Complex Variable II Answers

1. (a) Pole, order 3, residue 5 at 0;
pole, order 3, residue i at $-\frac{i}{2}$;
pole, order 4, residue 0 at $\frac{1}{3}$.

(b) Pole, order 2, residue 0 at i;
pole, order 2, residue 0 at $-i$.
(The residues must be 0 because $\frac{z}{(z^2+1)^2} = \frac{d}{dz}(-\frac{1}{2(z^2+1)})$ is a perfect derivative.)

(c) Pole, order 4, residue $\frac{1}{6}$ at 0.

(d) Essential singularity, residue 1 at 0.

(e) Pole, order 1, residue 1 at $2n\pi i$, for every $n \in \mathbb{Z}$.

(f) Removable singularity (residue 0) at 0;
pole, order 1, residue -1 at $2n\pi i$ for every $n \in \mathbb{Z} \setminus \{0\}$.

(g) (if $a \neq 0$) Pole, order 2, residue $-(\lambda a + 1)e^{-\lambda a} i/4a^3$ at ia;
pole, order 2, residue $(1 - \lambda a)e^{\lambda a} i/4a^3$ at $-ia$;
(if $a = 0$) pole, order 4, residue $-\lambda^3 i/6$ at 0.

(h) Essential singularity, residue 0 at $2n\pi$ for every $n \in \mathbb{Z}$. (To see that the residues are 0, observe that the function is even, so that any Laurent series for it with centre 0 has $a_n = b_n = 0$ for all odd n; in particular, $b_1 = 0$ and the residue at 0 is 0. Next, the function is periodic with period $2\pi i$, so all the residues are equal.)

2. (a) $2\pi i(\mathrm{Res}_0 + \mathrm{Res}_{-i/2} + \mathrm{Res}_{1/3}) = 2\pi i(5 + i) = 10\pi i - 2\pi$.

(b) $2\pi i \, \mathrm{Res}_i = 0$. (Or, observe that $\frac{z}{(z^2+1)^2} = \frac{d}{dz}(-\frac{1}{2(z^2+1)})$.)

(c) 0 (by Cauchy's theorem).

(d) (if $0 < |a| < 10$) $2\pi i \, \mathrm{Res}_{i|a|} = 2\pi i(-(\lambda|a| + 1)ie^{-\lambda|a|}/4|a|^3) = \pi(\lambda|a| + 1)e^{-\lambda|a|}/2|a|^3$.
If $a = 0$ or ± 10 the integral is ill-defined as Γ_2 passes through a singularity. If $|a| > 10$ the integral is 0 by Cauchy's theorem.

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3. (a)

$$\int_0^{2\pi} \frac{1}{k+\cos\theta} d\theta = \int_\Gamma \frac{1}{k+\frac{1}{2}(z+\frac{1}{z})} \frac{dz}{iz},$$

where Γ is the unit circle, taken anticlockwise. Set

$$f(z) = \frac{1}{(k+\frac{1}{2}(z+\frac{1}{z}))iz} = \frac{-2i}{(z+k+\alpha)(z+k-\alpha)}$$

where $\alpha = \sqrt{k^2-1}$. f has isolated singularities at $-k \pm \alpha$ of which $-k+\alpha$ lies within Γ. The residue there is

$$\lim_{z \to -k+\alpha} (z+k-\alpha) f(z) = \lim_{z \to -k+\alpha} \frac{-2i}{z+k+\alpha} = \frac{-2i}{2\alpha} = \frac{-i}{\alpha},$$

so the integral is

$$2\pi i \operatorname*{Res}_{-k+\alpha} f = 2\pi i \frac{-i}{\alpha} = \frac{2\pi}{\alpha} = \frac{2\pi}{\sqrt{k^2-1}}.$$

(b)

$$\int_0^{2\pi} \frac{\cos(n\theta)}{2-\cos\theta} d\theta = \mathcal{R}\mathrm{e}\left(\int_0^{2\pi} \frac{\exp(in\theta)}{2-\cos\theta} d\theta\right) = \mathcal{R}\mathrm{e}\left(\int_\Gamma \frac{z^n}{2-\frac{1}{2}(z+\frac{1}{z})} \frac{dz}{iz}\right),$$

where Γ is the unit circle, taken anticlockwise. Set

$$f(z) = \frac{z^n}{(2-\frac{1}{2}(z+\frac{1}{z}))iz} = \frac{2iz^n}{z^2-4z+1} = \frac{2iz^n}{(z-2-\sqrt{3})(z-2+\sqrt{3})}.$$

f has isolated singularities at $2 \pm \sqrt{3}$ of which $2-\sqrt{3}$ lies within Γ. The residue there is

$$\lim_{z \to 2-\sqrt{3}} (z-2+\sqrt{3}) f(z) = \lim_{z \to 2-\sqrt{3}} \frac{2iz^n}{z-2-\sqrt{3}} = \frac{2i(2-\sqrt{3})^n}{-2\sqrt{3}} = \frac{-i(2-\sqrt{3})^n}{\sqrt{3}}.$$

So the integral is

$$\int_\Gamma f(z) dz = 2\pi i \operatorname*{Res}_{2-\sqrt{3}} f = 2\pi i \frac{-i(2-\sqrt{3})^n}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}} (2-\sqrt{3})^n,$$

and this is also its real part, the integral $\int_0^{2\pi} \frac{\cos(n\theta)}{2-\cos\theta} d\theta$.

For $n \leq 0$ we have

$$\int_0^{2\pi} \frac{\cos(n\theta)}{2-\cos\theta} d\theta = \int_0^{2\pi} \frac{\cos(|n|\theta)}{2-\cos\theta} d\theta = \frac{2\pi}{\sqrt{3}} (2-\sqrt{3})^{|n|}.$$

4. (a) The singularities are at the zeroes of $1+z^4$, that is, at $\frac{1}{\sqrt{2}}(\pm 1 \pm i)$. At each singularity z_0 the residue is

$$\begin{aligned}\lim_{z\to z_0}(z-z_0)\frac{e^{imz}}{1+z^4} &= e^{imz_0}\lim_{z\to z_0}\frac{z-z_0}{1+z^4} = e^{imz_0}\lim_{z\to z_0}\frac{1}{4z^3}\\ &= e^{imz_0}\lim_{z\to z_0}\frac{z}{4z^4} = e^{imz_0}(-\frac{1}{4}z_0).\end{aligned}$$

So the residues are

at $\frac{1}{\sqrt{2}}(1+i)$,

$$-\tfrac{1}{4\sqrt{2}}e^{-m/\sqrt{2}}e^{im/\sqrt{2}}(1+i) = \tfrac{1}{4}e^{-m/\sqrt{2}}e^{im/\sqrt{2}}e^{-3\pi i/4};$$

at $\frac{1}{\sqrt{2}}(-1+i)$,

$$-\tfrac{1}{4\sqrt{2}}e^{-m/\sqrt{2}}e^{-im/\sqrt{2}}(-1+i) = \tfrac{1}{4}e^{-m/\sqrt{2}}e^{-im/\sqrt{2}}e^{-\pi i/4};$$

at $\frac{1}{\sqrt{2}}(1-i)$,

$$-\tfrac{1}{4\sqrt{2}}e^{m/\sqrt{2}}e^{im/\sqrt{2}}(1-i) = \tfrac{1}{4}e^{m/\sqrt{2}}e^{im/\sqrt{2}}e^{3\pi i/4};$$

at $\frac{1}{\sqrt{2}}(-1-i)$,

$$-\tfrac{1}{4\sqrt{2}}e^{m/\sqrt{2}}e^{-im/\sqrt{2}}(-1-i) = \tfrac{1}{4}e^{m/\sqrt{2}}e^{-im/\sqrt{2}}e^{\pi i/4}.$$

(b) If $R>1$, then for z lying on Γ_R, $|\frac{1}{1+z^4}| \le \frac{1}{R^4-1}$, while the length of Γ_R is πR. So

$$|\textstyle\int_{\Gamma_R}\frac{1}{1+z^4}dz| \le \frac{\pi R}{R^4-1} \to 0$$

as $R\to\infty$.

(c) By the Residue Theorem, if $R>1$,

$$\int_\Gamma \frac{1}{1+z^4}dz = 2\pi i(\operatorname*{Res}_{(1+i)/\sqrt{2}}\frac{1}{1+z^4} + \operatorname*{Res}_{(-1+i)/\sqrt{2}}\frac{1}{1+z^4}),$$

taking Γ to be Γ_R followed by the straight-line segment from $-R$ to R. Now the sum of the residues is (taking $m=0$ in part (a))

$$-\tfrac{1}{4\sqrt{2}}((1+i)+(-1+i)) = -\tfrac{i}{2\sqrt{2}}.$$

So the integral is $\frac{\pi}{\sqrt{2}}$. Next, parametrising the straight-line path by $z=x$, $-R\le x\le R$, $dz=dx$ we have

$$\int_{-R}^{R}\frac{1}{1+x^4}dx + \int_{\Gamma_R}\frac{1}{1+z^4}dz = \frac{\pi}{\sqrt{2}},$$

so using (b) we get

$$\int_{-\infty}^{\infty}\frac{1}{1+x^4}dx = \lim_{R\to\infty}\int_{R}^{R}\frac{1}{1+x^4}dx = \frac{\pi}{\sqrt{2}}.$$

(d) Start with $m>0$, and consider the integral of $\frac{e^{imz}}{1+z^4}dz$ around the same contour Γ as in part (c). By the Residue Theorem

$$\int_\Gamma \frac{e^{imz}}{1+z^4}dz = 2\pi i(\operatorname*{Res}_{(1+i)/\sqrt{2}}\frac{e^{imz}}{1+z^4} + \operatorname*{Res}_{(-1+i)/\sqrt{2}}\frac{e^{imz}}{1+z^4}).$$

Now the sum of the residues is

$$\begin{aligned}&-\frac{1}{4\sqrt{2}}e^{-m/\sqrt{2}}(e^{im/\sqrt{2}}(1+i)+e^{-im/\sqrt{2}}(-1+i))\\ &\quad= -\frac{1}{4\sqrt{2}}e^{-m/\sqrt{2}}((\cos\frac{m}{\sqrt{2}}+i\sin\frac{m}{\sqrt{2}})(1+i)+(\cos\frac{m}{\sqrt{2}}-i\sin\frac{m}{\sqrt{2}})(-1+i))\\ &\quad= -\frac{1}{4\sqrt{2}}e^{-m/\sqrt{2}}(2i\cos\frac{m}{\sqrt{2}}+2i\sin\frac{m}{\sqrt{2}})\\ &\quad= -\frac{i}{2\sqrt{2}}e^{-m/\sqrt{2}}(\cos\frac{m}{\sqrt{2}}+\sin\frac{m}{\sqrt{2}}).\end{aligned}$$

