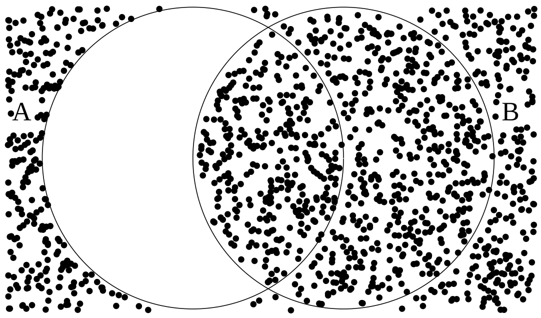
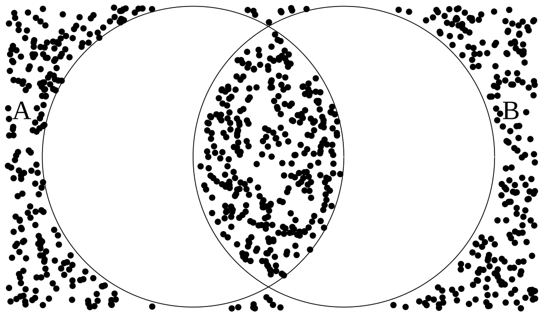
Propositional and Predicate Calculus Answers

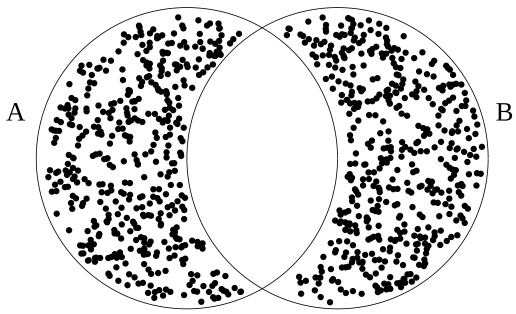
1. (i) $B \cup A^c$



(ii), (iv) $(B \cup A^c) \cap (A \cup B^c) = (A \cap B) \cup (A^c \cap B^c) = (A \triangle B)^c$



(iii), (v) $(A^c \cup B^c) \cap (A \cup B) = (A \cap B^c) \cup (A^c \cap B) = A \triangle B$



2.

p	q	$p \vee (\neg q)$	$q \vee (\neg p)$	$(q \vee \neg p) \wedge (p \vee \neg q)$
T	T	T	T	T
T	F	T	F	F
F	T	F	T	F
F	F	T	T	T

p	q	$(p \wedge (\neg q)) \vee (q \wedge (\neg p))$	$(p \wedge q) \vee ((\neg p) \wedge (\neg q))$
T	T	F	T
T	F	T	F
F	T	T	F
F	F	F	T

3.

p	q	$(p \vee q) \wedge ((\neg p) \vee (\neg q))$	$p \wedge (p \vee q)$	$q \vee (p \wedge q)$
T	T	F	T	T
T	F	T	T	F
F	T	T	F	T
F	F	F	F	F

p	q	$(p \vee q) \vee ((\neg p) \wedge (\neg q))$	$(p \wedge q) \wedge ((\neg p) \vee (\neg q))$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	T	F

4. If we take A to correspond to p and B to correspond to q, then under the standard matching we have

$$B \cup A^c \text{ and } q \vee (\neg p),$$

$$(B \cup A^c) \cap (A \cup B^c) \text{ and } (q \vee (\neg p)) \wedge (p \vee (\neg q)),$$

$$(A^c \cup B^c) \cap (A \cup B) \text{ and } (p \vee q) \wedge ((\neg p) \vee (\neg q))$$

(a mechanically strict translation would reverse one of these, so that $(A^c \cup B^c) \cap (A \cup B)$ would go with $((\neg p) \vee (\neg q)) \wedge (p \vee q)$, and $(A \cup B) \cap (A^c \cup B^c)$ with $(p \vee q) \wedge ((\neg p) \vee (\neg q))$),

$$(A \cap B) \cup (A^c \cap B^c) \text{ and } (p \wedge q) \vee ((\neg p) \wedge (\neg q)),$$

$$(A \cap B^c) \cup (A^c \cap B) \text{ and } (p \wedge (\neg q)) \vee (q \wedge (\neg p)).$$

5. There are infinitely many possibilities for each; the simplest seem to be

(a) $p \vee (\neg q)$

(b) $\neg q$

(c) $p \wedge (\neg q)$

(d) $q \wedge (\neg p)$

(e) $(\neg q) \wedge (\neg p)$ or $\neg(p \vee q)$.

Ask your class teacher about alternatives.

6.

p	q	r	s	$p \to q$	$r \to s$	$(p \to q) \to (r \to s)$
T	T	T	T	T	T	T
T	T	T	F	T	F	F
T	T	F	T	T	T	T
T	T	F	F	T	T	T
T	F	T	T	F	T	T
T	F	T	F	F	F	T
T	F	F	T	F	T	T
T	F	F	F	F	T	T
F	T	T	T	T	T	T
F	T	T	F	T	F	F
F	T	F	T	T	T	T
F	T	F	F	T	T	T
F	F	T	T	T	T	T
F	F	T	F	T	F	F
F	F	F	T	T	T	T
F	F	F	F	T	T	T

$(p \to q) \to (r \to s)$ is F only when $p \to q$ is T and $r \to s$ is F; that is, unless r is T, s is F and either p is F or q is T. So there are just 3 Fs in the right-hand column.

7. The final output of qslist will be such that whenever $j \neq k$ and qslist[j] R qslist[k], then $j < k$; that is, the total order qslist[1], qslist[2], ..., qslist[length$-$1] will extend the relation R.

To see this, look at what partition does. It puts into the 'high' section qslist[dp+1] ... qslist[right] exactly those numbers k it finds in the list qslist[left+1] ... qslist[right] such that goodorder(pivot,k,num) is T; that is, just those k such that pivot $R\,k$. This means that if k is put into the high section and j into the low section qslist[left] ... qslist[dp$-$1], then pivot $R\,k$ but pivot $\not R\; j$; since R is transitive, it follows that $k \not R\; j$.

Now we can argue by induction on n to show that

> if right $-$ left $= n$, then quicksort(left,right,qslist,num) arranges qslist[left] ... qslist[right] in such a way that qslist[j] $\not R$ qslist[i] whenever left $\leq i < j \leq$ right, and leaves the rest of qslist alone.

At the end we shall have

$$\texttt{qslist[}k\texttt{]} \not R \texttt{ qslist[}j\texttt{]} \text{ whenever } 0 \leq j < k < \texttt{length},$$

or, turning this round,

$$\text{if } j,\, k < \texttt{length} \text{ and } j \neq k \text{ and } \texttt{qslist[}j\texttt{]}\ R\ \texttt{qslist[}k\texttt{]}, \text{ then } j < k.$$

8. The final output of `qslist` will be such that whenever $j \neq k$ and `qslist[`j`]` R `qslist[`k`]`, then $j < k$; that is, the total order `qslist[1]`, `qslist[2]`, ..., `qslist[length−1]` will extend the relation R.

To see this, look at what `partition` does. It puts into the 'high' section `qslist[dp+1]` ... `qslist[right]` exactly those numbers k it finds in the list `qslist[left+1]` ... `qslist[right]` such that `goodorder(pivot,`k`,num)` is `T`; that is, just those k such that `pivot` $R\,k$. This means that if k is put into the high section and j into the low section `qslist[left]` ... `qslist[dp−1]`, then `pivot` $R\,k$ but `pivot` $\not R\ j$; since R is transitive, it follows that $k \not R\ j$.

Now we can argue by induction on n to show that

> if `right` $-$ `left` $= n$, then `quicksort(left,right,qslist,num)` arranges `qslist[left]` ... `qslist[right]` in such a way that `qslist[`j`]` $\not R$ `qslist[`i`]` whenever `left` $\leq i < j \leq$ `right`, and leaves the rest of `qslist` alone.

At the end we shall have

$$\texttt{qslist[}k\texttt{]}\ \not R\ \texttt{qslist[}j\texttt{]} \text{ whenever } 0 \leq j < k < \texttt{length},$$

or, turning this round,

$$\text{if } j,\ k < \texttt{length} \text{ and } j \neq k \text{ and } \texttt{qslist[}j\texttt{]}\ R\ \texttt{qslist[}k\texttt{]}, \text{ then } j < k.$$

9. (a) If the air isn't fresh, John hasn't opened the window.

(b) If Epimenides isn't a liar he isn't a Cretan.

(c) If I'm not a Dutchman then Clinton is a liar.

(d) If I go out it isn't dark.

(e) If Britain doesn't become more prosperous, interest rates haven't been lowered.

