Propositional and Predicate Calculus Problems

1. Draw Venn diagrams to represent (i) $B \cup A^c$ (ii) $(B \cup A^c) \cap (A \cup B^c)$ (iii) $(A^c \cup B^c) \cap (A \cup B)$ (iv) $(A \cap B) \cup (A^c \cap B^c)$ (v) $(A \cap B^c) \cup (A^c \cap B)$. [AA]

2. Give truth tables for
(i) $p \vee (\neg q)$ (ii) $q \vee (\neg p)$
(iii) $(q \vee (\neg p)) \wedge (p \vee (\neg q))$ (iv) $(p \wedge (\neg q)) \vee (q \wedge (\neg p))$
(v) $(p \wedge q) \vee ((\neg p) \wedge (\neg q))$.
[ABA]

3. Give truth tables for
(i) $(p \vee q) \wedge ((\neg p) \vee (\neg q))$ (ii) $p \wedge (p \vee q)$
(iii) $q \vee (p \wedge q)$ (iv) $(p \vee q) \vee ((\neg p) \wedge (\neg q))$
(v) $(p \wedge q) \wedge ((\neg p) \vee (\neg q))$.
[ABB]

4. Which of the formulae in question 1 match which of the formulae in questions 2 and 3? [ABC]

5. Find expressions which will yield each of the five truth tables (a)-(e):

p	q	a	b	c	d	e
T	T	T	F	F	F	F
T	F	T	T	T	F	F
F	T	F	F	F	T	F
F	F	T	T	F	F	T

[AC]

6. Give the truth table for $(p \to q) \to (r \to s)$. [AD]

7. In the version of `quicksort` described in lectures, what will happen if `goodorder` is amended so that, for some transitive relation R, `T` is returned iff `num[i]` R `num[j]`? [AE]

8. In the version of `quicksort` described in lectures, what will happen if `goodorder` is amended so that, for some transitive relation R, `T` is returned iff `num[i]` R `num[j]`? [AE]

9. Find contrapositives for the following sentences:
(a) If John opens the window we'll have fresh air.
(b) If Epimenides is a Cretan he's a liar.
(c) If Clinton isn't a liar I'm a Dutchman.
(d) If it's dark I don't go out.
(e) If interest rates are lowered Britain will be more prosperous. [BA]

10. Give truth tables for the following expressions:
(i) $p \to (q \vee r)$
(ii) $p \to (q \wedge r)$
(iii) $(p \to q) \vee (p \to r)$
(iv) $(p \to q) \wedge (p \to r)$
Are any of these expressions logically equivalent to each other? [BBA]

11. Give truth tables for the following expressions:
(i) $(p \to q) \to r$
(ii) $(p \wedge q) \to r$
(iii) $(p \vee q) \to r$
(iv) $(p \to r) \wedge (q \to r)$
Are any of these expressions logically equivalent to each other? [BBB]

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12. Give truth tables for the following expressions:
(i) $(p \to r) \vee (q \to r)$
(ii) $(q \to r) \wedge (p \vee r)$
(iii) $(q \vee (\neg p)) \to r$.
Are any of these expressions logically equivalent to each other? [BBC]

13. Give contrapositives for the following statements:
(i) If I work hard I shall pass the examination.
(ii) If I work hard I shall fail the examination.
(iii) If I don't work hard I shall pass the examination.
(iv) If I don't work hard I shall fail the examination. [BC]

14. Give truth tables for the following expressions:
(i) $(p \wedge q) \vee (\neg(p \vee q))$.
(ii) $(\neg p) \wedge (q \vee (\neg r))$. [BD]

15. Writing p for 'John is healthy', q for 'John is wealthy' and r for 'John is wise', put into symbolic form the sentences
(i) 'John is healthy, wealthy and wise'.
(ii) 'John is not healthy but he is wealthy and wise'.
(iii) 'John is not wealthy but he is either healthy or wise'.
(iv) 'John is neither healthy nor wealthy nor wise'. [BE]

16. Writing p for 'interest rates are higher in the UK than in Euroland', q for 'inflation is higher in the UK than in Euroland' and r for 'the economy is growing faster in the UK than in Euroland', put into symbolic form the sentences
(i) 'Interest rates and inflation are higher in the UK than in Euroland, but the economy is growing no faster'.
(ii) 'Interest rates are no higher in the UK than in Euroland, but either the economy is growing faster or inflation is higher'. [BF]

17. Find sentences which are negations of the following:
(i) 'Sam swims on Thursdays and Kate plays tennis on Saturdays'.
(ii) 'Either this program has an error in the first ten lines or it is being run with an incomplete data set'.
(iii) 'Either the train is late or my watch is fast.'
(iv) 'n is an integer greater than 0.' [BG]

18. Give a truth table for $(p \vee q) \wedge (p \vee r) \wedge ((\neg p) \vee (r \wedge q))$. Is there a simpler expression which is logically equivalent to this one? [BH]

19. Give a truth table for $(p \vee q) \vee (p \wedge r)$. Is there a simpler expression which is logically equivalent to this one? [BI]

20. Show directly from a truth table that the argument form

$p \to (\neg p)$
therefore $\neg p$

is valid. [CA]

21. For the Knights and Knaves dialogue

Amos Barnabas is a knight.
Barnabas Amos and I are of different types,

draw up the full truth table for the propositions p, 'Amos is a knight', and q, 'Barnabas is a knight', representing each sentence spoken as an appropriate implication, and determining which line of the table is the only possibility. [CB]

22. On the Island of Knights and Knaves everyone is either a Knight (and always tells the truth) or a Knave (and always lies). Moreover, everyone knows of each of the others whether he or she is a knight or a knave.