So the integral is $\frac{\pi}{\sqrt{2}}e^{-m/\sqrt{2}}(\cos\frac{m}{\sqrt{2}}+\sin\frac{m}{\sqrt{2}})$. Next, by Jordan's Lemma, because $m>0$ and $|\frac{1}{1+z^4}| \le \frac{1}{R^4-1}$ for z on Γ_R and $\lim_{R\to\infty}\frac{1}{R^4-1}=0$,

$$\lim_{R\to\infty}\int_{\Gamma_R}\frac{e^{imz}}{1+z^4}dz = 0.$$

So, just as in (c) above, we get

$$\int_{-\infty}^{\infty}\frac{e^{imx}}{1+x^4}dx = \frac{\pi}{\sqrt{2}}e^{-m/\sqrt{2}}(\cos\frac{m}{\sqrt{2}}+\sin\frac{m}{\sqrt{2}}).$$

Taking the real part we get

$$\int_{-\infty}^{\infty}\frac{\cos mx}{1+x^4}dx = \frac{\pi}{\sqrt{2}}e^{-m/\sqrt{2}}(\cos\frac{m}{\sqrt{2}}+\sin\frac{m}{\sqrt{2}}),$$

so that

$$\int_{0}^{\infty}\frac{\cos mx}{1+x^4}dx = \frac{\pi}{2\sqrt{2}}e^{-m/\sqrt{2}}(\cos\frac{m}{\sqrt{2}}+\sin\frac{m}{\sqrt{2}}).$$

This deals with $m>0$. The case $m=0$ has already been done in (c), and the same formula is of course valid. But for negative m we need to observe that

$$\begin{aligned}\int_0^\infty\frac{\cos mx}{1+x^4}dx &= \int_0^\infty\frac{\cos(-mx)}{1+x^4}dx\\ &= \frac{\pi}{2\sqrt{2}}e^{-(-m/\sqrt{2})}(\cos(-\frac{m}{\sqrt{2}})+\sin(-\frac{m}{\sqrt{2}})) = \frac{\pi}{2\sqrt{2}}e^{m/\sqrt{2}}(\cos\frac{m}{\sqrt{2}}-\sin\frac{m}{\sqrt{2}}).\end{aligned}$$

To get a formula valid for all m we could write

$$\int_0^\infty\frac{\cos mx}{1+x^4}dx = \frac{\pi}{2\sqrt{2}}e^{-|m|/\sqrt{2}}(\cos\frac{m}{\sqrt{2}}+\sin\frac{|m|}{\sqrt{2}}).$$

The argument given for $m>0$ fails for $m<0$ because if $m<0$ then $\lim_{R\to\infty}\int_{\Gamma_R}\frac{e^{imz}}{1+z^4}dz$ is not zero.

5. (a) Singularities at 0, residue 1; i, residue $-\frac{1}{2}e^{-m}$; and $-i$, residue $-\frac{1}{2}e^{m}$.

(b) $\frac{1}{1+z^2} = 1-z^2+z^4-\ldots$, $\frac{e^{imz}}{1+z^2} = (1+imz-\frac{m^2}{2!}z^2-\ldots)(1-z^2+z^4-\ldots) = 1+imz-(1+\frac{m^2}{2})z^2-\ldots$.

(c) By the Residue Theorem, the value of the integral will be $2\pi i\,\mathrm{Res}_{z=i}\frac{e^{iz}}{z(1+z^2)} = -\frac{\pi i}{e}$, if we take the small semicircle (let us call it Δ_ϵ) to lie above the axis. Now, writing Γ_R for the large semicircle, the integral is

$$\int_{\Gamma_R}\frac{e^{iz}}{z(1+z^2)}dz+\int_{-R}^{-\epsilon}\frac{e^{ix}}{x(1+x^2)}dx+\int_{\Delta_\epsilon}\frac{e^{iz}}{z(1+z^2)}dz+\int_{\epsilon}^{R}\frac{e^{ix}}{x(1+x^2)}dx.$$

The second integral, substituting $\xi=-x$, is

$$-\int_{\epsilon}^{R}\frac{e^{-i\xi}}{\xi(1+\xi^2)}d\xi,$$

so the sum of the second and fourth integrals is

$$\int_{\epsilon}^{R}\frac{e^{ix}-e^{-ix}}{x(1+x^2)}dx = 2i\int_{\epsilon}^{R}\frac{\sin x}{x(1+x^2)}dx.$$

Next, Jordan's Lemma tells us that the first integral tends to 0 as $R\to\infty$, because $\frac{1}{z(1+z^2)} \le \frac{1}{R(R^2-1)}$ for all z on Γ_R, and $\lim_{R\to\infty}\frac{1}{R(R^2-1)} = 0$. So we get

$$2i\int_{\epsilon}^{\infty}\frac{\sin x}{x(1+x^2)}+\int_{\Delta_\epsilon}\frac{e^{imz}}{z(1+z^2)}dz = -\frac{\pi i}{e}.$$

The next step is to estimate the integral around the small semicircle. We know from (b) that $\frac{e^{iz}}{z(1+z^2)}$ has a Laurent series, valid near 0, of the form $\frac{1}{z}+i+\ldots = \frac{1}{z}+g(z)$ say; because g is continuous at 0, $\int_{\Delta_\epsilon}g(z)dz\to 0$ as $\epsilon\downarrow 0$, and we are left with

$$\int_{\Delta_\epsilon}\frac{1}{z}dz = \int_{-\pi}^{0}\frac{1}{\epsilon e^{-i\theta}}(-i\epsilon e^{-i\theta})d\theta = \int_{-\pi}^{0}(-i)d\theta = -i\pi.$$

Accordingly

$$\lim_{\epsilon\downarrow 0}\int_{\Delta_\epsilon}\frac{e^{iz}}{z(1+z^2)}dz = -i\pi.$$

and

$$2i\int_{0}^{\infty}\frac{\sin x}{x(1+x^2)}dx - i\pi = -\frac{\pi i}{e}.$$

Finally

$$\int_{0}^{\infty}\frac{\sin x}{x(1+x^2)}dx = \frac{\pi}{2}(1-\frac{1}{e}).$$

6. (a)(i) $\mathbb{C} \setminus \{z : \mathcal{I}\mathrm{m}(z) = 0,\, \mathcal{R}\mathrm{e}(z) \geq 0\}$; $\{z : |\mathcal{I}\mathrm{m}(z)| < \pi\}$.

(ii) $\{z : z \text{ is not real}\}$; $\{i\pi, -i\pi\}$.

(iii) $\mathbb{C}$; $\mathbb{C}$.

(iv) $\mathbb{C} \setminus \{z : \mathcal{R}\mathrm{e}(z) = 0,\, |\mathcal{I}\mathrm{m}(z)| \geq 1\}$; $\{z : |\mathcal{I}\mathrm{m}(z)| < \pi\}$.

(v) $\mathbb{C} \setminus \{z : \mathcal{I}\mathrm{m}(z) = 0,\, |\mathcal{R}\mathrm{e}(z)| \leq 1\}$; $\{z : |\mathcal{I}\mathrm{m}(z)| < \pi\} \setminus \{0\}$.

(vi) $\mathbb{C} \setminus \{1\}$; $\mathbb{C} \setminus \{-1\}$.

(vii) $\mathbb{C} \setminus \{z : \mathcal{I}\mathrm{m}(z) = 0,\, |\mathcal{R}\mathrm{e}(z)| \geq 1\}$; $\{z : |\mathcal{I}\mathrm{m}(z)| < \frac{\pi}{2}$.

(viii) $\mathbb{C} \setminus \{z : \mathcal{I}\mathrm{m}(z) = 0,\, \mathcal{R}\mathrm{e}(z) \leq 0\}$; $\{z : |\arg(z)| < \frac{\pi}{3}\}$.

(ix) $\{z : \mathcal{R}\mathrm{e}(z) \neq 0\}$; $\{z : \mathcal{R}\mathrm{e}(z) > 0\}$;

(x) $\mathbb{C} \setminus \{z : \mathcal{I}\mathrm{m}(z) = 0,\, \mathcal{R}\mathrm{e}(z) \leq 0\}$; $\{z : \exp(-\pi) < |z| < \exp(\pi)\}$.

(b) The functions in (a)(ii), (iii), (iv), (v), (ix) and (x) are not injective. For the others, the inverses f^{-1} have domains the sets-of-values described above, and formulae as follows:

(i) $f^{-1}(w) = -\exp w$.

(vi) $f^{-1}(w) = \frac{w-1}{w+1}$.

(vii) $f^{-1}(w) = \frac{\exp(2w)-1}{\exp(2w)+1} = \tanh w$.

(viii) $f^{-1}(w) = \frac{1}{w^3}$.

7. (a) Singularities at $\pm i$, residues both $\frac{\operatorname{Ln} i}{2i} = \frac{\pi}{4}$.

(b) Taking the left-hand straight part of the curve to lie just above the axis, and the small almost-semicircle to lie in the upper half-plane, we get a contour lying entirely within the domain of $\frac{\operatorname{Ln} z}{1+z^2}$.

By the Residue Theorem, the value of the integral will be $2\pi i \operatorname{Res}_i \frac{\operatorname{Ln} z}{1+z^2} = \frac{\pi^2 i}{2}$. Now, writing Γ_R for the large almost-semicircle, the integral is approximately

$$\int_{\Gamma_R} \frac{\operatorname{Ln} z}{1+z^2} dz + \int_{-R}^{-\epsilon} \frac{\ln(-x) + i\pi}{1+x^2} dx + \int_{\Delta_\epsilon} \frac{\operatorname{Ln} z}{1+z^2} dz + \int_{\epsilon}^{R} \frac{\ln x}{1+x^2} dx.$$

The second integral, substituting $\xi = -x$, is

$$\int_{\epsilon}^{R} \frac{\ln \xi + i\pi}{1+\xi^2} d\xi,$$

so the sum of the second and fourth integrals is

$$\int_{\epsilon}^{R} \frac{2\ln x + i\pi}{1+z^2} dx = 2\int_{\epsilon}^{R} \frac{\ln x}{1+x^2} dx + i\pi \int_{\epsilon}^{R} \frac{1}{1+x^2} dx.$$

To estimate the first integral, observe that on Γ_R we have

$$\left|\frac{\operatorname{Ln} z}{1+z^2}\right| = \left|\frac{\ln R + i \arg z}{1+z^2}\right| \leq \frac{\ln R + \pi}{R^2 - 1},$$

while the length of Γ_R is (approximately) πR; so the integral is in modulus at most

$$\frac{\ln R + \pi}{R^2 - 1} \pi R \to 0 \text{ as } R \to \infty.$$

The next step is to estimate the integral around the small semicircle. Here we have

$$\left|\frac{\operatorname{Ln} z}{1+z^2}\right| = \left|\frac{\ln \epsilon + i \arg z}{1+z^2}\right| \leq \frac{|\ln \epsilon| + \pi}{1 - R^2},$$

while the length of Δ_ϵ is (approximately) $\pi\epsilon$; so the integral is in modulus at most

$$\frac{|\ln \epsilon| + \pi}{1 - \epsilon^2} \pi\epsilon \to 0 \text{ as } \epsilon \to 0.$$

Accordingly

$$2\int_0^\infty \frac{\ln x}{1+x^2} dx + i\pi \int_0^\infty \frac{1}{1+x^2} dx = \frac{\pi^2 i}{2}.$$

Finally, taking the real part,

$$\int_0^\infty \frac{\ln x}{1+x^2} = 0.$$

(Of course it is easy to show, using the substitution $\xi = \frac{1}{x}$, that

$$\int_0^1 \frac{\ln x}{1+x^2} dx = -\int_1^\infty \frac{\ln x}{1+x^2} dx.)$$

8. (a) $\frac{\pi\sqrt{2}}{4}$, $\frac{\pi}{4}$, $\frac{\pi\sqrt{2}}{4}$.

