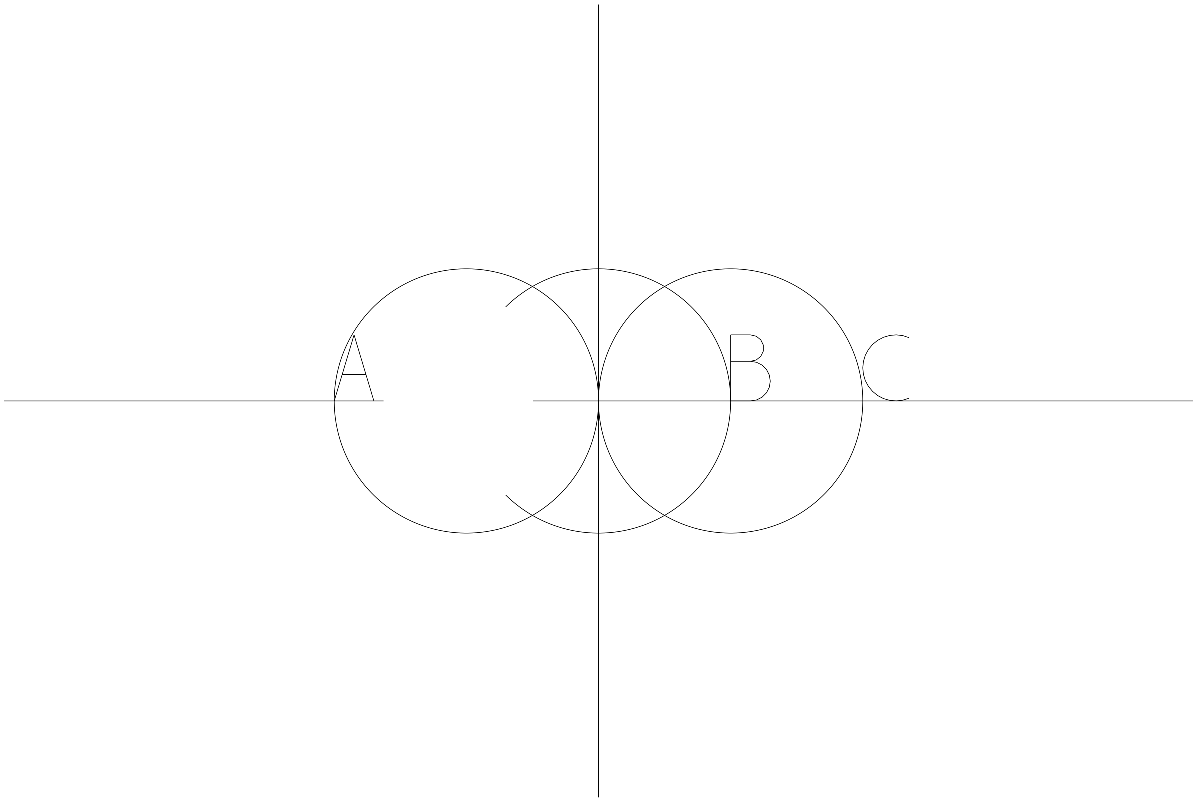


$w = \frac{3z-2}{2z+2}$ transforms



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