You meet a group of three people, and ask the first 'How many knights are there among you?' She replies, but you don't catch the answer. The second says 'She said there's just one'. The third person says 'That's a lie'.

What can you say about the three? [CC]

23. On the Island of Knights and Knaves, you meet two people. The first says 'At least one of us is a knave'.

(a) What can you conclude?

(b) Formalize your reasons in terms of the propositions p, 'The first person is a knight' and q, 'The second person is a knight'. (You should (i) write down the expressions, in terms of p and q, corresponding to the premises you are using and the conclusion you have reached (ii) validate the argument form which derives the conclusion from the premises with a truth table.) [CD]

24. As question 23, but the first speaker says 'Either I am a knave or he is a knight'. [CE]

25. As question 23, but this time there are three people, and the first says 'We are all knaves' and the second says 'Exactly one of us is a knight'. (Use p for 'the first speaker is a knight', q for 'the second speaker is a knight', r for 'the person who didn't speak is a knight', ϕ for $(\neg p) \wedge (\neg q) \wedge (\neg r)$ and ψ for $(p \wedge (\neg q) \wedge (\neg r)) \vee ((\neg p) \wedge q \wedge (\neg r)) \vee ((\neg p) \wedge (\neg q) \wedge r)$.) [CF]

26. On the Island of Knights and Knaves you meet someone who says 'Either I am a knave or the moon is made of green cheese'. What can you conclude? [CG]

27. Find sentences equivalent to the negations of the following:
(i) All mathematicians wear glasses.
(ii) Some mathematicians wear glasses.
(iii) Every prime number is odd.
(iv) Every odd number is prime. [DA]

28. Use truth tables to determine which of the following are valid argument forms:

(a)

	$p \to q$
	$q \to p$
therefore	$p \vee q$

(b)

	p
	$p \to q$
	$(\neg q) \vee r$
therefore	r

(c)

	$p \vee q$
	$p \to (\neg q)$
	$p \to r$
therefore	r

(In each case, draw up a complete truth table with four or eight lines corresponding to the possibilities for p, q and r and columns corresponding to the compound propositions listed; mark clearly which lines are permitted by the premises and the truth values of the conclusion which tell us whether the argument is valid.) [EA]

29. As question 28, with

(a)

	$p \to q$
	$p \to r$
therefore	$p \to (q \wedge r)$

(b)

	$(p \wedge (\neg q)) \to r$
	$p \vee q$
	$q \to p$
therefore	r

[EB]

30. For each of the following arguments,

(i) express it in symbolic form, stating explicitly which propositions correspond to each symbol you are using,

(ii) say whether it is valid,

(iii) if it is valid, give a full truth table establishing this, indicating which lines are permitted by the premises,

(iv) if the argument is *not* valid, give a combination of truth values for the atomic propositions involved which makes each premise `T` while the conclusion is `F`.

(a) If Tom is not on team A, then Abdul is not on team B,
If Abdul is not on team B, then Tom is on team A,
therefore either Tom is not on team A or Abdul is not on team B.

(b) If Alice solved the problem correctly, then Alice got the answer 2,
Alice got the answer 2,
therefore Alice solved the problem correctly.

(c) If Tom is not on team A, then Abdul is not on team B,
If Abdul is not on team B, then Tom is on team A,
therefore Tom is on team A.

[EC]

31. As question 30, with

(a) If Tom is clever, he is rich,
either Tom is clever or Tom is rich,
either Tom is not clever or he is not rich,
therefore Tom is not clever.

(b) If Tom is clever, he is rich,
either Tom is clever or Tom is rich,
either Tom is not clever or he is not rich,
therefore Tom is not rich.

[ED]

32. As question 30, with

(a) If Stephen went to the bar every night, he didn't do all the problem sheets,
if Stephen passed the test, he did all the problem sheets,
therefore if Stephen failed the test he went to the bar every night.

(b) If Tom is on the team, so is Abdul,
if Stephen is on the team, Abdul is not on the team,
either Tom is on the team or Abdul is on the team,
therefore Stephen is not on the team.