(b)(i) First let us calculate the integral. $\frac{z^\alpha}{1+z^4}$ has singularities at the fourth roots of -1, that is, at $\frac{1}{\sqrt{2}}(\pm 1 \pm i)$. At each singularity z_0 the residue is $\lim_{z\to z_0} z^\alpha \frac{z-z_0}{1+z^4} = \lim_{z\to z_0} z^\alpha \frac{1}{4z^3} = \lim_{z\to z_0} z^{\alpha+1}\frac{1}{4z^4} = -\frac{1}{4}z_0^{\alpha+1}$. So the sum of the residues is

$$\begin{aligned} -\frac{1}{4}\big(\exp(\frac{(\alpha+1)\pi i}{4}) + \exp(\frac{3(\alpha+1)\pi i}{4}) + \exp(-\frac{(\alpha+1)\pi i}{4}) + \exp(-\frac{3(\alpha+1)\pi i}{4})\big) \\ = -\frac{1}{2}\big(\cos\frac{\pi(\alpha+1)}{4} + \cos\frac{3\pi(\alpha+1)}{4}\big), \end{aligned}$$

and the integral is

$$-\pi i\big(\cos\frac{\pi(\alpha+1)}{4} + \cos\frac{3\pi(\alpha+1)}{4}\big).$$

(ii) Now let us examine the four sections of the integral. The first of the two straight-line segments must be regarded as lying just above the axis and the second must be regarded as lying just below the axis. So on the first we have $\arg z \doteq \pi i$ and on the second we have $\arg z \doteq -\pi i$, so that $\operatorname{Ln} z$ must be interpreted as $\ln|z| + \pi i$ and $\ln|z| - \pi i$ respectively, and z^α as $|z|^\alpha \exp i\pi\alpha$ and $|z|^\alpha \exp -i\pi\alpha$. Now parametrising the first as $z = x$, $-R \le x \le -\epsilon$ $dz = -dx$ we get an integral

$$\int_{-R}^{-\epsilon} \frac{|x|^\alpha e^{i\pi\alpha}}{1+x^4}dx = \int_\epsilon^R \frac{|x|^\alpha e^{i\pi\alpha}}{1+x^4}dx,$$

while the second (taking $z = -x$, $\epsilon \le x \le R$, $dz = -dx$) is

$$-\int_\epsilon^R \frac{|x|^\alpha e^{-i\pi\alpha}}{1+x^4}dx.$$

Summing these we get

$$\int_\epsilon^R \frac{|x|^\alpha(e^{i\pi\alpha} - e^{-i\pi\alpha})}{1+x^4}dx = 2i\sin\pi\alpha \int_\epsilon^R \frac{x^\alpha}{1+x^4}dx.$$

So if the two circles are Γ_R and Δ_ϵ we have

$$\begin{aligned} 2i\sin\pi\alpha \int_\epsilon^R \frac{x^\alpha}{1+x^4}dx + \int_{\Gamma_R}\frac{z^\alpha}{1+z^4}dz + \int_{\Delta_\epsilon}\frac{z^\alpha}{1+z^4}dz \\ = -\pi i\big(\cos\frac{\pi(\alpha+1)}{4} + \cos\frac{3\pi(\alpha+1)}{4}\big). \end{aligned}$$

(iii) Next let us examine the two circles Γ_R (anticlockwise) and Δ_ϵ (clockwise). The length of Γ_R is (approximately) $2\pi R$, and the greatest value of the integrand on Γ_R is at most

$$\frac{R^\alpha}{R^4-1},$$

so

$$\Big|\int_{\Gamma_R}\frac{z^\alpha}{1+z^4}dz\Big| \le \frac{2\pi R^{\alpha+1}}{R^4-1} \to 0$$

as $R \to \infty$ because $\alpha + 1 < 4$. Thus we have

$$2i\sin\pi\alpha \int_\epsilon^\infty \frac{x^\alpha}{1+x^4}dx + \int_{\Delta_\epsilon}\frac{z^\alpha}{1+z^4}dz = -\pi i\big(\cos\frac{\pi(\alpha+1)}{4} + \cos\frac{3\pi(\alpha+1)}{4}\big).$$

As for the inner circle, its length is approximately $2\pi\epsilon$ and the greatest value of the integrand on it is at most

$$\frac{\epsilon^\alpha}{1-\epsilon^4},$$

so

$$\Big|\int_{\Delta_\epsilon}\frac{z^\alpha}{1+z^4}dz\Big| \le \frac{2\pi\epsilon^{\alpha+1}}{1-\epsilon^4} \to 0$$

as $\epsilon \downarrow 0$ because $\alpha + 1 > 0$. Accordingly we get

$$2i\sin\pi\alpha \int_0^\infty \frac{x^\alpha}{1+x^4}dx = -\pi i\big(\cos\frac{\pi(\alpha+1)}{4} + \cos\frac{3\pi(\alpha+1)}{4}\big)$$

and, because $\sin\pi\alpha \neq 0$,

$$\int_0^\infty \frac{x^\alpha}{1+x^4}dx = -\frac{\pi}{2\sin\pi\alpha}\big(\cos\frac{\pi(\alpha+1)}{4} + \cos\frac{3\pi(\alpha+1)}{4}\big)$$

as required.

(c) The hypothesis '$\alpha > -1$' was used in showing that $\lim_{\epsilon\downarrow 0}\int_{\Delta_\epsilon}\frac{z^\alpha}{1+z^4}dz = 0$; the hypothesis '$\alpha < 4$' was used in showing that $\lim_{R\to\infty}\int_{\Gamma_R}\frac{z^\alpha}{1+z^4}dz = 0$; and the hypothesis '$\alpha$ is not an integer' was used when dividing by $\sin\pi\alpha$.

(d) $-\frac{\pi\sqrt{2}}{4}$, $-\frac{\pi}{4}$, $-\frac{\pi\sqrt{2}}{4}$.
As before, but with the signs reversed.
(Note that if you expand the formula in (b) in terms of $\cos\frac{\pi\alpha}{4}$ and $\sin\frac{\pi\alpha}{4}$, you get

$$\begin{aligned} \frac{\pi}{2\sin\pi\alpha}\big(\cos\tfrac{\pi(\alpha+1)}{4} + \cos\tfrac{3\pi(\alpha+1)}{4}\big) &= -\frac{\pi\sqrt{2}}{4(\cos\frac{\pi\alpha}{4} + \sin\frac{\pi\alpha}{4})} \\ &= \frac{\pi}{4\cos\frac{\pi(\alpha-1)}{4}}.) \end{aligned}$$

9. Integrate $\frac{z^\alpha}{z^2-2z+2}$ around a curve which runs from $-R$ to $-\epsilon$ just above the axis, from $-\epsilon$ (just above the axis) to $-\epsilon$ (just below the axis) around a clockwise circle centred on the origin, from $-\epsilon$ to $-R$ just below the axis, and from $-R$ to itself by a circle centred on 0, where $0 < \epsilon \ll 1 \ll R$.

The singularities are when $0 = z^2 - 2z + 2 = (z-1)^2 + 1$, that is, $z = 1 \pm i$. The residue at each is $\frac{z_0^\alpha}{2z_0 - 2}$; at $z_0 = 1 + i = \sqrt{2}\exp(\frac{\pi i}{4})$ this is

$$\frac{2^{\alpha/2}\exp(\frac{\alpha\pi i}{4})}{2i} = -\frac{i}{2}(1+i)^\alpha,$$

at $z_0 = 1 - i$ it is

$$\frac{2^{\alpha/2}\exp(\frac{-\alpha\pi i}{4})}{-2i} = \frac{i}{2}(1-i)^\alpha,$$

so the total integral is

$$\begin{aligned} 2\pi i \sum \mathrm{Res} &= 2\pi i \cdot 2^{\alpha/2}\tfrac{e^{\alpha\pi i/4} - e^{-\alpha\pi i/4}}{2i} \\ &= 2\pi i \cdot 2^{\alpha/2}\sin(\tfrac{\alpha\pi}{4}). \end{aligned}$$

Now for the four pieces. From $-R$ to $-\epsilon$ above the axis, parametrise as $z = x$ for $-R \le x \le -\epsilon$, $z^\alpha = (-x)^\alpha e^{i\alpha\pi}$, to get

$$\begin{aligned} I_1 &= \int_{-R}^{-\epsilon} \tfrac{(-x)^\alpha e^{i\alpha\pi}}{x^2-2x+2}\,dx \\ &= e^{i\alpha\pi}\int_R^\epsilon \tfrac{y^\alpha}{y^2+2y+2}(-dy) \end{aligned}$$

(substituting $y = -x$)

$$\begin{aligned} &= e^{i\alpha\pi}\int_\epsilon^R \tfrac{x^\alpha}{x^2+2x+2}\,dx \\ &\to e^{i\alpha\pi}J(\alpha) \end{aligned}$$

as $\epsilon \downarrow 0$ and $R \to \infty$, where $J(\alpha)$ is the integral we seek.

For the small circle, we have

$$\begin{aligned} |I_2| &\le (\text{length of curve}) \times (\text{maximum modulus}) \\ &\le 2\pi\epsilon\tfrac{\epsilon^\alpha}{2-2\epsilon-\epsilon^2} \\ &\to 0 \text{ as } \epsilon \downarrow 0 \end{aligned}$$

provided $\alpha + 1 > 0$, that is, $\alpha > -1$.