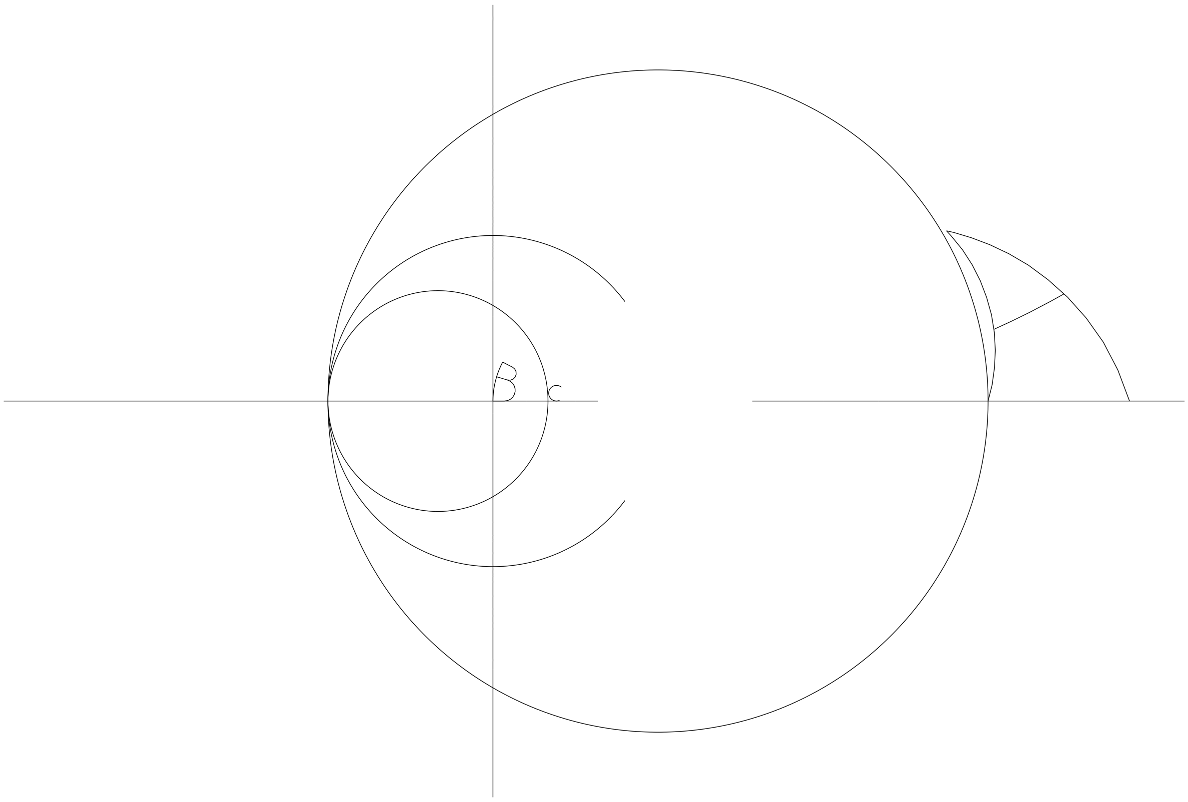
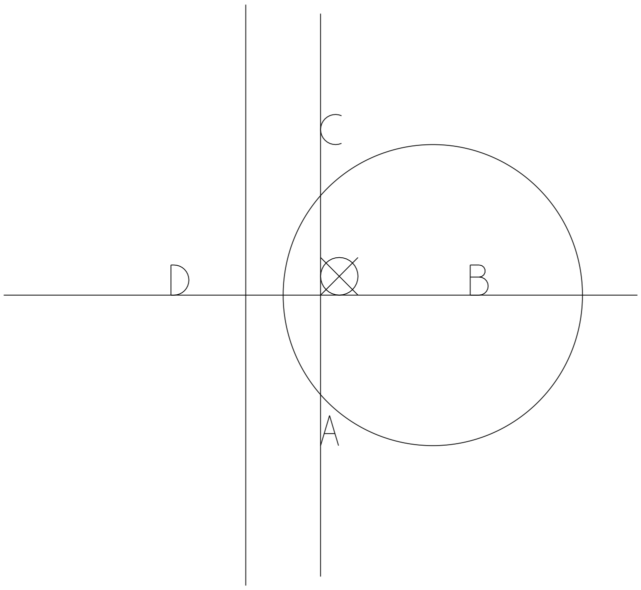