[EE]

33. Is the following a valid argument form?

$$(p \to r) \vee (p \to s)$$
$$(p \wedge r) \to q$$
$$s \to q$$
$$\text{therefore } p \to q$$

Justify your answer. If the argument form is *not* valid, set out an argument using it to reach an absurd conclusion from true premises. [FA]

34. Is either of the following a valid argument form?

(a)

	$(p \to q) \to p$
therefore	$\neg p$

(b)

	$(p \to q) \to p$
therefore	p

[FB]

35. Which of the following are valid argument forms?

(a)

	$p \to (\neg q)$
	$p \to (\neg(q \to r))$
therefore	$\neg p$

(b)

	$p \to q$
	$p \to (q \to r)$
therefore	p

(c)

	$p \leftrightarrow q$
	$q \leftrightarrow (p \to r)$
therefore	$p \wedge q \wedge r$

(d)

	$q \to p$
	$(p \to r) \to q$
therefore	q

For each which is *not* a valid form, describe an assignment of truth values to p, q and r which renders each premise `T` but the conclusion `F`. [FC]

36. Which of the following are valid argument forms?

(a)

	$(p \wedge q) \to (r \vee s)$
	$s \to (\neg p)$
	$q \vee r \vee s$
therefore	$p \to r$

(b)

	$(p \wedge q) \to (r \vee s)$
	$(p \to r) \to q$
	$(p \to s) \vee r$
therefore	$p \vee s$

(c)

	$(p \wedge q) \to (r \vee s)$
	$s \to (\neg p)$
	$q \vee r \vee s$
therefore	$p \to q$

(d)

	$(p \wedge q) \to (r \vee s)$
	$(p \to r) \to q$
	$(p \to s) \vee r$
therefore	$q \vee s$

For each which is *not* a valid form, describe an assignment of truth values to p, q, r and s which renders each premise `T` but the conclusion `F`. [FD]

37. In each of the following argument forms, justify each line beginning '$\therefore$' in terms of previous lines and one of the 'rules of inference' in the list below:

(a)

1.		$p \vee q$
2.		$q \to r$
3.		$(p \wedge s) \to t$
4.		$\neg r$
5.		$(\neg q) \to (u \wedge s)$
6.	$\therefore$	$\neg q$
7.	$\therefore$	$u \wedge s$
8.	$\therefore$	p
9.	$\therefore$	s
10.	$\therefore$	$p \wedge s$
	$\therefore$	t

(b)

1.		$p \to q$
2.		$r \vee s$
3.		$(\neg s) \to (\neg t)$
4.		$(\neg q) \vee s$
5.		$\neg s$
6.		$((\neg p) \wedge r) \to u$
7.		$w \vee t$
8.	$\therefore$	$\neg q$
9.	$\therefore$	$\neg p$
10.	$\therefore$	r
11.	$\therefore$	$(\neg p) \wedge r$
12.	$\therefore$	u
13.	$\therefore$	$\neg t$
14.	$\therefore$	w
	$\therefore$	$u \wedge w$

[FE]

38. Show that the following argument form is valid. (*Hint*: consider what an invalidating line in the truth table would have to look like.)

$(p \wedge q) \to r$
$(\neg r) \to (p \vee s)$
$q \vee t$
$t \to s$
therefore $r \vee s$.

[FF]

39. Write down the negations of the statements
(i) $\forall\, m \in \mathbb{Z}, \exists\, n \in \mathbb{Z}, m + n = m,$
(ii) $\exists\, n \in \mathbb{Z}, \forall\, m \in \mathbb{Z}, m + n = m,$
(iii) $\forall\, m \in \mathbb{Z}, \exists\, n \in \mathbb{Z}, m + n = 0,$
(iv) $\exists\, n \in \mathbb{Z}, \forall\, m \in \mathbb{Z}, m + n = 0,$
in each case starting your formula with $\forall \ldots \exists$ or $\exists \ldots \forall$. For each of these *negations*, say whether it is true or false. [GA]

40. As question 39, but with every $\mathbb{Z}$ replaced by $\mathbb{N}$. [GB]

41. As question 39, but with every $\mathbb{Z}$ replaced by $\mathbb{P}$, the set $\{1, 2, 3, \ldots\}$ of strictly positive integers. [GC]

42. As question 42, but with the sentences
(i) Every correct computer program terminates.
(ii) Some incorrect computer programs terminate.
(iii) All flying pigs have wings.
(iv) There is a wingless flying pig.
and an extra formula type
$\exists\, x,\ (\neg P(x)) \wedge Q(x).$ [GDB]

43. Consider the following English sentences:
(i) Buttercups are yellow.
(ii) Some yellow flowers are buttercups.
(iii) Every integer is either positive or negative.
(iv) There is an ugly parrot.
(v) Only intelligent people like music.
(vi) You can fool all the people some of the time.
(vii) You can fool all the people some of the time, you can fool some of the people all the time, but you can't fool all the people all the time.

Match each of these with one of the formula types
$\forall\, x,\ P(x) \to Q(x),$
$\exists\, x,\ P(x) \wedge Q(x),$
$\forall\, x \in A,\ P(x) \vee Q(x),$
$\forall\, x \in A,\ P(x) \to Q(x),$
$\exists\, x \in A,\ \forall\, y \in B,\ P(x, y),$
$(\exists\, x \in A,\ \forall\, y \in B,\ P(x, y)) \wedge (\exists\, y \in B,\ \forall\, x \in A,\ P(x, y))$
$\quad \wedge(\neg(\forall\, x \in A,\ \forall\, y \in B,\ P(x, y)))$
in each case saying what $P(x)$, $Q(x)$, A etc. stand for.