For the section from $-\epsilon$ to $-R$ below the axis, parametrise as $z = -x$ for $\epsilon \le x \le R$, $dz = -dx$, $z^\alpha = x^\alpha e^{-i\pi\alpha}$ and get

$$\begin{aligned} I_3 &= \int_\epsilon^R \tfrac{x^\alpha e^{-i\pi\alpha}}{x^2+2x+2}(-dx) \\ &\to -e^{-i\pi\alpha}J(\alpha) \end{aligned}$$

as $\epsilon \downarrow 0$ and $R \to \infty$.

As for the large circle, we again have

$$\begin{aligned} |I_4| &\le (\text{length of curve}) \times (\text{maximum modulus}) \\ &\le 2\pi R\tfrac{R^\alpha}{R^2-2R-2} \\ &\to 0 \text{ as } R \to \infty \end{aligned}$$

provided $1 + \alpha < 2$, that is, $\alpha < 1$.

Putting these together,

$$\begin{aligned} 2\pi i \cdot 2^{\alpha/2}\sin(\tfrac{\alpha\pi}{4}) &= I_1 + I_2 + I_3 + I_4 \\ &\to (e^{i\pi\alpha} - e^{-i\pi\alpha})J(\alpha) \\ &= 2i\sin(\pi\alpha)J(\alpha) \end{aligned}$$

as $\epsilon \downarrow 0$ and $R \to \infty$, provided $-1 < \alpha < 1$. So we get

$$J(\alpha) = \frac{2^{\alpha/2}\pi\sin(\frac{\alpha\pi}{4})}{\sin\pi\alpha}$$

provided $-1 < \alpha < 1$ and $\alpha \ne 0$.

10. (no solution written out)

11. We integrate $\frac{\cot \pi z}{z^2+a^2}$ around a large square with vertices at $(m+\frac{1}{2})(\pm 1 \pm i)$.

(a) The singularities of $\frac{\cot \pi z}{z^2+a^2}$ are at $\pm ia$ and at all integers n. At ia the residue is

$$\begin{aligned}
\lim_{z\to ia} \cot \pi z \frac{z-ia}{z^2+a^2} &= \frac{\cot i\pi a}{2ia} \\
&= \frac{\cos i\pi a}{2ia \sin i\pi a} \\
&= \frac{e^{-\pi a}+e^{\pi a}}{2a(e^{-\pi a}-e^{\pi a})} \\
&= \frac{\cosh \pi a}{2a \sinh \pi a}.
\end{aligned}$$

Because this is an even function of a the residue at $-ia$ is the same. At each integer n the residue is

$$\begin{aligned}
\lim_{z\to n} \frac{z-n}{\sin \pi z} \frac{\cos \pi z}{z^2+a^2} &= \frac{\cos \pi n}{n^2+a^2} \lim_{z\to n} \frac{1}{\pi \cos \pi z} \\
&= \frac{1}{\pi(n^2+a^2)}.
\end{aligned}$$

So, for $m > |a|$, the integral is

$$2\pi i\Big(\sum_{n=-m}^{m} \frac{1}{\pi(n^2+a^2)} - 2\frac{\cosh \pi a}{2a \sinh \pi a}\Big).$$

(b) Now everywhere on the contour we have $|\cot \pi z| \leq 4$, so that $|\frac{\cot \pi z}{z^2+a^2}| \leq \frac{4}{|z|^2-a^2} \leq \frac{4}{m^2-a^2}$; while the length of the contour is $4(2m+1)$, so the integral is at most $\frac{16(2m+1)}{m^2-a^2}$, which tends to 0 as $m \to \infty$. Accordingly we get

$$2\pi i\Big(\sum_{n=-\infty}^{\infty} \frac{1}{\pi(n^2+a^2)} - 2i\frac{\cosh \pi a}{2a \sinh \pi a}\Big) = 0,$$

and

$$\sum_{n=-\infty}^{\infty} \frac{1}{\pi(n^2+a^2)} = \frac{\cosh \pi a}{a \sinh \pi a},$$

that is,

$$\frac{1}{a^2} + 2\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi \cosh \pi a}{a \sinh \pi a},$$

so

$$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{1}{2}\Big(\frac{\pi \cosh \pi a}{a \sinh \pi a} - \frac{1}{a^2}\Big),$$

as required.

(c) The limit is $\frac{\pi^2}{6}$.

12. Integrate $\frac{\cot \pi z}{(z+\frac{1}{2})^2}$ around a large square with corners at $(m+\frac{1}{2})(\pm 1 \pm i)$. The value of the integral is bounded by

$$\begin{aligned}(\text{length of curve}) \times (\text{maximum modulus}) &\le 4(2m+1).\frac{2}{m^2} \\ &\to 0\end{aligned}$$

as $m \to \infty$, because for every z on the square we have $|z+\frac{1}{2}| \ge |z| - \frac{1}{2} \ge m$. The integral is equal to $2\pi i \sum \text{Res}$; now there are residues at each integer n with $-m-\frac{1}{2} \le n \le m+\frac{1}{2}$, that is, for $-m \le n \le m$, and the residue at n is $\frac{1}{\pi}\frac{1}{(n+\frac{1}{2})^2}$. In addition there is a residue when $z+\frac{1}{2}=0$, that is, when $z=-\frac{1}{2}$; this is a pole of order 2 and the residue is therefore

$$\begin{aligned}\lim_{z\to z_0} \frac{d}{dz}(z-z_0)^2 f(z) &= \lim_{z\to -\frac{1}{2}} \frac{d}{dz} \cot \pi z \\ &= \lim_{z\to -\frac{1}{2}} \frac{-\pi}{\sin^2 \pi z} \\ &= -\pi.\end{aligned}$$

Accordingly we must have

$$2\pi i(-\pi + \textstyle\sum_{-m}^{m} \frac{1}{\pi(n+\frac{1}{2})^2}) \to 0$$

as $m \to \infty$, that is,

$$\textstyle\sum_{-\infty}^{\infty} \frac{1}{\pi(n+\frac{1}{2})^2} = \pi,$$

that is,

$$\textstyle\sum_{-\infty}^{\infty} \frac{1}{(n+\frac{1}{2})^2} = \pi^2.$$

Now the required sum was

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^2} &= \frac{1}{(\frac{1}{2})^2} + \frac{1}{(1\frac{1}{2})^2} + \frac{1}{(2\frac{1}{2})^2} + \dots \\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{1}{(n+\frac{1}{2})^2} \\ &= \frac{\pi^2}{2}.\end{aligned}$$

13. (a) $3wz + 4w - 2z - 1 = 0$ so $z = \frac{1-4w}{3w-2}$.

(b)(i) $|z| = |\frac{1-4w}{3w-2}| = 1$ iff $|1-4w|^2 = |3w-2|^2$ iff $(1-4w)(1-4\bar{w}) = (3w-2)(3\bar{w}-2)$ iff $1-4\bar{w}-4w+16w\bar{w} = 9w\bar{w} - 6w - 6\bar{w} + 4$ iff $7w\bar{w} + 2w + 2\bar{w} - 3 = 0$ iff $w\bar{w} + \frac{2}{7}w + \frac{2}{7}\bar{w} - \frac{3}{7} = 0$ iff $(w + \frac{2}{7})(\bar{w} + \frac{2}{7}) = \frac{25}{49}$; this is the circle centre $-\frac{2}{7}$ radius $\frac{5}{7}$.

(ii) z is real iff w is real; so the image of the real axis (omitting $-\frac{4}{3}$) is the real axis (omitting $\frac{2}{3}$).

(iii) $z = \frac{1-4w}{3w-2}$ is purely imaginary iff $(1-4w)(3\bar{w}-2)$ is purely imaginary iff $(1-4w)(3\bar{w}-2) + (1-4\bar{w})(3w-2) = 0$ iff $3(w+\bar{w}) - 24w\bar{w} - 4 + 8(w+\bar{w}) = 0$ iff $0 = w\bar{w} - \frac{11}{24}(w+\bar{w}) + \frac{1}{6} = (w - \frac{11}{24})(\bar{w} - \frac{11}{24}) = \frac{25}{576}$; this is the circle centre $\frac{11}{24}$, radius $\frac{5}{24}$ (omitting $\frac{2}{3}$).

(iv) Observe that if $z = i$ then $w = \frac{2i+1}{3i+4}$ has positive imaginary part; so the upper half of the z-plane is transformed into the upper half of the w-plane.

(v) Observe that if $z = 0$ then $w = \frac{1}{4}$ is inside the circle of part (i); so the open unit disc is transformed into the interior of that circle.

14. (a) Because the real axis is the perpendicular bisector of the line from i to $-i$, the map $z \mapsto \dfrac{z-i}{z+i}$ will take the real axis to the unit circle. It moves 0 to -1, so if we take

$$w = -\frac{z-i}{z+i}$$

we'll get a transformation with the required properties.

[Alternatives: $w = \dfrac{i-z}{i+z}$, $w = \dfrac{1-iz}{1+iz}$.]

(b) The transformation of part (a) had its pole at $-i$, so must move the unit circle to a straight line; to find out which, observe that i moves to 0 and 1 moves to $-\dfrac{1-i}{1+i} = i$, so it must be the straight line through 0 and i, that is, the imaginary axis. Accordingly, if we rotate through 90°, we'll get a suitable transformation; the formula (multiplying by $-i$) is

$$w = i\frac{z-i}{z+i}.$$

[Alternatives: $w = i\dfrac{z+i}{z-i}$, $w = i\dfrac{i-z}{i+z}$.]

15. (a) Using the formula $\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_1-w_2)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_1-z_2)}$ we have

$$\frac{2w}{-(w+1)} = \frac{(z-1)(\omega-\omega^2)}{(z-\omega^2)(1-\omega)} = \frac{\omega(z-1)}{z-\omega^2};$$

hence

$$w = \frac{\omega z - \omega}{-z(2+\omega) + 2\omega^2 + \omega} = -\frac{i}{\sqrt{3}}\frac{z-1}{z+1}.$$

(b) We want to rotate the plane through 90° anticlockwise about a centre 1; we can do this by taking $w - 1 = i(z-1)$, that is, $w = iz - i + 1$. $w = \frac{1}{z}$ works in a sense, but brutalizes 0.

