Finite Mathematics Problems

1. If $\mathcal{U} = \{1, 2, 3, \dots, 10\}$, $A = \{1, 4, 7, 10\}$, $B = \{1, 2, 3, 4, 5\}$, $C = \{2, 4, 6, 8\}$, find the sets
(i) $A \cup B$ (ii) $A \cap C$ (iii) $A \setminus B$ (iv) $\mathcal{U} \setminus \mathcal{U}$ (v) $(A \cup B) \setminus (C \setminus B)$.
(In each case, list the members of the set.) [AA]

2. Consider the following nine expressions:
(i) $A \cap B = A$, (ii) $A \cup B = A$, (iii) $A \setminus B \subseteq B$, (iv) $\mathcal{U} \setminus (A \cap B) = \mathcal{U} \setminus B$, (v) $A \setminus B = B \setminus A$,
(a) $B \subseteq A$, (b) $A \subseteq B$, (c) $A = B$, (d) $A \cap B = \emptyset$.
Assuming that A and B are both subsets of $\mathcal{U}$, each of the expressions (i)-(v) is equivalent to one of the expressions (a)-(d); which? [AB]

3. Declare each of the following true or false, and be prepared to explain your reasoning. (Here x might be anything, and A, B and C might be any sets.)
(i) $\{x\} \subseteq \{x\}$
(ii) $\{x\} \in \{x, \{x\}\}$
(iii) $\{x\} \in \{x\}$
(iv) $\{x\} \subseteq \{x, \{x\}\}$
(v) $A \times B = B \times A$
(vi) $\mathcal{U} \setminus (A \setminus B) = \mathcal{U} \setminus (B \setminus A)$
(vii) $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$
(viii) $\mathcal{P}\emptyset = \{\emptyset\}$
(ix) $A \setminus (B \cup C) = (A \setminus B) \cup C$.
(In (vi), assume that A and B are both subsets of $\mathcal{U}$.) [AC]

4. Draw a Venn diagram to illustrate $A \triangle (B \triangle C)$. Is this the same as $(A \triangle B) \triangle C$? [AD]

5. Use the Laws of Algebra of Sets to prove that

$$(\mathcal{U} \cap A) \cup (B \cap A) = A,$$

annotating each step in your argument with the appropriate number of the Law being used from the list given in lectures.
State the dual identity. [AE]

6. Use the Laws of Algebra of Sets to prove that

$$(A^c \cap B)^c \cap (A \cup B) = A,$$

annotating each step in your argument with the appropriate number of the Law being used from the list given in lectures.
State the dual identity. [AF]

7. Let $A = \{1, a, x\}$, $B = \{1, b\}$. List the members of the sets
(i) $A \times B$ (ii) $B \times A$ (iii) $(A \times B) \setminus (B \times A)$.
[AG]

8. Which of the following are true for all sets A, B, C?
(i) $(A \cap B) \times C = (A \times C) \cap (B \times C)$,
(ii) $C \times (A \cup B) = (C \times A) \cup (C \times B)$,
(iii) $(A \setminus B) \times C = (A \times C) \setminus (B \times C)$,
(iv) $C \times (A \triangle B) = (C \times A) \triangle (C \times B)$,
(v) $(A \triangle B) \times (B \triangle A) = (A \times B) \triangle (B \times A)$.
[AH]

9. (a) If $\#(A) = 12$, $\#(B) = 11$, $\#(C) = 7$, $\#(A \cap B) = 5$, $\#(B \cap C) = 3$, $\#(A \cap C) = 5$, $\#(A \cap B \cap C) = 2$ what is $\#(A \cup B \cup C)$?
(b) If $\#(A) = m$, $\#(B) = n$ what is $\#(\mathcal{P}(A \times B))$? [BA]

10. How many ways are there of selecting a set of 2 or more people from a group of 12 people? (*Hint*: how many ways are there of selecting a set of 1 or fewer people?) [BB]

11. (i) For arbitrary sets A and B, is $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$? Give reasons for your answer.
(ii) For arbitrary sets A and B, is $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$? Give reasons for your answer.
(iii) Does the Principle of Duality apply to these questions? If not, why not? [BC]

12. Let R be a relation on a finite set A.
(a) What does it mean, in terms of the digraph of R, to say that R is (i) reflexive (ii) symmetric (iii) transitive?
(b) What does it mean, in terms of the incidence matrix $\mathbf{M}$ of R, to say that R is (i) reflexive (ii) symmetric (iii) transitive? [BD]

13. Let A be the set $\{1, 2, 3, 4\}$ and R the relation $\{(1,3), (1,4), (3,2), (3,3), (3,4)\}$.
(a)(i) Draw the arrow diagram for R.
(ii) Draw the digraph of R.
(iii) Write down the incidence matrix of R.
(b)(i) Write down the inverse R^{-1} of R.
(ii) Draw the arrow diagram for R^{-1}.
(iii) Draw the digraph of R^{-1}.
(iv) Write down the incidence matrix of R^{-1}. [BE]