[GD]

44. As question 43, with the sentences
(i) There is a prime number which is less than any other prime number.
(ii) Every non-negative real number has at least one real square root.
(iii) Nobody, except perhaps John, loves Jane.
(iv) All dogs chase all rabbits.
and the formula types
$\forall x \in A,\, P(x) \to Q(x)$,
$\exists x,\, (\neg P(x)) \wedge Q(x)$,
$\forall x \in A,\, \forall y \in B,\, P(x,y)$,
$\exists x \in A,\, \forall y \in B,\, P(x,y)$,
$\exists x \in A,\, \exists y \in B,\, P(x,y)$,
$\forall x,\, (\exists y \in A, P(x,y)) \to Q(x)$,
$\forall x,\, (\forall y \in A,\, P(x,y)) \to Q(x)$,
$\forall x \in A,\, P(x) \to (\exists y \in A,\, Q(x,y))$,
$\exists x \in A,\, \forall y \in A,\, (\neg P(x,y)) \to Q(x,y)$. [GE]

45. As question 44, with the sentences
(i) Some dogs chase all rabbits.
(ii) Some dogs chase some rabbits.
(iii) Only a dog will chase all rabbits.
(iv) Only a dog will chase any rabbit at all.
and the formula types as listed. [GF]

46. The following appeared in the *Times*, 25.1.99:

> Tomorrow John Battle, Energy Minister, will be told by power companies and advisers on pool reform that the deadline of April next year is impossible. He will also be told that legislation will be necessary because all the parties involved in the pool are unlikely to endorse reform.

Explain carefully the differences, in this context, between the phrases
(i) all the parties are unlikely to endorse the reform
(ii) there is a party which is unlikely to endorse the reform
(iii) it is unlikely that all the parties will endorse the reform.
Which do you suppose the author intended, and why? [KZ]

47. For each of the following arguments, decide whether it is valid. (*Remember!* this depends on the form of the argument, not the truth of its conclusion.) If it is *not* valid, then
(i) express it in symbolic form, stating explicitly which propositions correspond to each symbol you are using,
(ii) give an example of an argument in the same form in which the premises are clearly true and the conclusion is equally clearly false.

(a) John is a clumsy oaf,
clumsy oafs spill porridge,
$\therefore$ John spills porridge.
(b) John spills porridge,
clumsy oafs spill porridge,
$\therefore$ John is a clumsy oaf.
(c) John spills porridge,
only clumsy oafs spill porridge,
$\therefore$ John is a clumsy oaf.
(d) not all real numbers have squares less than 0
$\therefore$ all real numbers have squares not less than 0.

[LA]

48. As question 47, with
(a) If x is a real number and $x > 2$, then $x^2 > 4$,
$\therefore$ if x is a real number and $x^2 \not> 4$, $x \not> 2$.
(b) If x and y are real numbers, then either $x \leq y$ or $y \leq x$,
π is a real number,
$\therefore$ if x is a real number, either $x \leq \pi$ or $\pi \leq x$.
(c) All human beings are mortal,
Zeus is not mortal,
$\therefore$ Zeus is not a human being. [LBA]

49. As question 47, with
(a) There is no insect with eight legs,
all spiders have eight legs,
$\therefore$ no insect is a spider.
(b) No insect is a spider,
$\therefore$ no spider is an insect.
(c) For every integer m there is an integer n such that $m < n$,
$\therefore$ for every integer n there is an integer m such that $m < n$. [LBB]

50. As question 47, with
(a) There is a real number x such that $x \geq 0$,
there is a real number x such that $x \leq 0$,
$\therefore$ there is a real number x such that $x \geq 0$ and $x \leq 0$.
(b) For every real number x, $x^2 \geq 0$,
for every real number x, $x^4 \geq 0$,
$\therefore$ for every real number x, $x^2 \geq 0$ and $x^4 \geq 0$. [LBC]

51. As question 47, with
(a) For every real number x, $x^2 \geq 0$,
there is a real number x such that $x^3 \geq 0$,
$\therefore$ there is a real number x such that $x^2 \geq 0$ and $x^3 \geq 0$.
(b) For every real number x, if $x^3 < 0$ then $x^5 < 0$,
$\therefore$ there is a real number x such that $x^5 < 0$. [LBD]

52. As question 47, with
(a) All human beings are mortal,
Felix is mortal,
$\therefore$ Felix is a human being.

53. What conclusions may be drawn from the following sets of premises?
(a) All healthy people eat apples;
Harry does not eat apples.
(b) All healthy people eat apples;
Felix does not eat apples;
Felix is healthy.
(c) For all real numbers x and y, if neither x nor y is negative, nor is xy;
for all real numbers z, $z + 1$ and $z - 1$ are real numbers;
a is a real number and $(a + 1)(a - 1)$ is negative. [LD]

54. For each of the following argument forms,

(i) say whether it is valid,

(ii) if it is *not* valid, find an argument in the given form in which the premises are clearly true and the conclusion is equally clearly false.

(a) $\forall x \in A, \forall y \in B, P(x) \to Q(x,y)$
$\therefore \forall x \in A, P(x) \to (\forall y \in B, Q(x,y))$.