(c) If $z = x + iy$ then $\mathcal{R}\mathrm{e}((1+i)z) = x - y$, so we are looking at the half-plane $y < x - 1$. To move the line ℓ on which $y = x - 1$ to the unit circle, we can use a transformation of the form $w = \frac{z-a}{z-b}$ where a, b are complex numbers which are reflections of each other in ℓ. Since we also need $w = 2$ when $z = 0$, we must have $2 = \frac{-a}{-b}$, that is, $a = 2b$. So the line through a and b also passes through 0, and since it is at right angles to ℓ it is the line $x + y = 0$. This crosses ℓ at $d = \frac{1}{2}(1-i)$, and a and b are multiples of d; say $b = td$ and $a = 2td$. Since also $|a-d| = |b-d|$, we must have $|1-t| = |2t-1|$, so that $t = \frac{2}{3}$ and $a = \frac{2}{3}(1-i)$, $b = \frac{1}{3}(1-i)$, $w = \frac{3z-2+2i}{3z-1+i}$.

[There are other solutions, corresponding to $w = c\frac{z-a}{z-b}$ where c is an arbitrary complex number of modulus 1, since this also takes ℓ to the unit circle if a and b reflect each other in ℓ. Starting from $c = -1$, in particular, we get the solution $w = -\frac{z-2+2i}{z+1-i} = \frac{4-(1+i)z}{2+(1+i)z}$.]

(d) This time, running the idea backwards, we see that if $b = \bar{a}$ is the complex conjugate of a, then the transformation $z = e^{i\gamma}\frac{w-a}{w-b}$ will make the real axis in the w-plane correspond to the unit circle in the z-plane. So the transformation $z - z^* = e^{i\gamma}\frac{w-a}{w-b}$ will make the real axis in the w-plane correspond to the circle centre z^* radius 1 in the z-plane. Now $z - z^* = e^{i\gamma}\frac{w-a}{w-b}$ iff $w = \frac{b(z-z^*)-e^{i\gamma}a}{z-z^*-e^{i\gamma}}$. Since we are also asking for $z = 0$ to correspond to $w = i$, we need $bz^* + e^{i\gamma}a = i(z^* + e^{i\gamma})$, that is, $e^{-i\gamma}bz^* + a = i(e^{-i\gamma}z^* + 1)$.

Now we had free choice of γ, so we may, if it helps, take γ such that $e^{-i\gamma}z^*$ is real and positive; that is, $\gamma = \arg z^*$ (if $z^* \neq 0$), and $e^{-i\gamma}z^* = r = |z^*|$. In this case, our equation becomes

$$rb + a = i(1+r).$$

If $a = x + iy$, then $b = x - iy$ and, taking real and imaginary parts,

$$rx + x = 0, \quad -ry + y = 1 + r,$$

so that $x = 0$, $y = \frac{1+r}{1-r}$, $a = \frac{1+r}{1-r}i$, $b = -\frac{1+r}{1-r}i$,

$$w = -i\frac{1+r}{1-r} \cdot \frac{z-z^*+e^{i\gamma}}{z-z^*-e^{i\gamma}}.$$

Recalling that $e^{i\gamma} = \frac{1}{r}z^*$,

$$w = -i\frac{1+r}{1-r} \cdot \frac{rz+(1-r)z^*}{rz-(1+r)z^*}$$

where $r = |z^*|$.

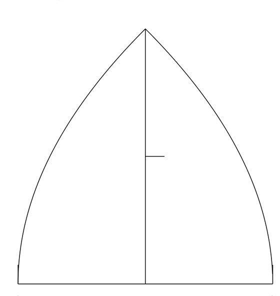
If z^* is actually 0, this formula of course breaks down at this point; but recalling that in this case we can take $e^{i\gamma}$ to be 1, we have the formula

$$w = -i\frac{z+1}{z-1}$$

as one of many possibilities.

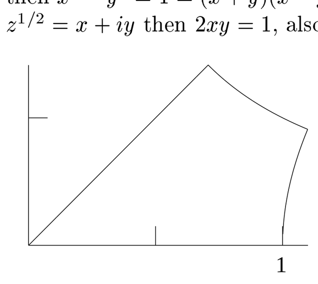
[Alternative solutions: $w = -i\frac{1-r}{1+r} \cdot \frac{rz-(1+r)z^*}{rz+(1-r)z^*}$ if $z^* \neq 0$, $w = i\frac{1-z}{1+z}$ if $z^* = 0$.]

16. (a) If $z = 1 + it$ then $z^2 = (1 - t^2) + 2it$ lies on the parabola $x = 1 - \frac{1}{4}y^2$; if $z = i + t$ then $z^2 = (t^2 - 1) + 2it$ lies on the parabola $x = \frac{1}{4}y^2 - 1$.



(b) If $(x + iy)^2 = a + ib$ then $x^2 - y^2 = a$, $2xy = b$; but also $x^2 + y^2 = \sqrt{a^2 + b^2}$ so $x^2 = \frac{1}{2}(a + \sqrt{a^2 + b^2})$ and $x = \pm\frac{1}{\sqrt{2}}\sqrt{a + \sqrt{a^2 + b^2}}$, $y = \frac{b}{2x}$, except when $x = 0$, in which case $b = 0$ and $a \le 0$ and $y = \pm\sqrt{-a}$.

(c) If $z = 1 + it$ and $z^{1/2} = x + iy$ then $x^2 - y^2 = 1 = (x + y)(x - y)$ so this side of the image is part of a hyperbola; similarly, if $z = i + t$ and $z^{1/2} = x + iy$ then $2xy = 1$, also a hyperbola.



17. If $f(z_1) = f(z_2)$ then $z_1(z_2^2 + 1) = z_2(z_1^2 + 1)$, i.e., $(z_1 - z_2)(1 - z_1 z_2) = 0$. So either $z_1 = z_2$ or $z_1 z_2 = 1$; but the latter is impossible if both z_1, z_2 belong to the domain of f.

We have $w = f(z)$ iff $wz^2 - 2z + w = 0$, that is, iff $z = \frac{1 \pm \sqrt{1 - w^2}}{w}$ or $w = z = 0$. To see which square root we must take, observe that if $z = \frac{1+v}{w}$ belongs to the domain of f, where v is one of the square roots of $1 - w^2$, then we must have $|1 + v| < |w|$, while $|1 + v||1 - v| = |1 - v^2| = |w^2| = |w|^2$; so we shall need $|1 + v| < |1 - v|$ and the real part of v must be strictly negative, that is, v must be $-(1 - w^2)^{1/2}$. This formula produces a z of modulus less than 1 iff $1 - w^2$ has a square root with strictly negative real part, that is, iff $1 - w^2$ is not real and negative, that is, unless w is real and of modulus at least 1. Thus we get the domain of f^{-1}, the set of values of f, to be

$$\{w : \mathcal{I}\mathrm{m}(w) \neq 0 \, or \, |w| < 1\},$$

with $f^{-1}(w) = \frac{1 - (1 - w^2)^{1/2}}{w}$ if $w \neq 0$, 0 if $w = 0$.

In particular, $f^{-1}(i) = (\sqrt{2} - 1)i$.

To find the Taylor series of f^{-1} around 0, recall that

$$(1 + v)^\alpha = 1 + \alpha v + \frac{\alpha(\alpha - 1)}{2}v^2 + \ldots$$

if $|v| < 1$, so that

$$(1 - w^2)^{1/2} = 1 - \tfrac{1}{2}w^2 - \tfrac{1}{8}w^4 \ldots$$

if $|w| < 1$, and

$$f^{-1}(w) = \tfrac{1}{w}(1 - 1 + \tfrac{1}{2}w^2 + \tfrac{1}{8}w^4 \ldots) = \tfrac{1}{2}w + \tfrac{1}{8}w^3 \ldots$$

if $|w| < 1$.

18. (a) Taking $A(z)$ to be the Z-transform of $\langle a_n \rangle_{n\geq 0}$, the Z-transform of $\langle a_{n+1} \rangle_{n\geq 0}$ is $zA(z) - z$ and the Z-transform of the constant sequence 1 is $\frac{z}{z-1}$. So the Z-transform of the given difference equation is

$$zA - z - 2A = \tfrac{3z}{z-1},$$

giving

$$A = \tfrac{1}{z-2}(z + \tfrac{3z}{z-1}) = \tfrac{z}{z-2} + \tfrac{3z}{(z-2)(z-1)}.$$

Now

$$\tfrac{3}{(z-2)(z-1)} = \tfrac{\alpha}{z-1} + \tfrac{\beta}{z-2}$$

where

$$3 = \alpha(z-2) + \beta(z-1)$$

so that

$$0 = \alpha + \beta,\ 3 = -2\alpha - \beta$$

and $\alpha = -3$, $\beta = 3$. Thus

$$A = \tfrac{z}{z-2} - \tfrac{3z}{z-1} + \tfrac{3z}{z-2} = 4\tfrac{z}{z-2} - 3\tfrac{z}{z-1}$$

and

$$a_n = 4.2^n - 3.$$

(b) Taking $Y(s)$ to be the Laplace transform of $y(t)$, the Laplace transform of y' is $sY - y(0) = sY - 1$ and the Laplace transform of the constant function 1 is $\frac{1}{s}$. So the Laplace transform of the given differential equation is

$$sY - 1 - 2Y = \tfrac{3}{s},$$

giving

$$Y = \tfrac{1}{s-2}(1 + \tfrac{3}{s}) = \tfrac{1}{s-2} + \tfrac{3}{(s-2)s}.$$

Now

$$\tfrac{3}{s(s-2)} = \tfrac{\alpha}{s} + \tfrac{\beta}{s-2}$$

where

$$3 = \alpha(s-2) + \beta s$$

so that

$$0 = \alpha + \beta,\ 3 = -2\alpha$$

and $\alpha = -\frac{3}{2}$, $\beta = \frac{3}{2}$. Thus

$$A = \tfrac{1}{s-2} - \tfrac{\frac{3}{2}}{s} + \tfrac{\frac{3}{2}}{s-2} = \tfrac{5}{2}\tfrac{1}{s-2} - \tfrac{3}{2}\tfrac{1}{s}$$

and

$$y(t) = \tfrac{5}{2}e^{2t} - \tfrac{3}{2}.$$

19. (a) Taking $A(z)$ to be the Z-transform of $\langle a_n\rangle_{n\geq 0}$, the Z-transform of $\langle a_{n+1}\rangle_{n\geq 0}$ is $z(A(z)-a_0) = zA - z$, the Z-transform of $\langle a_{n+2}\rangle_{n\geq 0}$ is $z(zA - z - a_1) = z^2A - z^2 + z$, and the Z-transform of the constant sequence 1 is $\frac{z}{z-1}$. So the Z-transform of the given difference equation is

$$z^2A - z^2 + z + 3(zA - z) + 2A = \tfrac{3z}{z-1},$$

giving

$$A = \tfrac{1}{z^2+3z+2}(z^2 + 2z + \tfrac{3z}{z-1}) = \tfrac{z^3+z^2+z}{(z+2)(z+1)(z-1)}.$$

Now

$$\tfrac{z^3+z^2+z}{(z+2)(z+1)(z-1)} = \alpha\tfrac{z}{z+2} + \beta\tfrac{z}{z+1} + \gamma\tfrac{z}{z-1}$$

where

$$\alpha(z+1)(z-1) + \beta(z+2)(z-1) + \gamma(z+2)(z+1) = z^2 + z + 1;$$

equating coefficients we have

(z^2) $\alpha + \beta + \gamma = 1$,
(z) $\beta + 3\gamma = 1$,
(1) $-\alpha - 2\beta + 2\gamma = 1$.