10. (i), (ii)

p	q	r	$q \vee r$	$q \wedge r$	$p \to (q \vee r)$	$p \to (q \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	T	F
T	F	T	T	F	T	F
T	F	F	F	F	F	F
F	T	T	T	T	T	T
F	T	F	T	F	T	T
F	F	T	T	F	T	T
F	F	F	F	F	T	T

(iii), (iv)

p	q	r	$p \to q$	$p \to r$	$(p \to q) \vee (p \to r)$	$(p \to q) \wedge (p \to r)$
T	T	T	T	T	T	T
T	T	F	T	F	T	F
T	F	T	F	T	T	F
T	F	F	F	F	F	F
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

$p \to (q \wedge r)$ and $(p \to q) \wedge (p \to r)$ are logically equivalent.
$p \to (q \vee r)$ and $(p \to q) \vee (p \to r)$ are logically equivalent.

11. (i), (ii)

p	q	r	$p \to q$	$p \wedge q$	$(p \to q) \to r$	$(p \wedge q) \to r$
T	T	T	T	T	T	T
T	T	F	T	T	F	F
T	F	T	F	F	T	T
T	F	F	F	F	T	T
F	T	T	T	F	T	T
F	T	F	T	F	F	T
F	F	T	T	F	T	T
F	F	F	T	F	F	T

(iii), (iv)

p	q	r	$p \vee q$	$p \to r$	$q \to r$	$(p \vee q) \to r$ $(p \to r) \wedge (q \to r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	T	T	T	T
T	F	F	T	F	T	F
F	T	T	T	T	T	T
F	T	F	T	T	F	F
F	F	T	F	T	T	T
F	F	F	F	T	T	T

$(p \vee q) \to r$ and $(p \to r) \wedge (q \to r)$ are logically equivalent.

12. (i)

p	q	r	$p \to r$	$q \to r$	$(p \to r) \vee (q \to r)$
T	T	T	T	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

(ii)

p	q	r	$q \to r$	$p \vee r$	$(q \to r) \wedge (p \vee r)$
T	T	T	T	T	T
T	T	F	F	T	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	F	F	F
F	F	T	T	T	T
F	F	F	T	F	F

(iii)

p	q	r	$q \vee (\neg p)$	$(q \vee (\neg p)) \to r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	T	F
F	F	T	T	T
F	F	F	T	F

(ii) and (iii) are logically equivalent.

13. (i) If I fail the examination I didn't work hard.

(ii) If I pass the examination I didn't work hard.

(iii) If I fail the examination I worked hard.

(iv) If I pass the examination I worked hard.

(Many variations are acceptable – ask the class teacher.)

14. (i)

p	q	$p \wedge q$	$p \vee q$	$\neg(p \vee q)$	$(p \wedge q) \vee (\neg(p \vee q))$
T	T	T	T	F	T
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	T

(ii)

p	q	r	$\neg p$	$\neg r$	$q \vee (\neg r)$	$(\neg p) \wedge (q \vee (\neg r))$
T	T	T	F	F	T	F
T	T	F	F	T	T	F
T	F	T	F	F	F	F
T	F	F	F	T	T	F
F	T	T	T	F	T	T
F	T	F	T	T	T	T
F	F	T	T	F	F	F
F	F	F	T	T	T	T

15. (i) $(p \wedge q) \wedge r$ or $p \wedge (q \wedge r)$ or $p \wedge q \wedge r$ (etc.) (The order of the terms doesn't matter, and we can safely omit the brackets.)

(ii) $(\neg p) \wedge q \wedge r$.

(iii) $(\neg q) \wedge (p \vee r)$.

(iv) $(\neg p) \wedge (\neg q) \wedge (\neg r)$.
Alternative solution: $\neg(p \vee q \vee r)$.

16. (i) $(p \wedge q) \wedge (\neg r)$.
(ii) $(\neg p) \wedge (q \vee r)$.

17. (i) Either Sam doesn't swim on Thursdays or Kate doesn't play tennis on Saturdays.

(ii) This program has no error in the first ten lines and is being run with a complete data set.

(iii) The train isn't late and my watch isn't fast.

(iv) Either n isn't an integer or it's an integer less than or equal to 0.

18.

p	q	r	$p \vee q$	$p \vee r$	$(\neg p) \vee (r \wedge q)$	$(p \vee q) \wedge (p \vee r) \wedge$ $((\neg p) \vee (r \wedge q))$
T	T	T	T	T	T	T
T	T	F	T	T	F	F
T	F	T	T	T	F	F
T	F	F	T	T	F	F
F	T	T	T	T	T	T
F	T	F	T	F	T	F
F	F	T	F	T	T	F
F	F	F	F	F	T	F

It is logically equivalent to $q \wedge r$.

19.

p	q	r	$p \vee q$	$p \wedge r$	$(p \vee q) \vee (p \wedge r)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	T	T	T
T	F	F	T	F	T
F	T	T	T	F	T
F	T	F	T	F	T
F	F	T	F	F	F
F	F	F	F	F	F

It is logically equivalent to $p \vee q$.

20. (No solution written out.)

21. (No solution written out.)

22. The second is a knave and the third is a knight, but the first might be either.

To see this, start with the third person's statement. If this is true, then the third person is a knight and the second is a knave; if it's false, then the third person is a knave and the second is a knight. So in either case, just one of them is a knight. Now ask: did the first person say 'there is just one knight'? If so, it was true iff the first person is a knave; that is, it's true iff the first person lied and the sentence was false. This is ridiculous. So the first person did *not* say 'there is just one knight'. But this means that the second is a knave and the third is a knight.

We can't, however, decide what the first is. Because if the first was a knight, she'll have said 'there are two knights among us', and the second lied and the third told the truth. While if the first was a knave, she might easily have said 'we are all knights', and again the second lied and the third told the truth, according to their natures.

23. (a) The speaker is a knight and the other is a knave.

(b) The speaker said 'One of us is a knave', which codes as '$(\neg p) \vee (\neg q)$'. This is true if, and only if, the speaker is a knight; so we have

$$p \leftrightarrow ((\neg p) \vee (\neg q)).$$

The conclusion is

$$p \wedge (\neg q).$$

To validate this with a truth table, we can argue

p	q	$(\neg p) \vee (\neg q))$	$p \leftrightarrow ((\neg p) \vee (\neg q))$	$p \wedge (\neg q)$	
T	T	F	F	F	
T	F	T	T	T	$<$
F	T	T	F	F	
F	F	T	F	F	

There are alternative versions possible. Note that $p \wedge (\neg q)$ is actually logically equivalent to $p \leftrightarrow ((\neg p) \vee (\neg q))$ or $p \leftrightarrow (\neg(p \wedge q))$ or $(\neg p) \leftrightarrow (p \wedge q)$.

24. (a) Both are knights.

(b) The premise is

$$p \leftrightarrow ((\neg p) \vee q)$$

and the conclusion is

$$p \wedge q.$$

The truth table is

p	q	$(\neg p) \vee q$	$p \leftrightarrow ((\neg p) \vee q)$	$p \wedge q$	
T	T	T	T	T	$<$
T	F	F	F	F	
F	T	T	F	F	
F	F	T	F	F	

Note that this is really the same question as question 0. The first speaker is saying 'Either I am lying or . . . '; from which you can conclude that he is telling the truth.

25. (a) The second speaker is a knight and the other two are knaves.

(b) The premises are

$$p \leftrightarrow \phi, \quad q \leftrightarrow \psi \ .$$

The conclusion is

$$(\neg p) \wedge q \wedge (\neg r).$$

The truth table is

p	q	r	ϕ	$p \leftrightarrow \phi$	ψ	$q \leftrightarrow \psi$	$(\neg p) \wedge q \wedge (\neg r)$	
T	T	T	F	F	F	F	F	
T	T	F	F	F	F	F	F	
T	F	T	F	F	F	T	F	
T	F	F	F	F	T	F	F	
F	T	T	F	T	F	F	F	
F	T	F	F	T	T	T	T	$<$
F	F	T	F	T	T	F	F	
F	F	F	T	F	F	T	F	

26. That he is a knight who believes that the moon is made of green cheese. (The inhabitants of this island are stronger in logic than in astronomy.)

27. (i) There is a mathematician who doesn't wear glasses.

(ii) No mathematicians wear glasses.

(iii) There is a prime number which isn't odd.

(iv) There is an odd number which isn't prime.

28. (a)

p	q	$p \to q$	$q \to p$	$p \vee q$	
T	T	T	T	T	$<$
T	F	F	T	T	
F	T	T	F	T	
F	F	T	T	F*	$<$

so the argument is invalid.

(b)

p	q	r	$p \to q$	$(\neg q) \vee r$	
T	T	T*	T	T	$<$
T	T	F	T	F	
T	F	T	F	T	
T	F	F	F	T	
F	T	T	T	T	
F	T	F	T	F	
F	F	T	T	T	
F	F	F	T	T	

so the argument is valid.

(c)

p	q	r	$p \vee q$	$p \to (\neg q)$	$p \to r$	
T	T	T	T	F	T	
T	T	F	T	F	F	
T	F	T	T	T	T	$<$
T	F	F	T	T	F	
F	T	T	T	T	T	$<$
F	T	F*	T	T	T	$<$
F	F	T	F	T	T	
F	F	F	F	T	T	

so the argument is invalid (we might have p F, q T and r F).