(b) $\forall x \in A, P(x) \to (\forall y \in B, Q(x,y))$
$\therefore \forall x \in A, \forall y \in B, P(x) \to Q(x,y)$.

(c) $\exists x \in A, P(x) \to Q(x)$
$\forall x \in A, P(x)$
$\therefore \exists x \in A, Q(x)$.

(d) $\forall x \in A, P(x) \to Q(x)$
$\forall x \in A, \neg Q(x)$
$\therefore \forall x \in A, \neg P(x)$.

(e) $\forall x \in A, P(x) \vee Q(x)$
$\forall x \in A, Q(x) \to R(x)$
$\exists x \in A, P(x) \vee R(x)$
$\therefore \exists x \in A, Q(x) \vee R(x)$.

(f) $\forall x \in A, P(x) \to Q(x)$
$\exists x \in A, \neg Q(x)$
$\therefore \exists x \in A, \neg P(x)$.

(g) $\exists x \in A, P(x) \to Q(x)$
$\forall x \in A, \neg Q(x)$
$\therefore \exists x \in A, \neg P(x)$.

(h) $\exists x \in A, P(x) \to Q(x)$
$\exists x \in A, Q(x) \to R(x)$
$\forall x \in A, P(x)$
$\therefore \exists x \in A, R(x)$.

[MA]

55. As question 54, but with

(a) $\exists x \in A, P(x)$
$\exists x \in A, Q(x)$
$\therefore \exists x \in A, P(x) \wedge Q(x)$

(b) $\exists x \in A, P(x)$
$\neg(\exists x \in A, P(x) \wedge Q(x))$
$\therefore \neg(\forall x \in A, Q(x))$.

(c) $\forall x \in A, P(x) \to Q(x)$
$\exists x \in A, P(x) \wedge R(x)$
$\therefore \exists x \in A, Q(x) \wedge R(x)$. [MBA]

56. As question 54, but with

(a) $\forall x \in A, \exists y \in B, P(x,y)$
$\forall y \in B, \exists x \in A, Q(x,y)$
$\therefore \forall x \in A, \forall y \in B, P(x,y) \vee Q(x,y)$.

(b) $(\forall x \in A, P(x)) \vee (\exists x \in A, Q(x))$
$\therefore \forall x \in A, P(x) \vee Q(x)$.

(c) $(\exists x \in A, P(x)) \vee (\exists x \in A, Q(x))$
$\therefore \exists x \in A, P(x) \vee Q(x)$. [MBB]

57. As question 54, but with
(a) $\forall\, x \in A,\ P(x) \vee Q(x)$
$\therefore\ (\forall\, x \in A,\ P(x)) \vee (\exists\, x \in A,\ Q(x))$.
(b) $(\forall\, x \in A,\ P(x)) \to (\forall\, x \in A,\ Q(x))$
$\therefore\ \forall\, x \in A,\ P(x) \to Q(x)$.
(c) $(\forall\, x \in A,\ P(x)) \to (\forall\, x \in A,\ Q(x))$
$\exists\, x \in A$
$\therefore\ \exists\, x \in A,\ P(x) \to Q(x)$. [MCA]

58. As question 54, but with
(a) $\exists\, x \in A,\ P(x) \to Q(x)$
$\forall\, x \in A,\ Q(x) \to R(x)$
$\forall\, x \in A,\ P(x)$
$\therefore\ \exists\, x \in A,\ R(x)$.
(b) $(\forall\, x \in A,\ P(x)) \vee (\forall\, x \in A,\ Q(x))$
$\therefore\ \forall\, x \in A,\ P(x) \vee Q(x)$.
(c) $(\exists\, x \in A,\ P(x)) \vee (\forall\, x \in A,\ Q(x))$
$\therefore\ \exists\, x \in A,\ P(x) \vee Q(x)$. [MCB]

59. As question 59, but with
$\forall\, x \in A,\ \exists\, y \in A,\ P(x, y)$
$\forall\, x \in A, \forall\, y \in A,\ \forall\, z \in A,\ (P(x, z) \wedge P(y, z)) \to (x = y))$
$\therefore\ \forall\, y \in A, \exists\, x \in A,\ P(x, y)$. [MD]

60. For each of the following arguments, (i) express it in symbolic form (ii) decide whether it is valid (iii) if it is *not* valid, give an example of an argument in the same form in which the premises are clearly true and the conclusion is equally clearly false.
(a) Some animals are furry
some animals have four legs
therefore there is a furry animal with four legs.
(In the first part of each answer, you are not required to explain what is represented by each symbol you use, but it will help your class teacher to adjudicate on unexpected responses if you provide the translations you have in mind.) [ME]

61. On the Island of Knights and Knaves, the children are as logical as their parents. You find a group of four of them disconsolate in front of a broken window.
ALEXANDER Charles did it.
BARBARA I didn't do it.
CHARLES Nor did I.
DAVID Alexander did it himself.
ALEXANDER More than one of us is a knave.
CHARLES Three of us are knaves.
Assuming that not more than one person was involved in breaking the window, and that they all know who it was, was it one of them, and if so which? [NA]

62. For each of the following argument forms, say whether it is valid or invalid. If it is valid, try to explain your reasons for thinking so; if it is invalid, give an example of an argument in that form leading from true premises to false conclusions.