Also, taking specific values for z, we get

$(z=-2)$ $3\alpha = 3$, $\alpha = 1$,
$(z=-1)$ $-2\beta = 1$, $\beta = -\frac{1}{2}$,
$(z=1)$ $6\gamma = 3$, $\gamma = \frac{1}{2}$,

which duly match the equations above.

$$A = \tfrac{z}{z+2} - \tfrac{1}{2}\tfrac{z}{z+1} + \tfrac{1}{2}\tfrac{z}{z-1}$$

and

$$a_n = (-2)^n - \tfrac{1}{2}(-1)^n + \tfrac{1}{2}.$$

(a) Taking $Y(s)$ to be the Laplace transform of $y(t)$, the Laplace transform of y' is $sY - y(0) = sY - 1$, the Laplace transform of y'' is $s(sY-1) - y'(0) = s^2Y - s + 1$ and the Laplace transform of the constant function 1 is $\frac{1}{s}$. So the Laplace transform of the given differential equation is

$$s^2Y - s + 1 + 3(sY - 1) + 2Y = \tfrac{3}{s},$$

giving

$$Y = \tfrac{1}{(s+2)(s+1)}(s + 2 + \tfrac{3}{s}) = \tfrac{s^2+2s+3}{s(s+1)(s+2)}.$$

Now

$$\tfrac{s^2+2s+3}{s(s+1)(s+2)} = \tfrac{\alpha}{s} + \tfrac{\beta}{s+1} + \tfrac{\gamma}{s+2}$$

where

$$\alpha(s+1)(s+2) + \beta s(s+2) + \gamma s(s+1) = s^2 + 2s + 3.$$

Equating coefficients we have

(s^2) $\alpha + \beta + \gamma = 1$,
(s) $3\alpha + 2\beta + \gamma = 2$,
(1) $2\alpha = 3$;

taking particular values of s we get

$(s=0)$ $2\alpha = 3$, $\alpha = \frac{3}{2}$,
$(s=-1)$ $-\beta = 2$, $\beta = -2$,
$(s=-2)$ $2\gamma = 3$, $\gamma = \frac{3}{2}$,

matching the equations above. Thus

$$Y = \tfrac{\frac{3}{2}}{s} - \tfrac{2}{s+1} + \tfrac{\frac{3}{2}}{s+2}$$

and

$$y(t) = \tfrac{3}{2} - 2e^{-t} + \tfrac{3}{2}e^{-2t}.$$

20. (a) Taking $A(z)$ to be the Z-transform of $\langle a_n \rangle_{n\ge 0}$, the Z-transform of $\langle a_{n+1} \rangle_{n\ge 0}$ is $z(A(z)-a_0) = zA$, the Z-transform of $\langle a_{n+2} \rangle_{n\ge 0}$ is $z(zA - a_1) = z^2 A$, and the Z-transform of the sequence n is $-z\frac{d}{dz}\frac{z}{z-1} = \frac{z}{(z-1)^2}$. So the Z-transform of the given difference equation is

$$z^2 A + 2zA + A = \frac{z}{(z-1)^2},$$

giving

$$A = \frac{z}{(z+1)^2(z-1)^2}.$$

Now

$$\frac{z}{(z+1)^2(z-1)^2} = \alpha\frac{z}{z+1} + \beta\frac{z}{(z+1)^2} + \gamma\frac{z}{z-1} + \delta\frac{z}{(z-1)^2}$$

where

$$\alpha(z+1)(z-1)^2 + \beta(z-1)^2 + \gamma(z-1)(z+1)^2 + \delta(z+1)^2 = 1;$$

equating coefficients we have

(z^3) $\quad \alpha + \gamma = 0,$
(z^2) $\quad -\alpha + \beta + \gamma + \delta = 0,$
(z) $\quad -\alpha - 2\beta - \gamma + 2\delta = 0,$
(1) $\quad \alpha + \beta - \gamma + \delta = 1.$

Also, taking specific values for z, we get

$(z = -1)$ $\quad 4\beta = 1$, $\beta = \frac{1}{4}$,
$(z = 1)$ $\quad 4\delta = 1$, $\delta = \frac{1}{4}$,

so that $\alpha = \frac{1}{4}$, $\gamma = -\frac{1}{4}$, which duly match the equations above. Consequently

$$A = \frac{1}{4}\frac{z}{z+1} + \frac{1}{4}\frac{z}{(z+1)^2} - \frac{1}{4}\frac{z}{z-1} + \frac{1}{4}\frac{z}{(z-1)^2}$$

and

$$a_n = \tfrac{1}{4}(-1)^n - \tfrac{1}{4}n(-1)^n - \tfrac{1}{4} + \tfrac{1}{4}n = \tfrac{1}{4}(n-1)(1-(-1)^n),$$

recalling that the Z-transform of na^n is $-z\frac{d}{dz}\frac{z}{z-a} = \frac{az}{(z-a)^2}$.

(b) Taking $Y(s)$ to be the Laplace transform of $y(t)$, the Laplace transform of y' is $sY - y(0) = sY$, the Laplace transform of y'' is $s(sY) - y'(0) = s^2 Y$ and the Laplace transform of the function t is $-\frac{d}{ds}\frac{1}{s} = \frac{1}{s^2}$. So the Laplace transform of the given differential equation is

$$s^2 Y + 2sY + Y = \frac{1}{s^2},$$

giving

$$Y = \frac{1}{s^2(s+1)^2}.$$

Now

$$\frac{1}{s^2(s+1)^2} = \frac{\alpha}{s} + \frac{\beta}{s^2} + \frac{\gamma}{s+1} + \frac{\delta}{(s+1)^2}$$

where

$$\alpha s(s+1)^2 + \beta(s+1)^2 + \gamma s^2(s+1) + \delta s^2 = 1.$$

Equating coefficients we have

(s^3) $\quad \alpha + \gamma = 0,$
(s^2) $\quad 2\alpha + \beta + \gamma + \delta = 0,$
(s) $\quad \alpha + 2\beta = 0,$
(1) $\quad \beta = 1;$

taking particular values of s we get

$(s = 0)$ $\quad \beta = 1,$
$(s = -1)$ $\quad \delta = 1,$

so that $\alpha = -2$, $\gamma = 2$, matching the equations above. Thus

$$Y = -\frac{2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{1}{(s+1)^2}$$

and

$$y(t) = -2 + t + 2e^{-t} + te^{-t}.$$

21. (a) Taking $A(z)$ to be the Z-transform of $\langle a_n \rangle_{n\geq 0}$, the Z-transform of $\langle a_{n+1} \rangle_{n\geq 0}$ is $z(A(z)-a_0) = zA$, the Z-transform of $\langle a_{n+2} \rangle_{n\geq 0}$ is $z(zA-a_1) = z^2A$, and the Z-transform of $\langle \cos \omega n \rangle_{n\geq 0}$ is $\frac{1}{2}(\frac{z}{z-e^{i\omega}}+\frac{z}{z-e^{-i\omega}}) = \frac{z(z-\cos\omega)}{z^2-2z\cos\omega+1}$. So the Z-transform of the given difference equation is

$$z^2A + A = \frac{z(z-\cos\omega)}{z^2-2z\cos\omega+1},$$

giving

$$A = \frac{z(z-\cos\omega)}{(z^2+1)(z^2-2z\cos\omega+1)}.$$

Now we must recall that the Z-transform of $\langle \sin \omega n \rangle_{n\geq 0}$ is $\frac{1}{2i}(\frac{z}{z-e^{i\omega}} - \frac{z}{z-e^{-i\omega}}) = \frac{z\sin\omega}{z^2-2z\cos\omega+1}$, so that, in particular, the Z-transform of $\langle \sin \frac{\pi}{2}n \rangle_{n\geq 0}$ is $\frac{z}{z^2+1}$, while the Z-transform of $\langle \cos \frac{\pi}{2}n \rangle_{n\geq 0}$ is $\frac{z^2}{z^2+1}$. We should therefore seek to express A as

$$\frac{z(z-\cos\omega)}{(z^2+1)(z^2-2z\cos\omega+1)} = \alpha\frac{z^2}{z^2+1} + \beta\frac{z}{z^2+1} + \gamma\frac{z(z-\cos\omega)}{z^2-2z\cos\omega+1} + \delta\frac{z\sin\omega}{z^2-2z\cos\omega+1}$$

where

$$\begin{aligned}\alpha z(z^2-2z\cos\omega+1) + \beta(z^2-2z\cos\omega+1) + \gamma(z-\cos\omega)(z^2+1) + \delta\sin\omega(z^2+1) \\ = z - \cos\omega;\end{aligned}$$

equating coefficients we have

$$\begin{aligned}
&(z^3) \quad \alpha+\gamma = 0, && (1)\\
&(z^2) \quad -2\alpha\cos\omega + \beta - \gamma\cos\omega + \delta\sin\omega = 0, && (2)\\
&(z) \quad \alpha - 2\beta\cos\omega + \gamma = 1, && (3)\\
&(1) \quad \beta - \gamma\cos\omega + \delta\sin\omega = -\cos\omega. && (4)
\end{aligned}$$

Putting (1) and (3) together, we get $\beta\cos\omega = -\frac{1}{2}$; putting (1) and (2) together we get $beta + \gamma\cos\omega + \delta\sin\omega = 0$; subtracting (4) we get $\gamma = \frac{1}{2}$, so $\alpha = -\frac{1}{2}$; hence, substituting into (2) or (4), $\delta = \frac{1}{2}\tan\omega$. So we have

$$A = -\frac{1}{2}\frac{z}{z^2+1} - \frac{1}{2\cos\omega}\frac{1}{z^2+1} + \frac{1}{2}\frac{z-\cos\omega}{z^2-2z\cos\omega+1} + \frac{1}{2}\tan\omega\frac{\sin\omega}{z^2-2z\cos\omega+1}$$

and

$$a_n = -\frac{1}{2}\cos\frac{\pi}{2}n - \frac{1}{2\cos\omega}\sin\frac{\pi}{2}n + \frac{1}{2}\cos\omega n + \frac{1}{2}\tan\omega\sin\omega n.$$