29. (a)

p	q	r	$p \to q$	$p \to r$	$p \to (q \wedge r)$	
T	T	T	T	T	T*	$<$
T	T	F	T	F	F	
T	F	T	F	T	F	
T	F	F	F	F	F	
F	T	T	T	T	T*	$<$
F	T	F	T	T	T*	$<$
F	F	T	T	T	T*	$<$
F	F	F	T	T	T*	$<$

so the argument is valid.

(b)

p	q	r	$(p \wedge (\neg q)) \to r$	$p \vee q$	$q \to p$	
T	T	T	T	T	T	$<$
T	T	F*	T	T	T	$<$
T	F	T	T	T	T	$<$
T	F	F	F	T	T	
F	T	T	T	T	F	
F	T	F	T	T	F	
F	F	T	T	F	T	
F	F	F	T	F	T	

so the argument is not valid. (We might have p T, q T and r F.)

30. (a) (i) Writing p for 'Tom is on team A', q for 'Abdul is on team B' then the argument form is

$(\neg p) \to (\neg q)$; $(\neg q) \to p$;
therefore $(\neg p) \vee (\neg q)$.

(ii) The argument is invalid.

(iv) p and q are both T.

(b)(i) Writing p for 'Alice solved the problem correctly' and q for 'Alice got the answer 2', the argument form is

$p \to q$; q;
therefore p.

(ii) The argument is invalid.

(iv) p is F and q is T.

(c)(i) Writing p for 'Tom is on team A', q for 'Abdul is on team B' then the argument form is

$(\neg p) \to (\neg q)$; $(\neg q) \to p$;
therefore p.

(ii) The argument is valid.

(iii)

p	q	$(\neg p) \to (\neg q)$	$(\neg q) \to p$	
T	T	T	T	$<$
T	F	T	T	$<$
F	T	F	T	
F	F	T	F	

31. (a)(i) Writing p for 'Tom is clever' and q for 'Tom is rich', the argument form is

$p \to q$; $p \vee q$; $(\neg p) \vee (\neg q)$;
therefore $\neg p$.

(ii) The argument is valid.

(iii)

p	q	$p \to q$	$p \vee q$	$(\neg p) \vee (\neg q)$	$\neg p$	
T	T	T	T	F	F	
T	F	F	T	T	F	
F	T	T	T	T	T	$<$
F	F	T	F	T	T	

(b)(i) Writing p for 'Tom is clever' and q for 'Tom is rich' , the argument form is

$p \to q$; $p \vee q$; $(\neg p) \vee (\neg q)$;
therefore $\neg q$.

(ii) The argument is not valid.

(iv) p F and q T.

32. (a)(i) Writing p for 'Stephen went to the bar every night', q for 'Stephen did every problem sheet' and r for 'Stephen passed the test', the argument form is

$p \to (\neg q)$; $r \to q$;
therefore $(\neg r) \to p$.

(ii) This is not valid.

(iv) p F, q T, r F *or* p F, q F, r F.

(b)(i) Writing p for 'Tom is on the team', q for 'Abdul is on the team' and r for 'Stephen is on the team', the argument form is

$p \to q$; $r \to (\neg q)$; $p \vee q$;
therefore $\neg r$.

(ii) This is valid.

(iii)

p	q	r	$p \to q$	$r \to (\neg q)$	$p \vee q$	$\neg r$	
T	T	T	T	F	T	F	
T	T	F	T	T	T	T	$<$
T	F	T	F	T	T	F	
T	F	F	F	T	T	T	
F	T	T	T	F	T	F	
F	T	F	T	T	T	T	$<$
F	F	T	T	T	F	F	
F	F	F	T	T	F	T	

33. This argument is valid. (If the conclusion $p \to q$ is F, then p is T and q is F. Since $s \to q$, s is F; similarly, $p \wedge r$ is F, so r is F. But in this case $p \to r$ and $p \to s$ are both F.)

34. (a) is not valid; possibly p and q are both T.

(b) is valid. For if p were F, then $p \to q$ would be T, so p would have to be T. In terms of the truth table we have

p	q	$p \to q$	$(p \to q) \to p$	
T	T	T	T	$<$
T	F	F	T	$<$
F	T	T	F	
F	F	T	F	

so in fact $(p \to q) \to p$ is actually logically equivalent to p.

35. (a) This is valid. (If p were T, q would have to be F, so $q \to r$ would have to be T and $p \to (\neg(q \to r))$ would be F.)

(b) This is invalid; we could have p F while q and r are both T (or anything else).

(c) This is valid. (If p were F, then $p \to r$ would be T, so q would have to be T and p would be T, which is absurd. So p is T, so q and $p \to r$ are both T, so $p \wedge q \wedge r$ is T.)

(d) This is invalid; we could have p T while q and r are both F.

36. (a) This is valid. (If the conclusion $p \to r$ is F, we must have p T and r F. Since $s \to (\neg p)$ is T, s is F. Since $q \vee r \vee s$ is T, q must be T, so $p \wedge q$ is T and $r \vee s$ is T; but r and s are both F, so this is impossible.)

(b) This is invalid; we might have p and s F while q is T and r is anything.

(c) This is invalid; we might have p T, q F, r T and s F.

(d) This is valid. (If the conclusion $q \vee s$ is F, then q and s must both be F, so $p \to r$ must be F, that is, p is T and r is F; but in this case $(p \to s) \vee r$ is F. Note that the premise $(p \wedge q) \to (r \vee s)$ is redundant.)

37. (a)

1.		$p \vee q$	
2.		$q \to r$	
3.		$(p \wedge s) \to t$	
4.		$\neg r$	
5.		$(\neg q) \to (u \wedge s)$	
6.	$\therefore$	$\neg q$	Modus Tollens from (2) and (4)
7.	$\therefore$	$u \wedge s$	Modus Ponens from (5) and (6)
8.	$\therefore$	p	Disjunctive Syllogism from (1) and (6)
9.	$\therefore$	s	Conjunctive Simplification from (7)
10.	$\therefore$	$p \wedge s$	Conjunctive Addition from (8) and (9)
	$\therefore$	t	Modus Ponens from (3) and (10)

(b)

1.		$p \to q$	
2.		$r \vee s$	
3.		$(\neg s) \to (\neg t)$	
4.		$(\neg q) \vee s$	
5.		$\neg s$	
6.		$((\neg p) \wedge r) \to u$	
7.		$w \vee t$	
8.	$\therefore$	$\neg q$	Disjunctive Syllogism from (4) and (5)
9.	$\therefore$	$\neg p$	Modus Tollens from (1) and (8)
10.	$\therefore$	r	Disjunctive Syllogism from (2) and (5)
11.	$\therefore$	$(\neg p) \wedge r$	Conjunctive Addition from (9) and (10)
12.	$\therefore$	u	Modus Ponens from (6) and (11)
13.	$\therefore$	$\neg t$	Modus Ponens from (3) and (5)
14.	$\therefore$	w	Disjunctive Syllogism from (7) and (13)
	$\therefore$	$u \wedge w$	Conjunctive Addition from (12) and (14)

38. If there were a line in which every premise was T but the conclusion was F, we should have to have
$r \vee s$ is F, so r and s are both F,
s is F and $t \to s$ is T, so t is F,
t is F and $q \vee t$ is T, so q is T,
$(p \wedge q) \to r$ is T and r is F, so $p \wedge q$ is F,
$p \wedge q$ is F and q is T, so p is F,
$\neg r$ and $(\neg r) \to (p \vee s)$ are T, so $p \vee s$ is T,
p and s are both F so $p \vee s$ is F.
So there is no such line, and the argument is valid.
(Other arguments are possible; ask your class teacher.)

39. (i) $\exists m \in \mathbb{Z}, \forall n \in \mathbb{Z}, m + n \neq m$, false.

(ii) $\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z}, m + n \neq m$, false.

(iii) $\exists m \in \mathbb{Z}, \forall n \in \mathbb{Z}, m + n \neq 0$, false.

(iv) $\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z}, m + n \neq 0$, true.

40. (i) $\exists m \in \mathbb{N}, \forall n \in \mathbb{N}, m + n \neq m$, false.

(ii) $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}, m + n \neq m$, false.

(iii) $\exists m \in \mathbb{N}, \forall n \in \mathbb{N}, m + n \neq 0$, true.

(iv) $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}, m + n \neq 0$, true.

41. (i) $\exists\, m \in P, \forall\, n \in P, m + n \neq m$, true.

(ii) $\forall\, n \in P, \exists\, m \in P, m + n \neq m$, true.

(iii) $\exists\, m \in P, \forall\, n \in P, m + n \neq 0$, true.

(iv) $\forall\, n \in P, \exists\, m \in P, m + n \neq 0$, true.