14. Set $A = \{2, 3, 4, \dots, 12\}$ and let R be the relation on A defined by saying that $a\,R\,b$ iff $a|(b+1)$. Find $\#(R)$. [BF]

15. Let R and S be the following relations on $B = \{a, b, c, d\}$:

$$R = \{(a,a), (a,c), (c,b), (c,d), (d,b)\}, \quad S = \{(b,a), (c,c), (c,d), (d,a)\}.$$

Find (i) $R \circ S$ (ii) $S \circ R$ (iii) $R \circ R$ (iv) $S \circ S$. [CA]

16. Set $R = \{(x,y) : x, y \in \mathbb{P}, x + 3y = 12\}$, where $\mathbb{P} = \{1, 2, \dots\} = \mathbb{N} \setminus \{0\}$ is the set of strictly positive integers.
(i) List R as a set of ordered pairs. What are the domain and range of R?
(ii) Find $R \circ R$, R^{-1}, $R \circ R^{-1}$, $R^{-1} \circ R$. [CB]

17. For each of the following relations R say whether it is, or is not, reflexive/symmetric/transitive, when regarded as a relation on the given set A:
(a) $A = \mathbb{Z}$, $a\,R\,b$ iff $ab = 1$;
(b) $A = \mathbb{P}$, $a\,R\,b$ iff the highest common factor of a and b is 1.
Note: you are being asked a total of six questions here; make sure your answer sheet shows clearly your answers ('yes' or 'no') to each. [CC]

18. Let M be the set of 2×2 matrices with entries 0 or 1. Define a relation R on M by the rule that $(\mathbf{A}, \mathbf{B}) \in R$ iff $\mathbf{A}$ can be transformed into $\mathbf{B}$ by exchanging rows or exchanging columns any number of times. Show that R is an equivalence relation on M. [CD]

19. Let M be the set of 2×2 matrices with entries 0 or 1. Define a relation R on M by the rule that $(\mathbf{A}, \mathbf{B}) \in R$ iff $\mathbf{A}$ can be transformed into $\mathbf{B}$ by exchanging rows or exchanging columns any number of times. Assuming that R is an equivalence relation on M, describe the corresponding partition. (You should give a complete list of the sets belonging to the partition, listing the matrices belonging to each set.) [CE]

20. Set $A = \{1, 2, \dots, 9\}$ and let R be the relation on $A \times A$ defined by the rule

$$(a,b)\,R\,(c,d) \text{ iff } a + d = b + c.$$

Show that R is an equivalence relation. [CF]

21. Set $A = \{1, 2, \dots, 9\}$ and let R be the relation on $A \times A$ defined by the rule

$$(a,b)\,R\,(c,d) \text{ iff } a + d = b + c.$$

Assuming that R is an equivalence relation, list the members of the equivalence class of $(2,5)$. [CG]

22. For each of the following relations R say whether it is, or is not, reflexive/symmetric/transitive, when regarded as a relation on the given set A:

(i) A is the set of students at the University of Essex, and $a\,R\,b$ if there is someone they both know;
(ii) $A = \mathbb{Z}$ is the set of integers, and $a\,R\,b$ if $a + b$ is odd;
(iii) $A = \mathbb{Z}$ and $a\,R\,b$ if $a \times b$ is odd;
(iv) $A = \mathbb{Z}$ and $a\,R\,b$ if $a + b$ is even. [CH]

23. For each of the following relations say whether it is, or is not, reflexive/transitive/a partial order/a total order of the given set A:
(i) $A = \mathbb{R} \times \mathbb{R}$, $(a, b)\,R\,(c, d)$ if $a \leq c$ and $d \leq b$;
(ii) $A = \mathbb{Z}$, $m\,R\,n$ if $|m| \leq |n|$;
(iii) $A = \mathcal{P}\mathbb{N}$, $X\,R\,Y$ if *either* $X = Y$ *or* $X \neq Y$ and the smallest element of $X \triangle Y$ belongs to X. (This seems to be hard!) [CI]

24. Write $\mathbb{R}^2$ for the set $\mathbb{R} \times \mathbb{R}$ of ordered pairs (x, y) where x, y are real numbers. Define a relation R on $\mathbb{R}^2$ by writing $(a, b)\,R\,(c, d)$ if $a^2 + b^2 = c^2 + d^2$. Show that R is an equivalence relation on $\mathbb{R}^2$ and interpret the equivalence classes as subsets of the plane. (Recall that a point in the plane can be described in terms of two real numbers, its horizontal and vertical coordinates.) [DA]

25. Prove that a relation R on a non-empty set A is simultaneously symmetric and anti-symmetric if and only if it is included in the equality relation on A. [DB]

26. The set $A = \{L, +, !, @\}$ has a total order (not necessarily the order written here) with respect to which the following 'words' have been ordered lexicographically.

$$@@!L, \quad @@++, \quad @@+L, \quad +L+!!, \quad +L+!@.$$

Find the total order on A. [DC]

27. If Γ is any digraph on a (finite) set A, the *outdegree* of a vertex in A is the number of arcs of Γ leaving that vertex; similarly, the *indegree* of a vertex is the number of arcs arriving at that vertex. (A loop, of course, arrives at the same vertex it leaves.) (i) Explain why a relation on A is a function defined on A iff every vertex of its digraph has outdegree 1. (ii) Characterize injectivity and surjectivity of a function in terms of the indegrees of the vertices. [DD]

28. In each of the following examples, determine whether the relation R is a function from the set A to the set B. If so, give its set of values, and state whether it is injective, surjective, neither or both.