(a) $\forall x \in A,\ P(x) \vee Q(x);\quad \forall x \in A,\ P(x) \to R(x);\quad \exists x \in A,\ Q(x) \to R(x);$
$\therefore\ \exists x \in A,\ R(x).$

(b) $\exists x \in A,\ P(x) \vee Q(x);\quad \forall x \in A,\ P(x) \to R(x);\quad \forall x \in A,\ Q(x) \to R(x);$
$\therefore\ \exists x \in A,\ R(x).$

(c) $\exists x \in A,\ P(x) \vee Q(x);\quad \exists x \in A,\ P(x) \to R(x);\quad \forall x \in A,\ Q(x) \to R(x);$
$\therefore\ \exists x \in A,\ R(x).$

(d) $\forall x \in A,\ P(x) \vee Q(x);\quad \exists x \in A,\ P(x) \to R(x);\quad \exists x \in A,\ Q(x) \to R(x);$
$\therefore\ \exists x \in A,\ R(x).$

(e) $\forall x \in A,\ P(x) \to Q(x);\quad \exists x \in A,\ Q(x) \to R(x);$
$\therefore\ \exists x \in A,\ R(x) \vee (\neg P(x)).$

(f) $(\forall x \in A,\ P(x)) \to r;\quad \forall x \in A,\ Q(x);$
$\therefore\ (\forall x \in A,\ P(x) \wedge Q(x)) \to r.$

(g) $(\forall x \in A,\ P(x) \to Q(x)) \to r;\quad \neg(\exists x \in A,\ P(x));$
$\therefore\ r.$

[NB]

63. As question 62, but with

(a) $\forall x \in A,\ P(x) \to q;\quad \exists x,\ x \in A;$
$\therefore\ q.$

(b) $\forall x \in A,\ (\forall y \in B,\ P(x,y)) \to Q(x);\quad \exists x \in A,\ \neg Q(x);$
$\therefore\ \exists y \in B,\ \forall x \in A,\ \neg P(x,y).$

(c) $\forall x \in A,\ (\forall y \in B,\ P(x,y)) \to Q(x);\quad \exists x \in A,\ \neg Q(x);$
$\therefore\ \exists y \in B,\ \exists x \in A,\ \neg P(x,y).$

(d) $\forall x \in A,\ (\forall y \in B,\ P(x,y)) \to Q(x);\quad \forall x \in A,\ \neg Q(x);$
$\therefore\ \exists y \in B,\ \forall x \in A,\ \neg P(x,y).$

(e) $\forall x \in A,\ \exists y \in A,\ P(x,y);\quad \forall x \in A,\ \exists z \in B,\ Q(x,z);$
$\forall x \in A,\ \forall y \in A,\ \forall z \in B,\ (P(x,y) \wedge Q(y,z)) \to R(x,z);$
$\therefore\ \forall x \in A,\ \exists z \in B,\ R(x,z).$

(f) $\exists x \in A,\ \forall y \in A,\ P(x,y);\quad \forall x \in A,\ \exists z \in B,\ Q(x,z);$
$\forall x \in A,\ \forall y \in A,\ \forall z \in B,\ (P(x,y) \wedge Q(y,z)) \to R(x,z);$
$\therefore\ \exists x \in A,\ \forall z \in B,\ R(x,z).$

(g) $\forall x \in A,\ \exists y \in B,\ P(x,y);\quad \forall x \in A,\ \exists y \in B,\ Q(x,y);$
$\forall x \in A,\ \forall y \in B,\ \forall z \in B,\ (P(x,y) \wedge R(y,z)) \to P(x,z);$
$\forall x \in A,\ \forall y \in B,\ \forall z \in B,\ (Q(x,y) \wedge R(y,z)) \to Q(x,z);$
$\forall y \in B,\ \forall z \in B,\ \exists w \in B,\ R(y,w) \wedge R(z,w);$
$\therefore\ \forall x \in A,\ \exists w \in B,\ P(x,w) \wedge Q(x,w).$ [NC]

64. (a) Explain the meanings of the symbols $\neg$, $\wedge$, $\vee$, $\to$ in the propositional calculus, giving examples of English sentences which can be naturally represented in the forms $\neg p$, $p \wedge q$, $p \vee q$, $p \to q$.

(b) What is meant by the *contrapositive* of an expression of the form $p \to q$? Give the contrapositive of the example you chose in part (a) to represent this formula. [OA]

65. (a) Put into symbolic form the sentences

(i) 'If John is wealthy, he is either healthy or wise',

(ii) 'John is wise but he is neither healthy nor wealthy',

(iii) 'John has just one of the three qualities of health, wealth and wisdom',

clearly stating what is represented by each symbol you use.

(b) Find sentences which are negations of the three sentences in part (a). (You should try to find expressions in direct and simple English.)