42. (i) $\forall\, x, P(x) \to Q(x)$,
$P(x)$ for 'x is a correct computer program',
$Q(x)$ for 'x terminates'.

(ii) $\exists\, x, (\neg P(x)) \wedge Q(x)$,
$P(x)$ for 'x is a correct computer program',
$Q(x)$ for 'x terminates'.
(iii) $\forall\, x, P(x) \to Q(x)$,
$P(x)$ for 'x is a flying pig',
$Q(x)$ for 'x has wings.

(iv) $\exists\, x, (\neg P(x)) \wedge Q(x)$,
$Q(x)$ for 'x is a flying pig',
$P(x)$ for 'x has wings'.

43. (i) $\forall\, x, P(x) \to Q(x)$,
$P(x)$ for 'x is a buttercup',
$Q(x)$ for 'x is yellow'.

(ii) $\exists\, x, P(x) \wedge Q(x)$,
$P(x)$ for 'x is a buttercup',
$Q(x)$ for 'x is a yellow flower' (or the other way round).

(iii) $\forall\, x \in A, P(x) \vee Q(x)$,
A for $\mathbb{Z}$,
$P(x)$ for 'x is positive',
$Q(x)$ for 'x is negative' (or the other way round).

(iv) $\exists\, x, P(x) \wedge Q(x)$,
$P(x)$ for 'x is ugly',
$Q(x)$ for 'x is a parrot' (or the other way round).

(v) $\forall\, x \in A, P(x) \to Q(x)$,
A for the set of all people,
$P(x)$ for 'x likes music',
$Q(x)$ for 'x is intelligent'.

(vi) $\exists\, x \in A, \forall\, y \in B, P(x, y)$,
A for the set of times,
B for the set of people,
$P(x, y)$ for 'you can fool y at time x'.

(vii) $(\exists\, x \in A, \forall\, y \in B, P(x, y)) \wedge (\exists\, y \in B, \forall\, x \in A, P(x, y)) \wedge (\neg(\forall\, x \in A, \forall\, y \in B, P(x, y)))$,
A for the set of times,
B for the set of people,
$P(x, y)$ for 'you can fool y at time x'.

44. (i) $\exists\, x \in A, \forall\, y \in A, (\neg P(x,y)) \to Q(x,y)$,
A is the set of prime numbers,
$P(x,y)$ for 'x equals y',
$Q(x,y)$ for 'x is less than y'.
(Allow '$x = y$', '$x < y$'.)

(ii) $\forall\, x \in A,\ P(x) \to (\exists\, y \in A,\ Q(x,y))$,
A for $\mathbb{R}$,
$P(x)$ for 'x is non-negative',
$Q(x,y)$ for 'y is a square root of x'.
(Allow '$x \geq 0$', '$y^2 = x$', but not '$y = \sqrt{x}$', because that formulation assumes that x has only one square root.)

(iii) $\forall\, x \in A,\ P(x) \to Q(x)$,
A for the set of people,
$P(x)$ for 'x loves Jane',
$Q(x)$ for 'x is John'.

(iv) $\forall\, x \in A, \forall\, y \in B,\ P(x,y)$,
A for the set of dogs,
B for the set of rabbits,
$P(x,y)$ for 'x chases y'.

45. (i) $\exists\, x \in A, \forall\, y \in B,\ P(x,y)$,
A for the set of dogs,
B for the set of rabbits,
$P(x,y)$ for 'x chases y'.

(ii) $\exists\, x \in A, \exists\, y \in B,\ P(x,y)$,
A for the set of dogs,
B for the set of rabbits,
$P(x,y)$ for 'x chases y'.

(iii) $\forall\, x, (\forall\, y \in A,\ P(x,y)) \to Q(x)$,
A for the set of rabbits,
$P(x,y)$ for 'x chases y',
$Q(x)$ for 'x is a dog'.

(iv) $\forall\, x, (\exists\, y \in A, P(x,y)) \to Q(x)$,
A for the set of rabbits,
$P(x,y)$ for 'x chases y',
$Q(x)$ for 'x is a dog'.

(Note the difficulty that the English language has in distinguishing between (iii) and (iv). Would
(iii) Only a dog will chase any rabbit *or* Only a dog will chase every rabbit
(iv) Only a dog will chase a rabbit
have been clearer? Can anyone find an alternative way of doing a better job?)

46. (i) means that for every one of the parties involved, the probability that they will refuse to endorse the reform is well over 50%. (ii) is very different; it says that for at least one party, it is worth betting that they will oppose the reform. And (iii) is different again; it says that it is worth betting that at least one party will oppose the reform. To see the difference clearly, a little probability theory helps. Suppose that there are three parties, and for each of them the chance that they will agree to the reform is 60%. In this case, you can't say that, for any particular party, there is any profit in betting that they will oppose the reform. But looking at all three, and supposing that their decisions are independent, the chance that 'all the parties will endorse the reform' is $\frac{60}{100} \times \frac{60}{100} \times \frac{60}{100} = 0{\cdot}216$, so it's well worth betting that 'not all the parties will endorse the reform'.

I expect the author meant (iii), because the Government will need to be able to order people about if there is anyone who won't go along with the agreed scheme; and since the author isn't pointing at any particularly difficult customer, the suggestion is that the problem arises because of the number of people involved.

The difference between (i) and (iii) is essentially the difference between $\forall\,\neg$ and $\neg\,\forall$. Most journalists are unable to make this distinction, so you should always consider the possibility that they've said something they didn't mean.

47. (a) Valid.

(b) Invalid.

(i) Writing $P(x)$ for 'x is a clumsy oaf', $Q(x)$ for 'x spills porridge' and a for 'John', the argument becomes

$Q(a)$; $\forall\, x,\ P(x) \to Q(x)$;
$\therefore\ P(a)$.

(The substitution of a for 'John' is not compulsory.)

(ii) Leaving a as 'John', we could try 'x is a woman' for $P(x)$ and 'x is a human being' for $Q(x)$; then $Q(a)$ and $\forall\, x,\ P(x) \to Q(x)$ are true, but $P(a)$ is false.
(Of course there are innumerable alternatives. Share any particularly enterprising ones with the class.)

(c) Valid.

(d) Invalid.

(i) Writing $\mathbb{R}$ for the set of real numbers, $P(x)$ for $x^2 < 0$ the argument is
$\neg(\forall\, x \in \mathbb{R},\ P(x))$
$\therefore\ \forall\, x \in \mathbb{R},\ \neg P(x)$.

(ii) If we just change $P(x)$ to mean '$x^2 = 0$', then the argument becomes
not all real numbers have squares equal to 0
$\therefore$ all real numbers have squares not equal to 0,
which is rubbish.

48. (a) Valid.

(b) Valid.

(c) Valid.

49. (a) Valid.

(b) Valid.

(c) Invalid.

Writing A for the set of integers, $P(x,y)$ for '$x < y$', the argument form is

$$\forall m \in A,\ \exists n \in A,\ P(m,n),$$
$$\therefore\ \forall n \in A,\ \exists m \in A,\ P(m,n).$$

An argument of the same form is

for every man x, there is a man y such that y is (or was) the father of x

(which is true)

therefore, for every man y, there is a man x such that y is (or was) the father of x

(which is false). (Or just take $\mathbb{N}$ in place of $\mathbb{Z}$ in the argument as given.)

50. (a) Invalid.

Writing $\mathbb{R}$ for the set of real numbers, $P(x)$ for '$x \geq 0$' and $Q(x)$ for '$x \leq 0$', the argument form is

$$\exists x \in \mathbb{R},\ P(x),$$
$$\exists x \in \mathbb{R},\ Q(x),$$
$$\therefore\ \exists x \in \mathbb{R},\ P(x) \wedge Q(x).$$

If we just take '$x > 0$' for $P(x)$ and leave the rest unchanged, then the argument becomes

there are real numbers greater than 0, there are real numbers less than or equal to 0,

therefore there is a real number which is both greater than 0 and less than or equal to 0,

which is absurd.

(b) Valid.

51. (a) Valid.

(b) Invalid.

Writing $\mathbb{R}$ for the set of real numbers, $P(x)$ for '$x^3 < 0$' and $Q(x)$ for '$x^5 < 0$', the argument form is

$$\forall x \in \mathbb{R},\ P(x) \to Q(x),$$
$$\therefore\ \exists x \in \mathbb{R},\ Q(x).$$

If we take '$x^2 < 0$' for $P(x)$ and '$x^4 < 0$' for $Q(x)$, then the argument becomes

whenever x is a real number such that $x^2 < 0$ then $x^4 < 0$

(which is true)

therefore there is a real number x such that $x^4 < 0$

(which is false).

52. (No solution written out.)

53. (a) Harry is not a healthy person.

(b) Felix is not a person.

(c) Either $a + 1$ is negative or $a - 1$ is negative.

54. (a) Valid.

(b) Valid.

(c) Valid.

(d) Valid.