$w = z^2$ transforms



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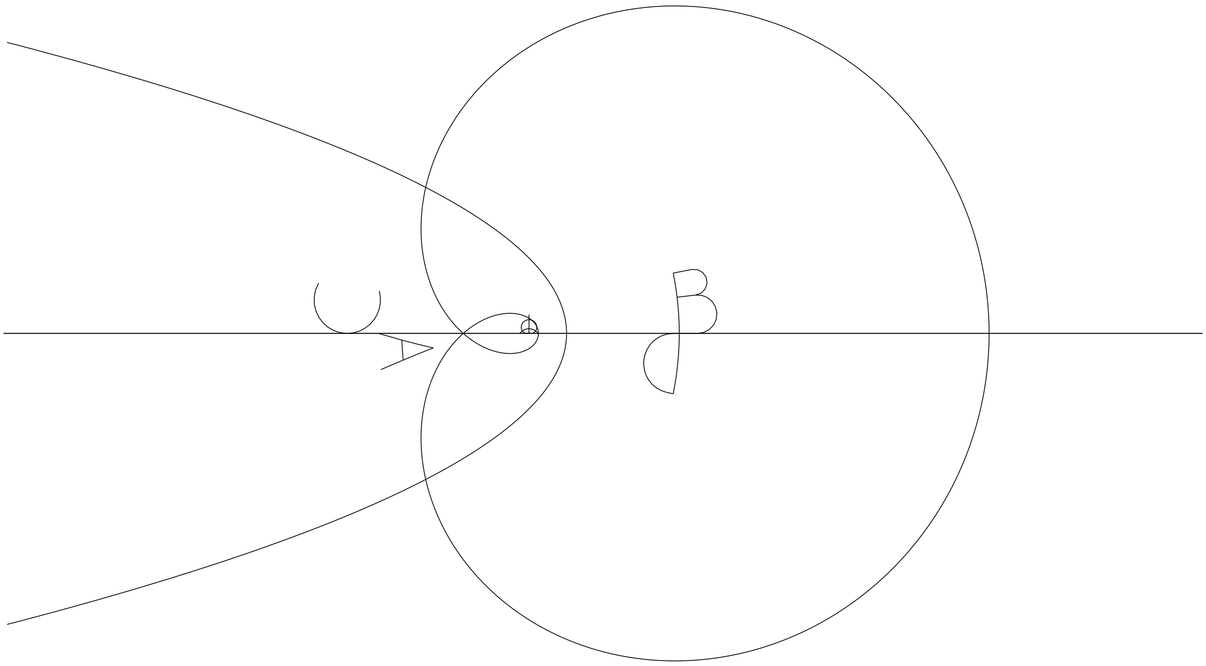
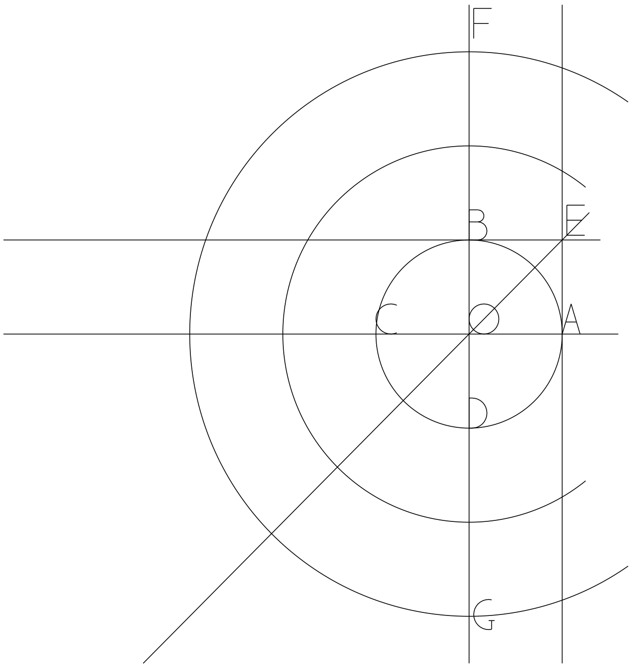