Taking the limit as $\omega \to \frac{\pi}{2}$ we have

$$\begin{aligned}
\lim_{\omega\to\frac{\pi}{2}} a_n &= \lim_{\omega\to\frac{\pi}{2}} -\frac{1}{2}\cos\frac{\pi}{2}n - \frac{1}{2\cos\omega}\sin\frac{\pi}{2}n + \frac{1}{2}\cos\omega n + \frac{1}{2}\tan\omega\sin\omega n \\
&= \lim_{\omega\to\frac{\pi}{2}} \frac{1}{2\cos\omega}(\sin\omega\sin\omega n - \sin\frac{\pi}{2}n) \\
&= \lim_{\omega\to\frac{\pi}{2}} \frac{1}{-2\sin\omega}(\cos\omega\sin\omega n + n\sin\omega\cos\omega n) \\
&= -\frac{1}{2}n\cos\frac{\pi}{2}n.
\end{aligned}$$

(a) Taking $Y(s)$ to be the Laplace transform of $y(t)$, the Laplace transform of y' is $sY - y(0) = sY$, the Laplace transform of y'' is $s(sY) - y'(0) = s^2Y$ and the Laplace transform of $\cos\omega t$ is $\frac{s}{s^2+\omega^2}$. So the Laplace transform of the given differential equation is

$$s^2Y + Y = \frac{s}{s^2+\omega^2},$$

giving

$$Y = \frac{1}{(s^2+1)(s^2+\omega^2)}.$$

Recalling that the Laplace transform of $\sin\omega t$ is $\frac{\omega}{s^2+\omega^2}$, we seek to express Y in the form

$$\frac{s}{(s^2+1)(s^2+\omega^2)} = \frac{\alpha s}{s^2+1} + \frac{\beta}{s^2+1} + \frac{\gamma s}{s^2+\omega^2} + \frac{\delta\omega}{s^2+\omega^2}$$

where

$$\alpha s(s^2+\omega^2) + \beta(s^2+\omega^2) + \gamma s(s^2+1) + \delta\omega(s^2+1) = s.$$

Equating coefficients we have

$$\begin{aligned}
&(s^3) \quad \alpha+\gamma = 0, && (1)\\
&(s^2) \quad \beta + \delta\omega = 0, && (2)\\
&(s) \quad \alpha\omega^2 + \gamma = 1, && (3)\\
&(1) \quad \beta\omega^2 + \delta\omega = 0; && (4)
\end{aligned}$$

putting (1) and (3) together we get $\alpha = \frac{1}{\omega^2-1}$, $\gamma = -\frac{1}{\omega^2-1}$; from (2) and (3) we get $\beta = \delta = 0$. Thus

$$Y = \frac{1}{\omega^2-1}(\frac{s}{s^2+1} - \frac{s}{s^2+\omega^2})$$

and

$$y(t) = \frac{1}{\omega^2-1}(\cos t - \cos\omega t).$$

Taking the limit as $\omega \to 1$ we have

$$\begin{aligned}
\lim_{\omega\to 1} y(\omega, t) &= \lim_{\omega\to 1} \frac{1}{\omega^2-1}(\cos t - \cos\omega t) \\
&= \lim_{\omega\to 1} \frac{1}{2\omega}(t\sin\omega t) \\
&= \frac{1}{2}t\sin t.
\end{aligned}$$

22. (a) Taking $A(z)$, $B(z)$ to be the Z-transforms of the sequences $\langle a_n\rangle_{n\ge0}$, $\langle b_n\rangle_{n\ge0}$ respectively, the Z-transform of $\langle a_{n+1}\rangle_{n\ge0}$ is $z(A(z)-a_0)=z(A-1)$ and the Z-transform of $\langle a_{n+2}\rangle_{n\ge0}$ is $z(z(A-1)-a_1)=z^2(A-1)$, so the Z-transform of the given difference equation is

$$z^2(A-1)+2z(A-1)+2A=B,$$

and

$$A=\tfrac{1}{z^2+2z+2}(B+z^2+2z)=C+B\times D,$$

setting $C=\frac{z^2+2z}{z^2+2z+2}$, $D=\frac{1}{z^2+2z+2}$. To find inverse Z-transforms (that is, Laurent series) for C and D, note that

$z^2+2z+2=(z+1)^2+1$;
the Z-transform of $\langle\cos\omega n\rangle_{n\ge0}$ is $\frac12(\frac{z}{z-e^{i\omega}}+\frac{z}{z-e^{-i\omega}})=\frac{z^2-z\cos\omega}{z^2-2z\cos\omega+1}$,
so the Z-transform of $\langle r^n\cos\omega n\rangle_{n\ge0}$ is $\frac{(z/r)^2-(z/r)\cos\omega}{(z/r)^2-2(z/r)\cos\omega+1}=\frac{z^2-rz\cos\omega}{z^2-2rz\cos\omega+r^2}$,
and the Z-transform of $\langle(\sqrt2)^n\cos\frac{3\pi}{4}n\rangle_{n\ge0}$ is $\frac{z^2+z}{z^2+2z+2}$;
the Z-transform of $\langle\sin\omega n\rangle_{n\ge0}$ is $\frac{1}{2i}(\frac{z}{z-e^{i\omega}}-\frac{z}{z-e^{-i\omega}})=\frac{z\sin\omega}{z^2-2z\cos\omega+1}$,
so the Z-transform of $\langle r^n\sin\omega n\rangle_{n\ge0}$ is $\frac{(z/r)\sin\omega}{(z/r)^2-2(z/r)\cos\omega+1}=\frac{rz\sin\omega}{z^2-2rz\cos\omega+r^2}$,
and the Z-transform of $\langle(\sqrt2)^n\sin\frac{3\pi}{4}n\rangle_{n\ge0}$ is $\frac{z}{z^2+2z+2}$.

Now we see that $C=\frac{z^2+z}{z^2+2z+2}+\frac{z}{z^2+2z+2}$ so

$$c_n=(\sqrt2)^n(\cos\tfrac{3\pi}{4}n+\sin\tfrac{3\pi}{4}n)=-(\sqrt2)^{n+1}\cos\tfrac{3\pi}{4}(n+1).$$

As for D, we have $D=\frac12(1-C)$; now the sequence with Z-transform 1 is just the sequence $e_0=1$, $e_n=0$ for $n\ge1$, so that we have

$$d_0=\tfrac12(1-c_0)=0,$$
$$d_n=-\tfrac12c_n=-(\sqrt2)^{n-2}(\cos\tfrac{3\pi}{4}n+\sin\tfrac{3\pi}{4}n)=(\sqrt2)^{n-1}\cos\tfrac{3\pi}{4}(n+1)$$

for $n\ge1$. Thus the final solution is

$$a_n=c_n-\tfrac12\textstyle\sum_{k=0}^{n-1}b_kc_{n-k}$$

where

$$c_n=(\sqrt2)^n(\cos\tfrac{3\pi}{4}n+\sin\tfrac{3\pi}{4}n)=-(\sqrt2)^{n+1}\cos\tfrac{3\pi}{4}(n+1)$$

and $a_0=c_0=1$.

(b) Taking Y to be the Laplace transform of y and F to be the Laplace transform of f, the Laplace transform of y' is $sY-y(0)=sY-1$ and the Laplace transform of y'' is $s(sY-1)-y'(0)=s^2Y-s$. So the Laplace transform of the given differential equation is

$$s^2Y-s+2(sY-1)+2Y=F,$$

and

$$Y=\tfrac{1}{s^2+2s+2}(F+s+2)=G+F\times H,$$

setting $G=\frac{s+2}{s^2+2s+2}$, $H=\frac{1}{s^2+2s+2}$. To find inverse Laplace transforms for G and H, note that

$s^2+2s+2=(s+1)^2+1$;
the Laplace transform of $\cos t$ is $\frac{s}{s^2+1}$,
so the Laplace transform of $e^{-t}\cos t$ is $\frac{s+1}{(s+1)^2+1}$;
the Laplace transform of $\sin t$ is $\frac{1}{s^2+1}$,
so the Laplace transform of $e^{-t}\sin t$ is $\frac{1}{(s+1)^2+1}$.

Accordingly we have $g(t)=e^{-t}(\cos t+\sin t)$ and $h(t)=e^{-t}\sin t$, and the solution is

$$y(t)=e^{-t}(\cos t+\sin t)+\textstyle\int_0^t f(u)e^{-(t-u)}\sin(t-u)du.$$

23. (a) Simple pole, residue 1.
(b) π.
(c) $d_1=1$, $d_n=0$ for $n\ge2$ (because cot has a simple pole of residue 1).
$c_0=c_2=c_4=c_6=0$ because cot is odd.
$c_1-\frac{1}{3!}=-\frac{1}{2!}$ so $c_1=-\frac13$;
$c_3-\frac{1}{3!}c_1+\frac{1}{5!}=\frac{1}{4!}$ so $c_3=-\frac{1}{45}$;
$c_5-\frac{1}{3!}c_3+\frac{1}{5!}c_1-\frac{1}{7!}=-\frac{1}{6!}$ so $c_5=-\frac{2}{945}$.

24. (i) $\mathrm{Ln}(1+\frac{1}{2z}) = \frac{1}{2z} - \frac{1}{8z^2} + \dots$, $\int = \pi i$.
(ii) $\arctan\frac{1}{z} = \frac{1}{z} - \frac{1}{3z^3} + \frac{1}{5z^5} - \dots$, $\int = -\frac{2\pi i}{3}$.
(iii) $\frac{\exp(\frac{1}{z})}{z^2-1} = (1 + \frac{1}{z} + \frac{1}{2z^2} + \dots)\frac{1}{z^2}(1 + \frac{1}{z^2} + \frac{1}{z^4} + \dots)$ so $\int = 0$.

25. (i) If $0 < |z| < 1$, then
$\exp\frac{1}{z} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \dots$, $\frac{1}{z-1} = -(1 + z + z^2 + \dots)$ so

$$f(z) = -(1 + z + z^2 + \dots)(1 + \frac{1}{z} + \frac{1}{2z^2} + \dots)$$

which has $\frac{1}{z}$ coefficient

$$-(1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \dots) = 1 - e.$$

(ii) If $|z| > 1$, $\frac{1}{z-1} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots$ so

$$f(z) = (\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots)(1 + \frac{1}{z} + \frac{1}{2z^2} + \dots)$$

which has $\frac{1}{z}$ coefficient 1.

(iii) Set $w = z - 1$ so that $f(z) = \frac{1}{w}\exp(\frac{1}{w+1})$. Now $\exp\frac{1}{w+1}$ is analytic in $\{w : |w| < 1\}$ so has a Taylor series with constant coefficient $\exp(\frac{1}{0+1}) = e$; this is the coefficient of $\frac{1}{w}$ in the Laurent expansion of $\frac{1}{w}\exp(\frac{1}{w+1})$ near 0, so is also the coefficient of $\frac{1}{z-1}$ in the Laurent expansion of f near 1.