(e) Invalid.
Try $A = \mathbb{N}$, '$x \in \mathbb{N}$' for $P(x)$, '$x = -1$' for both $Q(x)$ and $R(x)$.

(f) Valid.

(g) Valid.

(h) Invalid.
Take $A = \mathbb{N}$, '$x \in \mathbb{N}$' for $P(x)$, 'x is even' for $Q(x)$ (so that $P(2) \to Q(2)$ is $\mathtt{T}$), '$x \notin \mathbb{N}$' for $R(x)$ (so that $Q(3) \to R(3)$ is $\mathtt{T}$).

55. (a) Invalid.
Take $A = \mathbb{R}$, '$x > 0$' for $P(x)$, '$x < 0$' for $Q(x)$.
(Of course there are infinitely many other solutions; ask your class teacher!)

(b) Valid.
[A possible validation uses give-it-a-name. If we replace the first premise by $a \in A$ and $P(a)$ (a being a name for 'some $x \in A$ such that $P(x)$') and the second by the logically equivalent $\forall\, x \in A,\ \neg(P(x) \wedge Q(x))$, then instantiation gives us $\neg(P(a) \wedge Q(a))$. Now $P(a)$ and $\neg(P(a) \wedge Q(a))$ give us $\neg Q(a)$, and $a \in A$, $\neg Q(a)$ give us $\exists\, x \in A,\ \neg Q(x)$ or $\neg(\forall\, x \in A,\ Q(x))$.]

(c) Valid.
[Again we can use give-it-a-name. If we replace the second premise by $a \in A$ and $P(a) \wedge R(a)$, and instantiate in the first premise to get $P(a) \to Q(a)$, Modus Ponens gives us $Q(a)$, so we have $Q(a) \wedge R(a)$ and $\exists\, x \in A,\ Q(x) \wedge R(x)$.]

56. (a) Invalid.
Try $A = B = \mathbb{N}$, '$x = y$' for both $P(x, y)$ and $Q(x, y)$.

(b) Invalid.
Try $A = \mathbb{N}$, 'x is even' for both $P(x)$ and $Q(x)$.

(c) Valid.
[For this one I prefer argument-by-contradiction. If we start by negating the conclusion, we have $\neg(\exists\, x \in A,\ P(x) \vee Q(x))$, which is logically equivalent to $\forall\, x \in A,\ (\neg P(x)) \wedge (\neg Q(x))$. Now we know that

$$(\neg p) \wedge (\neg q); \quad \therefore\ \neg p$$

is valid, so by universal natural deduction

$$\forall\, x \in A,\ (\neg P(x)) \wedge (\neg Q(x)); \quad \therefore\ \forall\, x \in A,\ \neg P(x)$$

is valid, and we can deduce $\forall\, x \in A,\ \neg P(x)$. Similarly, we can deduce $\forall\, x \in A,\ \neg Q(x)$, and putting these together we have

$$(\forall\, x \in A,\ \neg P(x)) \wedge (\forall\, x \in A,\ \neg Q(x)).$$

But this is the negation of the premise, so we have a contradiction.]

57. (a) Valid.
[Try argument-by-contradiction. If we deny the conclusion, we have $\neg(\forall x \in A, P(x))$ and $\neg(\exists x \in A, Q(x))$. Shifting the $\neg$s to the right, these become $\exists x \in A, \neg P(x)$ and $\forall x \in A, \neg Q(x)$. Giving-a-name to the first of these, we have $a \in A, \neg P(a)$; instantiating in the second, we can add $\neg Q(a)$. Putting these together, we get $\exists x \in A, (\neg P(x)) \wedge (\neg Q(x))$, which is the negation of the premise.]

(b) Invalid.
Take $A = \mathbb{N}$, 'x is even' for $P(x)$, 'x is odd' for $Q(x)$.
[The point here is that because $\forall x \in A, P(x)$ is F, $(\forall x \in A, P(x)) \to (\forall x \in A, Q(x))$ is T. A counter-example has to be of this form, since in any example in which the premise is true and also $\forall x \in A, P(x)$ is T, we must have both $P(x)$ and $Q(x)$ true for every $x \in A$.]

(c) Valid.
[If the conclusion is false, then we must have

$$\forall x \in A, P(x) \wedge (\neg Q(x)).$$

It follows that $\forall x \in A, P(x)$ and therefore, from the first premise, that $\forall x \in A, Q(x)$. So $Q(x)$ and $\neg Q(x)$ are both true for every $x \in A$, and A must be the empty set, contradicting the second premise.]

58. (a) Valid. .
[Use give-it-a-name in the first premise to get $a \in A, P(a) \to Q(a)$; instantiate in the next two premises to get $Q(a) \to R(a)$ and $P(a)$; conclude $R(a)$ and $\exists x \in A, R(x)$.]

(b) Valid.
[Try denying the conclusion. We have $\exists x \in A, (\neg P(x)) \wedge (\neg Q(x))$; giving-it-a-name, we have $a \in A$ and $\neg P(a)$ and $\neg Q(a)$; so we get both $\exists x \in A, \neg P(x)$ and $\exists x \in A, \neg Q(x)$; putting these together, we have

$$(\exists x \in A, \neg P(x)) \wedge (\exists x \in A, \neg Q(x))$$

which is the negation of the premise.]

(c) Invalid.
$A = \emptyset$, $P(x)$ for 'x is a unicorn', $Q(x)$ for 'x is a real number such that $x^2 < 0$'.
[Then $\forall x \in A, Q(x)$ is true so the premise is true; but the conclusion is false. Only $A = \emptyset$ will work here.]

59. Invalid.
Take $A = \mathbb{N}$, '$y = x + 1$' for $P(x, y)$.
Remark The formula '$x = y$' is not normally taken to be a primitive notion in the predicate calculus.

60. (a)(i) $\exists x \in A, P(x)$; $\exists x \in A, Q(x)$; $\therefore \exists x \in A, P(x) \vee Q(x)$.
(ii) Invalid.
(iii) (for instance) some integers are even; some integers are odd; therefore there is an even integer which is odd.

(b)(i) $\exists x \in A, P(x)$; $\neg(\exists x \in A, P(x) \wedge Q(x)$; $\therefore \neg(\forall x \in A, Q(x))$

61. Charles.

First method Look at the other possibilities. (a) If Alexander is guilty, then he's a knave (because he said Charles did it), so lied again when he said there was more than one knave, and is the only knave; but Charles said there were three knaves, so is a second knave, which is impossible. (b) If Barbara is guilty, she's a knave, Charles is a knight, and David is a knave; so Alexander is a knight and Charles did it, which is ridiculous. (c) If David did it, he and Alexander are both knaves, and Alexander told the truth (the second time), which is impossible. (d) If none of them broke the window, then Charles and Barbara are knights, but Charles said three of them were knaves, so that's impossible. (e) So we're left with a guilty Charles and two knaves (Charles and David).

Second method Instead of looking for a window-breaker, count knaves. (a) At least one of Alexander and David is a knave, because they contradict one another. (b) At least one of Barbara and Charles is innocent, therefore a knight. (c) If there is just one knave, then Charles is a liar; but that makes two knaves. (d) If there are three knaves, Alexander and Charles are both telling the truth; but that makes two knights. (e) So there are just two knaves, Charles is a liar, and Charles broke the window.

62. (a) Valid.

We know that there is some $x \in A$ such that $Q(x) \to R(x)$, by the third premise. Give it a name; suppose that $a \in A$ is such that $Q(a) \to R(a)$. By the first two premises, $P(a) \vee Q(a)$ and $P(a) \to R(a)$. So $R(a)$ follows by Division into Cases.

(b) Valid.

Use the same idea, but starting with '$P(a) \vee Q(a)$'.

(c) Invalid.

Take $A = \{0, 1\}$ with $P(0)$ true, $Q(0)$ false, $R(0)$ false, $P(1)$ false, $Q(1)$ false, $R(1)$ false.

(d) Invalid.

Take $A = \{0, 1\}$ with $P(0)$ true, $Q(1)$ true and everything else false.

(e) Valid.

Take $a \in A$ such that $Q(a) \to R(a)$; then $P(a) \to Q(a)$ so $P(a) \to R(a)$.

(f) Valid.

Argue by contradiction. If the premises are true and the conclusion is false, then

1.		$(\forall\, x \in A,\, P(x)) \to r$	premise
2.		$\forall\, x \in A,\, Q(x)$	premise
3.		$\neg((\forall\, x \in A,\, P(x) \wedge Q(x)) \to r)$	denial of conclusion
4.	$\therefore$	$\neg r$	from (3)
5.	$\therefore$	$\forall\, x \in A,\, P(x) \wedge Q(x)$	from (3)
6.	$\therefore$	$\neg(\forall\, x \in A,\, P(x))$	from (1) and (4)
7.	$\therefore$	$\exists\, x \in A,\, \neg P(x)$	logically equivalent to (6)
8.	$\therefore$	$a \in A$	
9.	$\therefore$	$\neg P(a)$	give-it-a-name from (7)
10.	$\therefore$	$P(a) \wedge Q(a)$	instantiation from (5) and (8)
11.	$\therefore$	$\perp$	from (9) and (10).