$w = \exp z$ transforms



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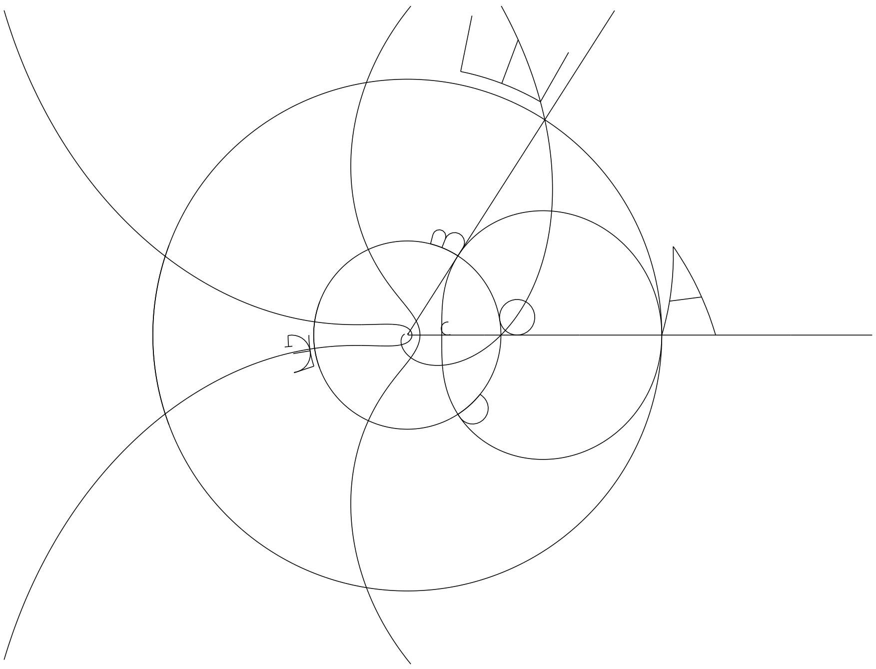
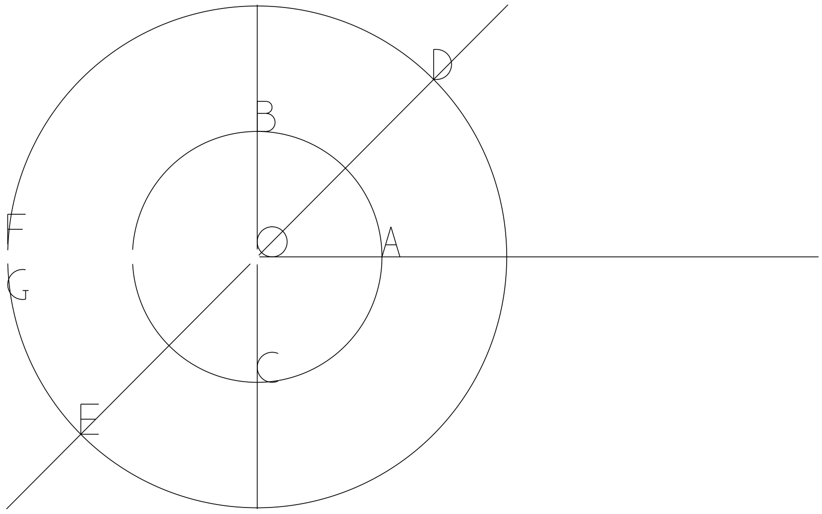