(c) Explain how the truth table for $p \vee q$ can be achieved by an expression involving only $\neg$ and $\wedge$. Hence show that any truth table can be achieved by an expression involving only $\neg$ and $\wedge$. [OB]

66. (a) Explain what is meant by a *tautology*. Show that one of the following is a tautology and the other is not:

(i) $(p \wedge ((p \vee q) \rightarrow r)) \rightarrow r$,

(ii) $(q \wedge ((p \wedge q) \rightarrow r)) \rightarrow r$.

(b) Show that if ϕ, ψ and χ are three expressions in the propositional calculus and $\phi \rightarrow \psi$, $\psi \rightarrow \chi$ are both tautologies, then $\phi \rightarrow \chi$ is also a tautology. [OC]

67. (a) Explain what is meant by saying that two expressions are *logically equivalent*. Find an expression which is logically equivalent to $(r \vee (\neg p)) \wedge (r \vee (\neg q))$ and which uses only two of the symbols $\vee$, $\wedge$, $\neg$ and $\rightarrow$.

(b) What is a *tautology*? Show that if ϕ, ψ are expressions of the propositional calculus such that $\phi \rightarrow \psi$ and $\psi \rightarrow \phi$ are logically equivalent, then $\phi \leftrightarrow \psi$ is a tautology. [OD]

68. Explain how to use truth tables to determine whether an argument form is valid. Use this method to decide which, if either, of the following forms is valid:

(i)
$$\begin{array}{l} p \vee q \\ p \rightarrow (\neg q) \\ p \rightarrow r \\ \therefore\ r. \end{array}$$

(ii)
$$\begin{array}{l} \neg(p \wedge q) \\ (\neg r) \rightarrow p \\ (\neg q) \rightarrow r \\ \therefore\ r. \end{array}$$

[OE]

69. (a) For each of the following expressions give a logically equivalent expression which uses only $\wedge$ and $\neg$, not $\vee$ or $\rightarrow$ or $\leftrightarrow$:

(i) $p \vee q$,

(ii) $p \rightarrow q$,

(iii) $p \leftrightarrow q$.

(b) Give the truth table of the expression $(p \vee r) \wedge (\neg(q \vee r))$. What is the simplest expression you can find which is logically equivalent to this?

(c) Show that two expressions ϕ, ψ are logically equivalent iff $\phi \leftrightarrow \psi$ is a tautology. [OF]

70. (a) Explain how to use truth tables to determine whether an argument is valid. Use this method to decide which, if either, of the following argument forms is valid:

(i)
$$\begin{array}{l} p \rightarrow (q \vee r) \\ q \rightarrow r \\ \neg r \\ \therefore\ \neg p. \end{array}$$

(ii)
$$\begin{array}{l} p \rightarrow (q \wedge r) \\ q \rightarrow r \\ p \vee q \\ \therefore\ p. \end{array}$$

[OG]

71. (a) Is the following a valid argument form?

$(\neg(r \vee s)) \to p$
$\neg(q \wedge s)$
$q \to (p \vee (\neg r))$
$\therefore\ q \to p.$

Justify your answer.

(b) Is the following a valid argument form?

$p \vee r$
$s \to p$
$q \to (r \vee s)$
$\therefore\ q \to p.$

Justify your answer. [OH]

72. (a) What is the principle 'Modus Ponens'? Express it in symbolic form.

(b) Explain how to use truth tables to decide whether an argument in the propositional calculus is valid. Illustrate your method by applying it to one valid argument and one invalid argument.

(c) Show that an argument form

$$\begin{array}{c} \phi_1 \\ \phi_2 \\ \dots \\ \phi_n \\ \therefore\ \psi \end{array}$$

is valid iff

$$(\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_n) \to \psi$$

is a tautology. [OI]

73. In the following argument form, explain how each line beginning '$\therefore$' can be deduced from one or more of the lines above, stating clearly a generally valid form (using not more than three atomic propositions) which you are employing. (You do not need to give names to the basic argument forms you use, which indeed need not have standard names.)

1.		$p \to (r \vee s)$
2.		$p \vee q$
3.		$q \to t$
4.		$s \to u$
5.		$t \to r$
6.		$\neg u$
7.	$\therefore$	$(\neg p) \vee r \vee s$
8.	$\therefore$	$\neg s$
9.	$\therefore$	$(\neg p) \vee r$
10.	$\therefore$	$p \to r$
11.	$\therefore$	$q \to r$
12.	$\therefore$	$r.$

[OJ]

74. On the Island of Knights and Knaves everyone is either a Knight (and always tells the truth) or a Knave (and always lies); and everyone knows of each of the others whether he or she is a Knight or a Knave.

You meet a group of four people, and ask Algernon 'How many knights are there among you?' He replies, but you don't catch the answer. Bertram says 'He said there are just two'. Charles says 'No, he didn't.' Daniel says 'But it's true, all the same.' Algernon says 'They're all lying'. Charles says 'Daniel and I are of the same type'.

What can you deduce, and how? [OK]

75. (a) Is the following argument form valid?

$$\begin{gathered} \neg(p \wedge q \wedge r \wedge t) \\ r \to (p \vee s) \\ q \vee t \\ t \to s \\ \therefore\ r \to s. \end{gathered}$$

Justify your answer.