So it is impossible for the premises to be true and the conclusion false, and the argument is valid.

(g) Valid.

$\neg(\exists\, x \in A,\, P(x))$ is logically equivalent to $\forall\, x \in A,\, \neg P(x)$, and $(\forall\, x \in A,\, \neg P(x)) \to (\forall\, x \in A,\, P(x) \to Q(x))$ is a tautology.

63. (a) Not valid.
Try $A = \{0\}$, $P(x)$ representing '$x = 1$', q representing '$0 = 1$'.

(b) Not valid.
Try $A = \{0, 1\}$, $B = \{0\}$, $Q(x)$ representing '$x = 0$', $P(x, y)$ representing '$x = y$'.

(c) Valid.
If the conclusion is false, then we have $\forall x \in A$, $(\forall y \in B,\ P(x, y))$ and also $\forall x \in A$, $(\forall y \in B,\ P(x, y)) \to Q(x)$, so can conclude $\forall x \in A$, $Q(x)$ by Universal Modus Ponens.

(d) Invalid.
Try $A = B = \{0, 1\}$, $Q(x)$ for '$x = 2$', $P(x, y)$ for '$x = y$'.

(e) Valid.
Starting from $x \in A$, take $y \in A$ such that $P(x, y)$ and $z \in B$ such that $Q(y, z)$; then $R(x, z)$ is true.

(f) Invalid.
Try $A = \{0\}$, $B = \{0, 1\}$, $P(x, y)$ and $Q(x, y)$ and $R(x, y)$ all representing '$x = y$'.

(g) Valid.
Starting from $x \in A$, we have y, $z \in B$ such that $P(x, y)$ and $Q(x, z)$; now there is a $w \in B$ such that $R(y, w)$ and $R(z, w)$, in which case $P(x, w)$ and $Q(x, w)$ are both true.

64. (a) $\neg p$, 'not-p', is the negation of the proposition p; it is true if p is false and false if p is true. Example: 'It's not raining', where p is the statement 'It's raining'.

$p \wedge q$ stands for 'p **and** q', for any two statements p and q; it is true if p and q are both true, false otherwise. Example: 'It's both cold and raining', where q is the statement 'It's cold'.

$p \vee q$ stands for 'p **or** q'; it is true if either p or q is true, false if both are false. Example: 'It's either cold or raining or both'.

$p \to q$ stands for 'p implies q' or 'if p, then q'; it is true if either p is false or q is true, and false only when p is true and q is false. Example: 'If it's raining, it's cold'.

(b) The contrapositive of $p \to q$ is $(\neg q) \to (\neg p)$. The contrapositive of 'If it's raining, it's cold' is 'If it isn't cold, it isn't raining'.

65. (a) Writing p for 'John is healthy', q for 'John is wealthy' and r for 'John is wise', the sentences are
(i) $q \to (p \vee r)$,
(ii) $r \wedge (\neg p) \wedge (\neg q)$,
(iii) $(p \wedge (\neg(q \vee r))) \vee (q \wedge (\neg(p \vee r))) \vee (r \wedge (\neg(p \vee q)))$.
[There are many alternatives.]

(b)(i) 'John is wealthy, but he is neither healthy nor wise'.
(ii) 'If John is wise, he is either healthy or wealthy'
or 'John is either healthy or wealthy or foolish'.
(iii) 'Either John is neither healthy nor wealthy nor wise, or he has at least two of these qualities'.

(c) $p \vee q$ has the same truth table as $\neg((\neg p) \wedge (\neg q))$.
For any truth table with variables $p_1, \ldots, p_n$, we have 2^n rows each either `T` or `F`. Label rows with `T`'s as $\ell_1, \ldots, \ell_k$, and set $q_{ij} = p_i$ if p_i is `T` on row ℓ_j, $\neg p_i$ if p_i is `F` on row ℓ_j. Then the expression $\phi_j = q_{1j} \wedge q_{2j} \wedge \ldots \wedge q_{nj}$ is `T` on row ℓ_j, `F` on all other rows. So $\psi = \phi_1 \vee \phi_2 \vee \ldots \vee \phi_k$ is `T` on exactly the rows we want. Of course the expression for ψ involves $\vee$ as well as $\wedge$ and $\neg$. But we get the same result from

$$\neg((\neg\phi_1) \wedge (\neg\phi_2) \wedge \ldots \wedge (\neg\phi_k)),$$

which uses only $\wedge$ and $\neg$.

66. (a) A *tautology* in the propositional calculus is an expression which takes the truth value T whatever truth values are assigned to its atomic propositions.

If we look at the truth tables for the given formulae

p	q	r	$p \vee q$	$(p \vee q) \to r$	$p \wedge ((p \vee q) \to r)$	$(p \wedge ((p \vee q) \to r)) \to r$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	T	T	T	T
T	F	F	T	F	F	T
F	T	T	T	T	F	T
F	T	F	T	F	F	T
F	F	T	F	T	F	T
F	F	F	F	T	F	T

p	q	r	$p \wedge q$	$(p \wedge q) \to r$	$q \wedge ((p \wedge q) \to r)$	$(q \wedge ((p \wedge q) \to r)) \to r$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	F	T	F	T
T	F	F	F	T	F	T
F	T	T	F	T	T	T
F	T	F	F	T	T	F<
F	F	T	F	T	F	T
F	F	F	F	T	F	T

we see that the expression in (i) is a tautology and the expression in (ii) is not.

[Note that the tautology (i) corresponds to the valid argument form

1.		p	premise
2.		$(p \vee q) \to r$	premise
3.	$\therefore$	$p \vee q$	disjunctive addition, from (1)
4.	$\therefore$	r	Modus Ponens from (2) and (3).]

67. (a) Two expressions ϕ and ψ of the propositional calculus are *logically equivalent* if they have the same truth table.

$(r \vee (\neg p)) \wedge (r \vee (\neg q))$ is logically equivalent to $r \vee ((\neg p) \wedge (\neg q))$ and to $r \vee (\neg(p \vee q))$.

(b) An expression in the propositional calculus is a *tautology* if its truth table has only T's in it, and no F's.

Consider the possible truth values of ϕ and ψ:

ϕ	ψ	$\phi \to \psi$	$\psi \to \phi$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Since we are told that $\phi \to \psi$ and $\psi \to \phi$ are logically equivalent, there can be no row in the full truth table in which one is T and the other is F. But that means that there can be no row in which ϕ is T and ψ is F, because on any such row $\phi \to \psi$ would be F and $\psi \to \phi$ would be T. Similarly there can be no row on which ψ is T and ϕ is F. So ϕ and ψ have the same truth value on every row, and must be logically equivalent.

68. Draw up a full truth table with a column for each atomic proposition $p, q, \ldots$ involved in the argument and (at least) a column for each premise and a column for the conclusion; if there are n atomic propositions the table will have 2^n rows. Look at those rows in which every premise is T. If the conclusion is T on all those rows, the argument is valid; otherwise it is invalid.

(i)

p	q	r	$p \vee q$	$p \to (\neg q)$	$p \to r$	
T	T	T	T	F	T	
T	T	F	T	F	F	
T	F	T	T	T	T	
T	F	F	T	T	F	
F	T	T	T	T	T	$<$
F	T	F*	T	T	T	$<$
F	F	T	F	T	T	
F	F	F	F	T	T	

so the argument is invalid.

(ii)

p	q	r	$\neg(p \wedge q)$	$(\neg r) \to p$	$(\neg q) \to r$	
T	T	T	F	T	T	
T	T	F	F	T	T	
T	F	T	T	T	T	$<$
T	F	F	T	T	F	
F	T	T	T	T	T	$<$
F	T	F	T	F	T	
F	F	T	T	T	T	$<$
F	F	F	T	F	F	

so the argument is valid. (If we reword it as

$$\begin{gathered}(\neg p) \vee (\neg q) \\ (\neg p) \to r \\ (\neg q) \to r \\ \therefore\ r\end{gathered}$$

we see that it's just a version of 'division into cases'.)

69. (a)(i) $\neg((\neg p) \wedge (\neg q))$
(ii) $\neg(p \wedge (\neg q))$
(iii) $(\neg(p \wedge (\neg q))) \wedge (\neg(q \wedge (\neg p)))$.

(b)

p	q	r	$p \vee r$	$\neg(q \vee r)$	$(p \vee r) \wedge (\neg(q \vee r))$
T	T	T	T	F	F
T	T	F	T	F	F
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	F	F	F
F	F	T	T	F	F
F	F	F	F	T	F

is logically equivalent to $p \wedge (\neg q) \wedge (\neg r)$ or $p \wedge (\neg(q \vee r))$.