$w = \mathrm{Ln}\, z$ transforms



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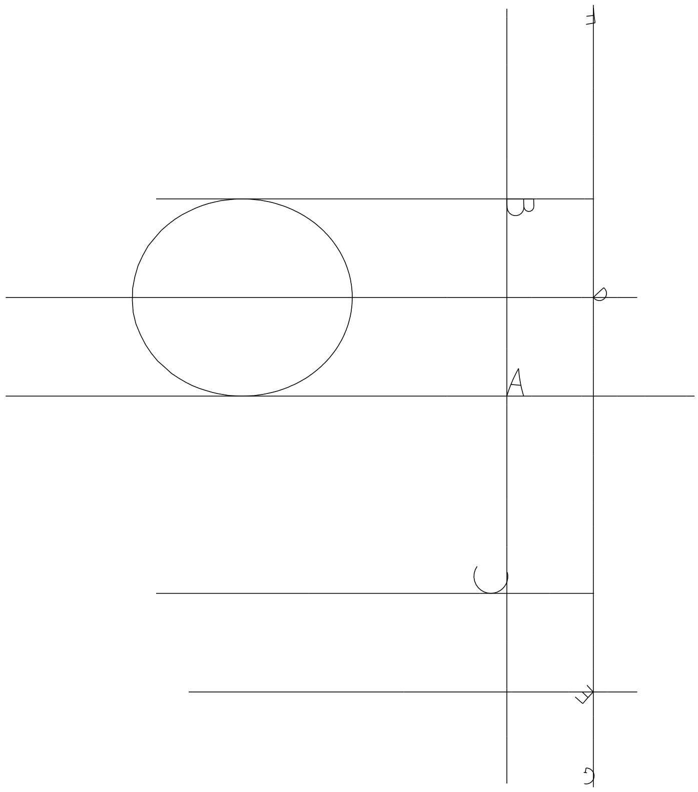
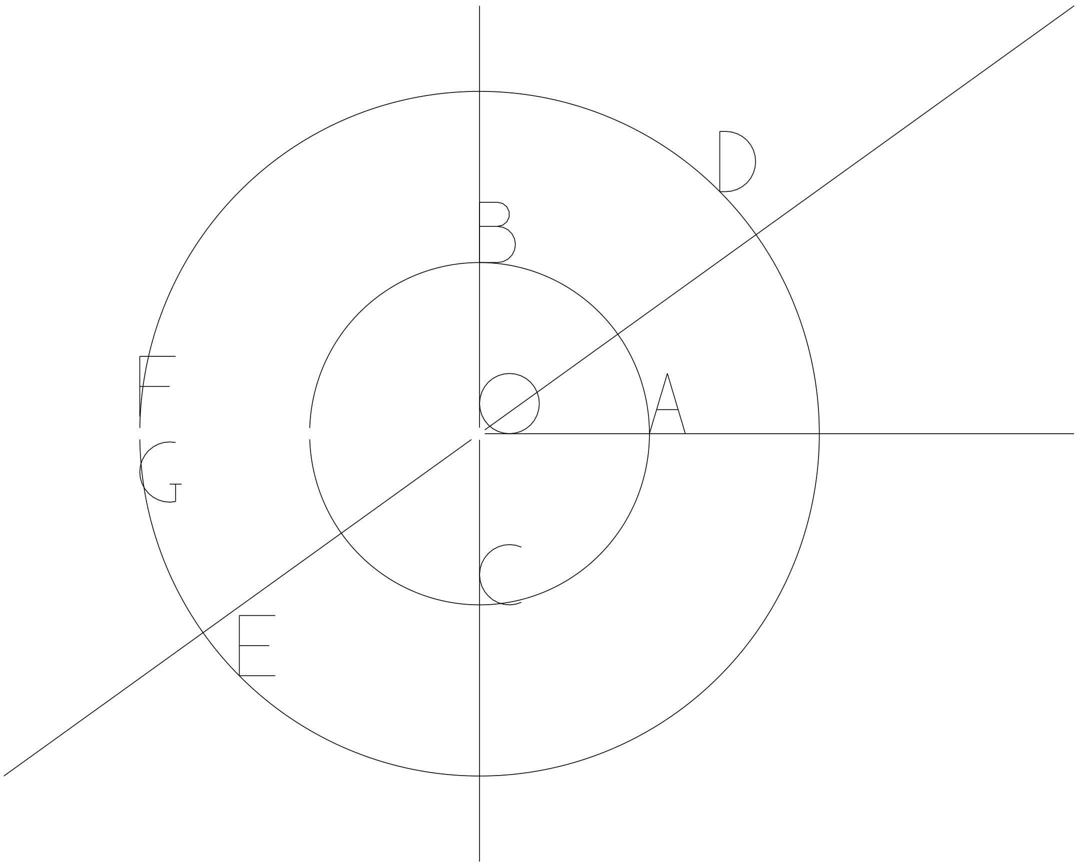
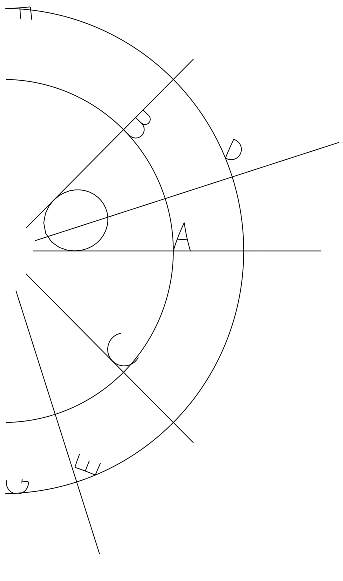