(b) Is the following argument form valid?

$$\begin{gathered} (s \to t) \to q \\ (p \wedge q) \to r \\ p \vee q \vee s \vee t \\ s \to p \\ p \to t \\ \therefore\ r. \end{gathered}$$

Justify your answer. [OL]

76. For each of the following English sentences

(α) express it in symbolic form, clearly stating what is represented by each symbol you use,

(β) find an English sentence representing its negation.

(You should look for sentences in clear and direct English.)

(i) All buttercups are yellow.
(ii) Some yellow flowers are buttercups.
(iii) Smith can outrun any man on the team.
(iv) Smith cannot outrun any man on the team.
(v) For every problem you can't solve, there's an easier problem which you can't solve.
(vi) Every problem you can solve leads to another problem.
(vii) There is an even number greater than 2 which is not the sum of two primes. [OM]

77. Consider the formulae

$$(\alpha)\ \forall\, x \in A,\ (\forall\, y \in B,\ P(x,y)) \to Q(x),$$

$$(\beta)\ \forall\, x \in A,\ (\exists\, y \in B,\ P(x,y)) \to Q(x).$$

(i) Find an example of sets A and B, and predicates $P(x,y)$ and $Q(x)$, which make (α) true and (β) false.

(ii) Find an example of sets A and B, and predicates $P(x,y)$ and $Q(x)$, which make (β) true and (α) false. [ON]

78. For each of the following arguments, (α) express it in symbolic form (clearly stating what is represented by each symbol you use) (β) say whether the argument is valid, giving reasons for your answer if you can.

(i)

For every real number x there is a real number y such that $x > y$,
therefore for every real number y there is a real number x such that $x > y$.

(ii)

$\frac{1}{3}$ is a rational number,
$\sqrt{2}$ is not a rational number,
$\sqrt{2}$ is the product of $\sqrt{18}$ and $\frac{1}{3}$,
the product of any two rational numbers is a rational number,
therefore $\sqrt{18}$ is not a rational number.

[OOA]

79. Explain what is meant by the principle of *universal natural deduction*. Illustrate your account by showing how this principle can be used in the proof that

$$A \subseteq B$$
$$B \subseteq C$$
$$\therefore\ A \subseteq C$$

is a valid argument. [OOB]

80. (a) What is meant by *instantiation*? Illustrate your account by explaining where this principle can be used when validating the classical syllogism

All men are mortal,
Socrates is a man,
therefore Socrates is mortal.

(b) For each of the following arguments, (α) express it in symbolic form (clearly stating what is represented by each symbol you use) (β) say whether the argument is valid, giving reasons for your answer if you can.

(i)
For all real numbers x, y and z, if $xy \neq 0$ and $yz \neq 0$ then $xz \neq 0$,
for every non-zero real number x there is a real number y such that $xy \neq 0$,
for every non-zero real number z there is a real number y such that $yz \neq 0$,
therefore $xz \neq 0$ for all non-zero real numbers x and z.

(ii) Writing P_2 for the set of natural numbers greater than or equal to 2,
for every $m \in P_2$ there is an $n \in P_2$ such that i is a factor of n whenever $i \in P_2$ and i is less than or equal to m,
for every $m \in P_2$ there is an $n \in P_2$ such that there is no $i \in P_2$ which is a factor of both m and n,
therefore for every $m \in P_2$ there is a $n \in P_2$ such that i is not a factor of n whenever $i \in P_2$ and i is less than or equal to m.

[OP]

81. Consider the argument form

$$\forall\, y \in A,\ \exists\, x \in A,\ P(x,y),$$
$$\forall\, x \in A,\ \exists\, y \in A,\ P(x,y),$$
$$\forall\, x \in A,\ \forall\, y \in A,\ \forall\, z \in A,\ (P(x,y) \wedge P(x,z)) \to (y = z),$$
$$\therefore\ \forall\, x \in A,\ \forall\, y \in A, \forall\, z \in A,\ (P(x,z) \wedge P(y,z)) \to (x = y).$$

Is this valid? Give reasons for your answer. [OQ]

82. On the Island of Knights and Knaves everyone is either a Knight (and always tells the truth) or a Knave (and always lies); and everyone knows of each of the others whether he or she is a knight or a knave. They also have two charitable organisations, the Hearts and the Diamonds; you do not know whether anyone belongs to both, but all the inhabitants know just who belongs to which group.

You meet a group of three people, who tell you

XERXES All Hearts are Diamonds.
YORICK No knight is a Diamond.
ZENO All knaves are Hearts.
XERXES I'm a Heart.
YORICK Xerxes is a Diamond.
ZENO That's not true.

Which charity, if either, does Xerxes support? [OR]

83. (a) State the First and Second Principles of Induction in the language of the predicate calculus.

(b) State the Well-Ordering Principle for the set $\mathbb{N}$ of natural numbers, and explain how it can be proved from the Second Principle of Induction. [OS]

84. Consider the premises
    1. Anyone who admires a tyrant later regrets it.
    2. Stalin was a dictator.
    3. Stalin was brutal.
    4. All brutal dictators are tyrants.
    5. John admired Stalin.

(a) What conclusion do you draw from these premises? (b) Express your argument in symbolic form. [PA]