(c) For any two expressions ϕ and ψ, $\phi \leftrightarrow \psi$ is `T` iff either ϕ and ψ are both `T` or are both `F`. So $\phi \leftrightarrow \psi$ is always `T` iff ϕ and ψ always have the same truth value; that is, $\phi \leftrightarrow \psi$ is a tautology iff ϕ and ψ are logically equivalent.

70. (No solution written out.)

71. In both cases, we look at what it would have to mean if the conclusion was `F` while every premise was `T`.

(a) If $q \to p$ is `F`, then q is `T` and p is `F`. Since $q \to (p \vee (\neg r))$ is `T`, $p \vee (\neg r)$ is `T`; but p is `F`, so $\neg r$ is `T` and r is `F`. Since $\neg(q \wedge s)$ is `T`, $q \wedge s$ is `F` and (because q is `T`) s is `F`. This means that $r \vee s$ is `F`, $\neg(r \vee s)$ is `T` and the premise $(\neg(r \vee s)) \to p$ is `F`, which is impossible.
So the argument is **valid**.

(b) Again, we are interested only in rows of the truth table in which q is `T` and p is `F`. Since $p \vee r$ is `T`, r is `T`; since $s \to p$ is `T`, s is `F`. Thus we have the assignment

p is `F`, q is `T`, r is `T`, s is `F`.

On this assignment, every premise is `T` and the conclusion is `F`, so the argument is **invalid**.

72. (a) Modus Ponens is the argument form

$$\begin{array}{c} p \to q \\ p \\ \therefore\ q. \end{array}$$

(b) Draw up a full truth table with a column for each atomic proposition p, q, … involved in the argument and (at least) a column for each premise and a column for the conclusion; if there are n atomic propositions the table will have 2^n rows. Look at those rows in which every premise is T. If the conclusion is T on all those rows, the argument is valid; otherwise it is invalid.

Applying this to the valid argument Modus Ponens, we have the table

p	q	$p \to q$	
T	T	T	$<$
T	F	F	
F	T	T	
F	F	T	

so that the premises $p \to q$ and p are both T only on the top row, where the conclusion q is also T.

Applying it to the invalid argument

$$\begin{array}{c} p \to q \\ q \\ \therefore\ p \end{array}$$

we get the table

p	q	$p \to q$	
T	T	T	$<$
T	F	F	
F	T	T	$<$
F	F	T	

in which the premises are T in both the first and third rows, while the conclusion p is F in the third row.

(c) If we look at the truth table for $(\phi_1 \wedge \ldots \wedge \phi_n) \to \psi$, we see that it is T exactly on those rows on which *either* ψ is T *or* $\phi_1 \wedge \ldots \wedge \phi_n$ is F, that is, at least one of ϕ_1, $\phi_2, \ldots$ is F. So $(\phi_1 \wedge \ldots \wedge \phi_n) \to \psi$ is a tautology if, and only if, on every row either ψ is T or at least one of the ϕ_i is F. But this says just that on any row on which every ϕ_i is T, then ψ is also T; which is our criterion for deciding whether the argument form is valid.

73. (7) is logically equivalent to (1).

(8) is a consequence of (4) and (6), by the argument form '$p \to q, \quad \neg q, \quad \therefore\ \neg p$' (Modus Tollens).

(9) is a consequence of (7) and (8), by the argument form '$p \vee q, \quad \neg q, \quad \therefore\ p$' (Disjunctive Syllogism) applied to $(\neg p) \vee r$ and s.

(10) is logically equivalent to (9).

(11) is a consequence of (3) and (5), by the argument form '$p \to q, \quad q \to r, \quad \therefore\ p \to r$' (Hypothetical Syllogism).

(12) is a consequence of (2), (10) and (11) (Division into Cases).

74. **case 1** Suppose C is a knight. Then D is also a knight, and there are just two knights, so A and B are knaves. We still don't know just what A said, but it was something other than 'two'.

case 2 Suppose, if possible, that C is a knave. Then D is again a knight (being of a different type from C), and again there are just two knights. But (since C contradicted him) B was telling the truth, and is a knight. But in this case A *did* say 'There are just two knights', and was telling the truth, so is a third knight, which is impossible.

Thus C cannot be a knave, and we are left in case 1: A and B are knaves and C and D are knights.

75. In each case we look for a row of the truth table in which the conclusion is F but the premises are all T.

(a) If $r \to s$ is F, r must be T and s must be F. So t is also F (because $t \to s$ is T) and q is T (because $q \vee t$ is T). At the same time, $p \vee s$ is T so p is T. Thus we should look at the line p T, q T, r T, s F, t F. Every premise is T on this line but the conclusion is F. Accordingly the argument is **invalid**.

(b) If r is F, then $p \wedge q$ is F (by the first premise), so either p is F or q is F. **case 1** If q is F, then $s \to t$ must be F (because $(s \to t) \to q$ is T), so s is T and t is F. Since $p \to t$ is T, p is F. But now $p \vee q \vee s \vee t$ is F, which is impossible. **case 2** If p is F then s is F (because $s \to p$ is T), so $s \to t$ is T and q must be T. But in this case every premise is T whatever t is.
Accordingly the argument is **invalid**.

76. (i)(α) $\forall\, x \in B,\ Y(x)$ where B is the set of buttercups and $Y(x)$ represents 'x is yellow'.
(β) There is a buttercup which isn't yellow.

(ii)(α) $\exists\, x \in F,\ Y(x) \wedge B(x)$ where F is the set of flowers and $B(x)$ represents 'x is a buttercup'.
(β) No yellow flowers are buttercups.

(iii)(α) $\forall\, x \in T,\ O(S,x)$ where T is the team, S represents Smith and $O(y,x)$ represents 'y can outrun x'.
(β) There is a man on the team whom Smith can't outrun.

(iv)(α) $\forall\, x \in T,\ \neg O(S,x)$.
(β) There is a man on the team whom Smith can outrun.

(v)(α) $\forall\, x \in P,\ \neg S(x) \to (\exists\, y \in P,\ E(y,x) \wedge (\neg S(y))$ where P is the set of problems, $S(x)$ represents 'you can solve x' and $E(y,x)$ represents 'y is easier than x'.
Alternatively: $\forall\, x \in P^*,\ \exists\, y \in P^*,\ E(y,x)$ where P^* is the set of problems you can't solve.
(β) There is a problem you can't solve for which there is no easier problem which you can't solve.

(vi)(α) $\forall\, x \in P,\ S(x) \to (\exists\, y \in P,\ L(x,y))$ where $L(x,y)$ stands for 'x leads to y'.
(β) There is a problem you can solve which doesn't lead to any other problem.

(vii)(α) $\exists\, x \in E,\ G(x,2) \wedge (\forall\, p \in P,\ \forall\, q \in P,\ x \neq p+q$ where E is the set of even numbers, $G(x,2)$ represents '$x > 2$' and P is the set of prime numbers.
(The replacement of '$x > 2$' by '$G(x,2)$' is not really essential; and it would be equally logical to replace '$x \neq p+q$' by a formula '$N(x,p,q)$'.)
(β) Every even number greater than 2 is the sum of two primes.[1]

77. (i) For (β) to be false, there must be an $a \in A$ such that $\exists\, y \in B,\ P(a,y)$ is T but $Q(a)$ is F. Since (α) is supposed to be true, $\forall\, y \in B,\ P(a,y)$ has to be F. So we had better have at least two things in B; say $B = \{b,c\}$, $A = \{a\}$, $P(a,b)$ is true, $P(a,c)$ is false and $Q(a)$ is false. For an absolutely concrete example: $A = \{0\}$, $B = \{0,1\}$, $P(x,y)$ if $x = y$, and $Q(x)$ if $x = 1$.

(ii) For (α) to be false, we again need an $a \in A$ such that $Q(a)$ is false; this time we need $\forall\, y \in B,\ P(a,y)$ to be true while $\exists\, y \in B,\ P(a,y)$ is false. This is impossible if B is not empty, but if $B = \emptyset$ it's easy. So take $A = \{0\}$, $B = \emptyset$, $Q(x)$ if $x = 1$, and P anything you like.

78. (i)(α)

$$\forall x \in A,\, \exists y \in A,\, G(x,y)$$
$$\therefore\ \forall y \in A, \exists x \in A,\, G(x,y)\ ,$$

where A represents the set of real numbers and $G(x,y)$ represents '$x > y$'.

(β) The argument is invalid; for instance, if we take $A = \mathbb{N}$ and take $G(x,y)$ to mean '$x < y$', then $\forall x \in A,\, \exists y \in A,\, G(x,y)$ is true but $\forall y \in A, \exists x \in A,\, G(x,y)$ is false.

(ii)(α)

$$Q(a)$$
$$\neg Q(b)$$
$$P(b,c,a)$$
$$\forall x,\, \forall y,\, \forall z,\, (Q(y) \wedge Q(z) \wedge P(x,y,z)) \to Q(x)$$
$$\therefore\ \neg Q(c),$$

where $Q(x)$ represents 'x is a rational number' and $P(x,y,z)$ represents 'x is the product of y and z', while a represents $\frac{1}{3}$, b represents $\sqrt{2}$ and c represents $\sqrt{18}$.