(iv) Setting $\zeta = \frac{1}{z-1}$, $z = 1 + \frac{1}{\zeta}$ we have $f(z) = \zeta\exp\frac{\zeta}{\zeta+1}$. Now $|z-1| > 1$ iff $|\zeta| < 1$, and for $|\zeta| < 1$ the function $\zeta\exp(\frac{\zeta}{\zeta+1})$ has a Taylor series in which the coefficient of ζ is 1; this is therefore the coefficient of $\frac{1}{z-1}$ in the Laurent expansion of f for $|z-1| > 1$.

(Alternatively, integrate f around a large circle to see that the answers to (ii) and (iv) must be the same.)

26. **(a)**

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-1/2}^{1/2} e^{-iyx} dx = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-iy} e^{-iyx} \right]_{-1/2}^{1/2}$$
$$= \frac{1}{-iy\sqrt{2\pi}} (e^{-iy/2} - e^{iy/2}) = \frac{2}{y\sqrt{2\pi}} \frac{e^{iy/2} - e^{-iy/2}}{2i} = \frac{2}{y\sqrt{2\pi}} \sin \frac{y}{2}.$$

(b)

$$(f * f)(x) = \int_{-\infty}^{\infty} f(x-t) f(t) dt.$$

Now $f(x-t)f(t) = 0$ unless $|t|$, $|x-t|$ are both less than or equal to $\frac{1}{2}$, that is, $t \in [-\frac{1}{2}, \frac{1}{2}] \cap [x - \frac{1}{2}, x + \frac{1}{2}]$; and for t in this intersection, $f(x-t)f(t) = 1$. Now the intersection is empty if $|x| > 1$; if $0 \le x \le 1$ it is the interval $[x - \frac{1}{2}, \frac{1}{2}]$ of length $1 - x$; and if $-1 \le x \le 0$ it is the interval $[-\frac{1}{2}, x + \frac{1}{2}]$ of length $1 + x$. So $(f * f)(x) = 0$ if $|x| > 1$ and $1 - |x|$ if $|x| \le 1$.

(c)

$$\hat{h}_1(y) = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-ixy} (1-x) dx$$
$$= \frac{1}{\sqrt{2\pi}} \Big(\left[(1-x) \frac{e^{-ixy}}{-iy} \right]_0^1 - \int_0^1 (-1) \frac{e^{-ixy}}{-iy} dx \Big)$$
$$= \frac{1}{\sqrt{2\pi}} \Big(\frac{1}{iy} - \frac{1}{iy} \int_0^1 e^{-ixy} dx \Big)$$
$$= \frac{1}{\sqrt{2\pi}} \Big(-\frac{i}{y} - \frac{1}{iy} \left[\frac{1}{-iy} e^{-ixy} \right]_0^1 \Big)$$
$$= \frac{1}{\sqrt{2\pi}} \Big(-\frac{i}{y} - \frac{1}{y^2} (e^{-iy} - 1) \Big) = \frac{1}{\sqrt{2\pi}} \frac{1 - iy - e^{-iy}}{y^2}.$$

(d) $\hat{h}_2(y) = \hat{h}_1(-y) = \frac{1}{\sqrt{2\pi}} \frac{1 + iy - e^{iy}}{y^2}$.

(e)

$$\hat{h}_1(y) + \hat{h}_2(y) = \frac{2(1 - \cos y)}{y^2 \sqrt{2\pi}}.$$

This is the Fourier transform of $h_1 + h_2$, which is equal (except at 0) to $f * f$. But the Fourier transform $\widehat{f * f}(y)$ will be

$$\sqrt{2\pi} \hat{f}(y)^2 = \frac{4}{y^2 \sqrt{2\pi}} \sin^2 \frac{y}{2} = \frac{2}{y^2 \sqrt{2\pi}} (1 - \cos y),$$

as above.

(f) Since $f * f$ has Fourier transform $\frac{4}{y^2\sqrt{2\pi}} \sin^2 \frac{y}{2}$, $\frac{4}{y^2\sqrt{2\pi}} \sin^2 \frac{y}{2}$ will have inverse Fourier transform $f * f$, and

$$f * f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} \frac{4}{y^2 \sqrt{2\pi}} \sin^2 \frac{y}{2} dy.$$

Setting $x = 0$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{4}{y^2\sqrt{2\pi}} \sin^2 \frac{y}{2} dy = 1,$$

that is,

$$\int_{-\infty}^{\infty} \frac{1}{y^2} \sin^2 \frac{y}{2} dy = \frac{2\pi}{4} = \frac{\pi}{2}.$$

27. **(a)**

$$\begin{aligned}\hat{f}(y) &= \frac{1}{\sqrt{2\pi}}\Big(\int_{-\infty}^{0} e^{-ixy}e^{ax}\,dx + \int_{0}^{\infty} e^{-ixy}e^{-ax}\,dx\Big)\\ &= \frac{1}{\sqrt{2\pi}}\Big(\left[\frac{1}{a-iy}e^{(a-iy)x}\right]_{-\infty}^{0} + \left[\frac{1}{-a-iy}e^{-(a+iy)x}\right]_{0}^{\infty}\Big)\\ &= \frac{1}{\sqrt{2\pi}}\Big(\frac{1}{a-iy} - \frac{1}{-a-iy}\Big) = \frac{2a}{\sqrt{2\pi}(a^2+y^2)}.\end{aligned}$$

(b)(i) The inverse Fourier transform of $\frac{2a}{\sqrt{2\pi}(a^2+y^2)}$ is $e^{-a|x|}$; as these are even functions, it is also the Fourier transform; so the Fourier transform of $\frac{1}{b^2+x^2}$ is $\frac{\sqrt{2\pi}}{2b}e^{-b|y|}$.

(ii) Integrate $\frac{e^{-iyz}}{z^2+b^2}$ around a large semicircle based on the interval $[-R, R]$. The total integral is

$$2\pi i\,\mathrm{Res}_{z=ib}\,\frac{e^{-iyz}}{(z+ib)(z-ib)} = 2\pi i\frac{e^{-iyib}}{2ib} = \pi\frac{e^{yb}}{b}.$$

By Jordan's Lemma, the integral around the curved section is small for large R *provided* $y<0$; so for $y<0$ we have

$$\hat{g}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-iyx}}{x^2+b^2}dx = \frac{1}{\sqrt{2\pi}}\frac{\pi e^{yb}}{b} = \frac{\sqrt{2\pi}}{2b}e^{yb}.$$

If $y>0$, we have to say

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{e^{-iyx}}{x^2+b+2} &= \int_{-\infty}^{\infty}\frac{e^{iyx}}{x^2+b^2} = 2\pi i\,\mathop{\mathrm{Res}}_{z=ib}\frac{e^{iyz}}{z^2+b^2}\\ &= 2\pi i\frac{e^{-yb}}{2ib} = \frac{\pi e^{-yb}}{b},\end{aligned}$$

so that

$$\hat{g}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-iyx}}{x^2+b^2}dx = \frac{1}{\sqrt{2\pi}}\frac{\pi e^{-yb}}{b} = \frac{\sqrt{2\pi}}{2b}e^{-yb}.$$

Putting these together we get

$$\hat{g}(y) = \frac{\sqrt{2\pi}}{2b}e^{-b|y|}$$

for all $y \neq 0$.

(For $y=0$, we get

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{x^2+b^2}dx &= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{b}\arctan\frac{1}{b}\right]_{-\infty}^{\infty}\\ &= \frac{\pi}{b\sqrt{2\pi}} = \frac{\sqrt{2\pi}}{2b}.)\end{aligned}$$

(c)

$$\begin{aligned}\widehat{g*h}(y) &= \sqrt{2\pi}\hat{g}(y)\hat{h}(y)\\ &= \sqrt{2\pi}\frac{\sqrt{2\pi}}{2b}e^{-b|y|}\frac{\sqrt{2\pi}}{2c}e^{-c|y|} = \frac{\pi\sqrt{2\pi}}{2bc}e^{-(b+c)|y|}.\end{aligned}$$

But the inverse Fourier transform of this is

$$\frac{\pi\sqrt{2\pi}}{2bc}\frac{2(b+c)}{\sqrt{2\pi}((b+c)^2+x^2)} = \frac{\pi(b+c)}{bc((b+c)^2+x^2)},$$

so this is $(g*h)(x)$.

28. **(a)** Using the rule

if $g(x) = f'(x)$ then $\hat{g}(y) = iy\hat{f}(y)$,

the Fourier transform of $xe^{-x^2/2}$ is $-iye^{-y^2/2}$.

Using the rule

if $g(x) = f(\gamma x)$, where $\gamma > 0$, then $\hat{g}(y) = \frac{1}{\gamma}\hat{f}(\frac{y}{\gamma})$,

the Fourier transform of $\frac{x}{\sigma}e^{-x^2/2\sigma^2}$ is $-\sigma{\cdot}i\sigma ye^{-\sigma^2y^2/2}$, and the Fourier transform of $xe^{-x^2/2\sigma^2}$ is $-i\sigma^3ye^{-\sigma^2y^2/2}$.

(b) The Fourier transform of g is $\frac{1}{\sqrt{2}}e^{-y^2/4}$. So the Fourier transform of $g*g$ is $\sqrt{2\pi}\hat{g}^2$, that is, $\frac{\sqrt{2\pi}}{2}e^{-y^2/2}$, and $(g*g)(x)$ must be $\frac{\sqrt{2\pi}}{2}e^{-x^2/2}$.

29. We have

$$\int_0^\infty \frac{\cos mx}{1+x^4}dx = \frac{\pi}{2\sqrt{2}}e^{-|m|/\sqrt{2}}(\cos\frac{m}{\sqrt{2}} + \sin\frac{m}{\sqrt{2}}).$$

Now

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \frac{e^{-iyx}}{1+x^4}dx &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \frac{\cos yx - i\sin yx}{1+x^4}dx = \frac{2}{\sqrt{2\pi}}\int_0^\infty \frac{\cos yx}{1+x^4}dx \\ &= \frac{2}{\sqrt{2\pi}}\frac{\pi}{2\sqrt{2}}e^{-|y|/\sqrt{2}}(\cos\frac{y}{\sqrt{2}} + \sin\frac{|y|}{\sqrt{2}}) \\ &= \frac{\sqrt{\pi}}{2}e^{-|y|/\sqrt{2}}(\cos\frac{y}{\sqrt{2}} + \sin\frac{|y|}{\sqrt{2}}).\end{aligned}$$