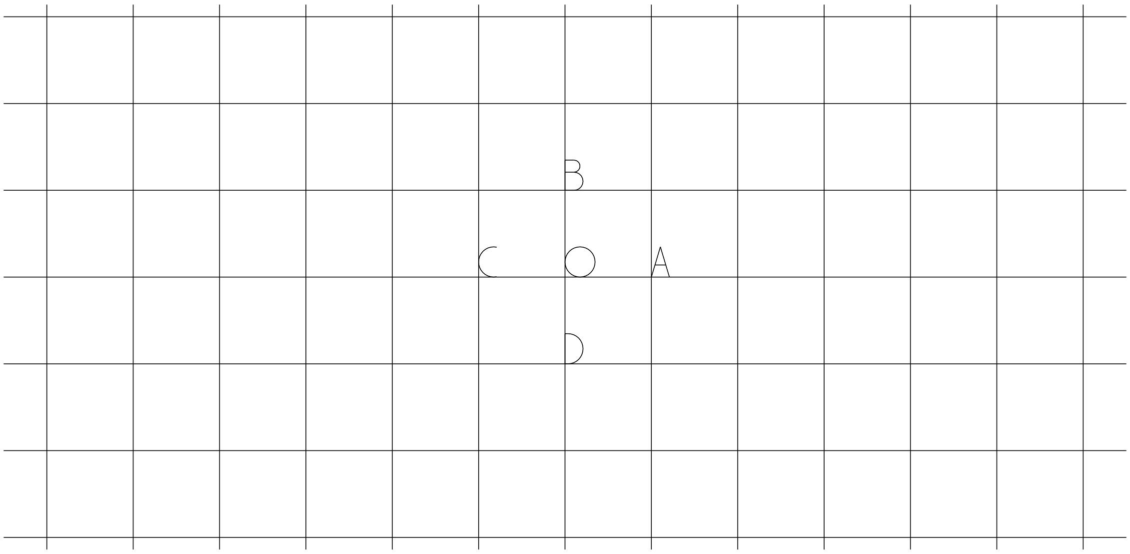
$w = z^{1/2}$ transforms



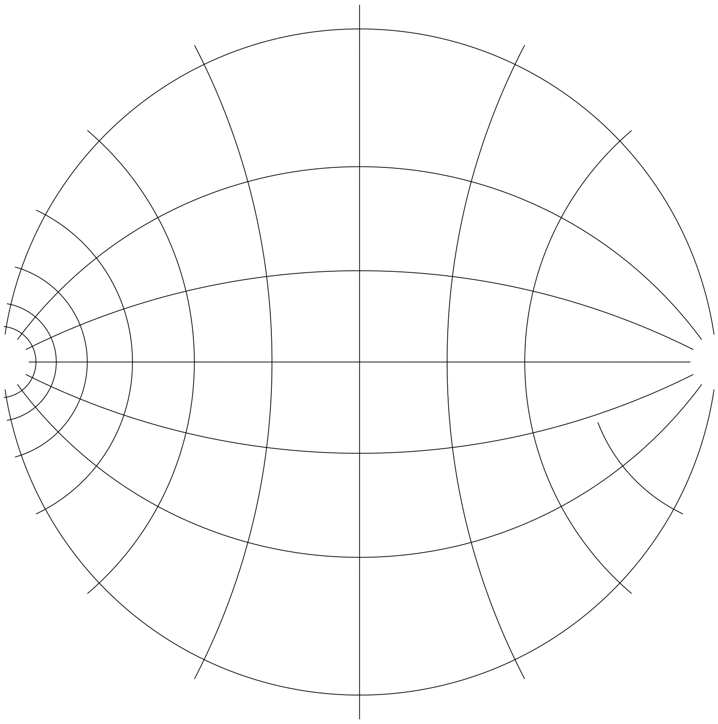
into



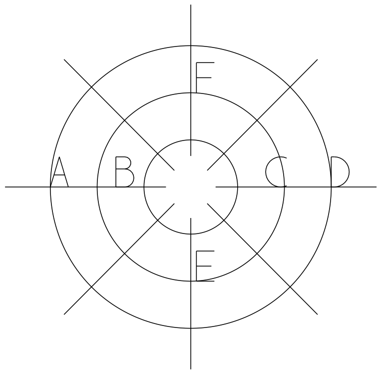
$w = \tanh \frac{z}{4}$ transforms



into



$w = \frac{1}{2}(z + \frac{1}{z})$ transforms



into