(β) The argument is valid. For by the principle of instantiation, we can argue

1. $Q(a)$	premise
2. $\neg Q(b)$	premise
3. $P(b,c,a)$	premise
4. $\forall x, \forall y, \forall z, (Q(y) \wedge Q(z) \wedge P(x,y,z)) \to Q(x)$	premise
5.$\therefore (Q(c) \wedge Q(a) \wedge P(b,c,a)) \to Q(b)$	instantiating in (4)
6.$\therefore \neg(Q(c) \wedge Q(a) \wedge P(b,c,a))$	Modus Tollens from (5) and (2)
7.$\therefore (\neg Q(c)) \vee (\neg Q(a)) \vee (\neg P(b,c,a))$	logically equivalent to (6)
8.$\therefore (\neg Q(c)) \vee (\neg Q(a))$	disjunctive syllogism from (7) and (3)
9.$\therefore \neg Q(c)$	disjunctive syllogism from (8) and (1).

79. *Universal Natural Deduction*: Given any valid argument form

$$\phi_1$$
$$\dots$$
$$\phi_n$$
$$\therefore\ \psi,$$

it remains a valid argument form if we put a quantifier $\forall x$ at the beginning of each line, thus:

$$\forall x,\, \phi_1$$
$$\forall x,\, \dots$$
$$\forall x,\, \phi_n$$
$$\therefore\ \forall x,\, \psi.$$

Thus, to prove that

$$A \subseteq B$$
$$B \subseteq C$$
$$\therefore\ A \subseteq C$$

is a valid argument, we can start with the argument

$$(x \in A) \to (x \in B)$$
$$(x \in B) \to (x \in C)$$
$$\therefore\ (x \in A) \to (x \in C),$$

which is a case of the valid form Hypothetical Syllogism, and convert it to

$$\forall x,\, (x \in A) \to (x \in B)$$
$$\forall x,\, (x \in B) \to (x \in C)$$
$$\therefore\ \forall x,\, (x \in A) \to (x \in C),$$

which translates immediately into the required form because '$A \subseteq B$' is just an abbreviation for '$\forall x, (x \in A) \to (x \in B)$'.

80. (a) *Instantiation* is the principle that the argument forms

$$\begin{gathered}\forall x,\, P(x)\\ \therefore\ P(a)\end{gathered}$$

and

$$\begin{gathered}\forall x \in A,\, P(x)\\ a \in A\\ \therefore\ P(a)\end{gathered}$$

are valid, for any predicate $P(x)$ and any a for which the formula $P(a)$ is grammatically correct.

The given syllogism can be expressed in the form

$$\begin{gathered}\forall x,\, P(x) \to Q(x),\\ P(S),\\ \therefore\ Q(S),\end{gathered}$$

where $P(x)$ represents 'x is a man', $Q(x)$ represents 'x is mortal' and S represents Socrates. Now we can argue

1.		$\forall x,\, P(x) \to Q(x)$	premise
2.		$P(S)$	premise
3.	$\therefore$	$P(S) \to Q(S)$	instantiation in (1)
4.	$\therefore$	$Q(S)$	Modus Ponens from (3) and (2).

(b)(i)(α)

$$\begin{gathered}\forall x \in A,\, \forall y \in A,\, \forall z \in A,\, (P(x,y) \wedge P(y,z)) \to P(x,z),\\ \forall x \in A,\, Q(x) \to (\exists\, y \in A,\, P(x,y)),\\ \forall z \in A,\, Q(z) \to (\exists\, y \in A,\, P(y,z)),\\ \therefore\ \forall x \in A,\, \forall z \in A,\, (Q(x) \wedge Q(z)) \to P(x,z),\end{gathered}$$

where A represents the set of real numbers, $P(x,y)$ represents '$xy \neq 0$' and $Q(x)$ represents '$x \neq 0$'.

(β) The argument is **invalid**. To see this, repeat the argument with the same set A and the same interpretation of $Q(x)$, but read $P(x,y)$ as '$xy > 0$'. Then the premises are all valid (for if x, y and z are real numbers and xy, yz are both greater than 0, then $xy^2z > 0$ so $xz > 0$, because $y^2 > 0$); but the conclusion is false.

(ii)(α)

$$\begin{gathered}\forall x \in A,\, \exists\, y \in A,\, \forall z \in A,\, P(z,x) \to Q(z,y),\\ \forall x \in A,\, \exists\, y \in A,\, \neg(\exists\, z \in A,\, Q(z,x) \wedge Q(z,y)),\\ \therefore\ \forall x \in A,\, \exists\, y \in A,\, \forall z \in A,\, P(z,x) \to (\neg Q(z,y)),\end{gathered}$$

where A represents the set P_2, $P(z,x)$ represents '$z \leq x$' and $Q(z,x)$ represents 'z is a factor of x'.

(β) The argument is **valid**. To see this, it is helpful to re-word the second premise, first by changing the names of the variables x and y to y and w, and then by converting the $\neg\exists$ to $\forall\neg$, so that it becomes

$$\forall y \in A,\, \exists\, w \in A,\, \forall z \in A,\, Q(z,y) \to (\neg Q(z,w)).$$

Now, starting from any $x \in A$, the first premise tells us that there is a $y \in A$ such that $P(z,x) \to Q(z,y)$ for every $z \in A$, and the second premise tells us that there is now a $w \in A$ such that $Q(z,y) \to (\neg Q(z,w))$ for every $z \in A$. Putting these together, a universal version of Hypothetical Syllogism tells us that (for these particular x, y and w) $P(z,x) \to (\neg Q(z,w))$ for every z. But since this works for every x, the conclusion is true.

81. The argument is not valid. To see this, try $A = \mathbb{N}$ and write 'either $y = x - 1$ or $x = y = 0$' for $P(x,y)$. (The second and third premises declare that there is a function $f : A \to A$ such that $P(x,y)$ iff $y = f(x)$, while the third premise says that f is surjective and the conclusion says that f is injective. So the whole argument says that 'any surjective function $f : A \to A$ is injective', which is certainly false if A is infinite.)

82. Since Yorick and Zeno contradict each other, one of them is a knight and the other is a knave. **case 1** If Zeno is a knight, then Xerxes is not a Diamond, so at least one of his statements 'All Hearts are Diamonds' and 'I'm a Heart' is false, and Xerxes also is a knave, therefore neither a Heart nor a Diamond; but honest Zeno has told us that all knaves are Hearts, so this can't be. **case 2** Accordingly Zeno is a knave and Yorick is a knight and Xerxes is a Diamond. But Yorick also tells us that no knight is a Diamond, so Xerxes is a knave and is not a Heart.

Thus Xerxes is a Diamond but not a Heart.

83. (a) The First Principle is the argument form

$$\begin{gathered} P(0) \\ \forall n \in \mathbb{N},\ P(n) \to P(n+1) \\ \therefore\ \forall n \in \mathbb{N},\ P(n). \end{gathered}$$

The Second Principle is

$$\begin{gathered} P(0) \\ \forall n \in \mathbb{N},\ \big(\forall i \in \mathbb{N},\ ((0 \le i) \wedge (i < n)) \to P(i)\big) \to P(n) \\ \therefore\ \forall n \in \mathbb{N},\ P(n). \end{gathered}$$

[Alternatively:

$$\begin{gathered} \forall n \in \mathbb{N},\ \big(\forall i \in \mathbb{N},\ (i < n) \to P(i)\big) \to P(n) \\ \therefore\ \forall n \in \mathbb{N},\ P(n).] \end{gathered}$$

(b) The **well-ordering principle** for $\mathbb{N}$ says

$$\forall A,\ ((A \subseteq \mathbb{N}) \wedge (A \neq \emptyset)) \to (\exists n \in A, \forall m \in A,\ n \le m).$$

To prove this from the Second Principle, suppose that $A \subseteq \mathbb{N}$ is any non-empty set. Write $P(n)$ for '$n \in \mathbb{N} \setminus A$'. Because A is non-empty, we know that '$P(n)$ for every $n \in \mathbb{N}$' is false. So one of the hypotheses of the Second Principle is false. (i) If the hypothesis '$P(0)$' is false, then $0 \in A$, and of course $0 \le m$ for every $m \in A$. (ii) If the hypothesis 'whenever $P(i)$ for $0 \le i < n$, then $P(n)$' is false, then there must be some $n \in \mathbb{N}$ such that $P(i)$ for every $i < n$ but $\neg P(n)$; that is, $n \in A$ but $i \notin A$ whenever $i < n$. But this means that if $m \in A$ then $m \not< n$, that is, $n \le m$. Thus in either case (i) or case (ii) we have an $n \in A$ such that $n \le m$ for every $m \in A$.

84. (No solution written out.)