(i) $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$, $R = \{(a, 1), (b, 1), (c, 1), (d, 1)\}$.
(ii) $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$, $R = \{(a, 1), (b, 1), (c, 2)\}$.
(iii) $A = B = \mathbb{R}$, $R = \{(x, y) : x \in \mathbb{R},\ y \in \mathbb{R},\ x^2 + y^2 = 4\}$.
(iv) $A = B = \mathbb{Z}$, $R = \{(m, n) : m \in \mathbb{Z}, n \in \mathbb{Z}, n = 2m + 1\}$. [DE]

29. Let f, $g : \mathbb{Z} \to \mathbb{Z}$ be functions defined by the rules

$$f(n) = 5n + 2, \quad g(n) = 7n + k \text{ for every } n \in \mathbb{Z},$$

where $k \in \mathbb{Z}$. Find $(f \circ g)(n)$, $(g \circ f)(n)$. If $f \circ g = g \circ f$, what is k? [EA]

30. Which of the following functions are injective? For those which are, describe their ranges and the inverse functions.

(a) $f : \mathbb{Z} \to \mathbb{Z}$ defined by saying that $f(n) = 2n$ for every $n \in \mathbb{Z}$.
(b) $g : \mathbb{Z} \to \mathbb{Z}$ defined by saying that $g(n) = \sqrt{n^2}$ for every $n \in \mathbb{Z}$.
(c) $h : \mathcal{P}\mathbb{Z} \to \mathcal{P}\mathbb{Z}$ defined by saying that $h(A) = A \triangle \mathbb{N}$ for every $A \subseteq \mathbb{Z}$. [EB]

31. (a) Give an example of a bijective function $f : \mathbb{N} \to \mathbb{N}$ such that $f = f^{-1}$ but $f(n) \neq n$ for every n.
(b) Show that if $f : \{0, 1, 2, 3, 4\} \to \{0, 1, 2, 3, 4\}$ is a bijective function such that $f = f^{-1}$ then there is some $n \in \{0, 1, 2, 3, 4\}$ such that $f(n) = n$. [EC]

32. Set $A = \{a, b, c, d\}$ and let R be the relation on A with incidence matrix $M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$.
Find the incidence matrix of the transitive closure R^* of R. [FA]

33. Set $A = \{a, b, c\}$ and let R be the relation on A with incidence matrix $M_R = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$. Find the incidence matrix M_{R^*} of the transitive closure R^* of R. Find the smallest equivalence relation $R^\sim$ including R, and describe the corresponding partition of A. [FB]

34. Set $A = \{a, b, c\}$ and let R be the relation on A with incidence matrix $M_R = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$. Find the incidence matrix M_{R^*} of the transitive closure R^* of R. Find the smallest equivalence relation $R^\sim$ including R, and describe the corresponding partition of A. [FC]

35. Set $A = \{a, b, c, d, e\}$, $R = \{(a, b), (b, b), (b, d), (d, c), (e, e)\}$. Find the transitive closure R^* of R, and list the ordered pairs belonging to it. [FD]

36. Which of the following sets are finite, and which are infinite?
(a) The set of prime numbers.
(b) The set of electrons in the solar system.
(c) The set of numbers which your pocket calculator can represent in its memory.
(d) The set of real numbers between 0 and 1.
(e) The set of rational numbers between -2 and -1.
(f) The set of real numbers x such that $x^2 \leq 0$.
(g) The set of correct programs written in the computer language C.
(h) The set of correct programs written in the computer language C which can be correctly compiled by some computer in the University of Essex.
(i) The set of functions from $\{0, 1, \ldots, 10^{10}\}$ to itself.
(j) The set of functions from $\{0, 1\}$ to $\mathbb{N}$. [FE]

37. Which of the following sets are finite, and which are infinite?
(a) The set of 'Pythagorean triples', that is, $\{(x, y, z) : x,\ y,\ z \in \mathbb{P},\ x^2 + y^2 = z^2\}$ (*recall*: $\mathbb{P} = \{1, 2, \ldots\}$).
(b) The set $\{(x, y, z, n) : x,\ y,\ z,\ n \in \mathbb{P},\ n \geq 3,\ x^n + y^n = z^n\}$.
(c) The set of injective functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) \leq n$ for every $n \in \mathbb{N}$.
(d) The set of injective functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(n) \leq n$ for every $n \in \mathbb{Z}$. [FF]

38. Set $A = \{p, q, r, s\}$.
(a) How many different relations are there on A?
(b) How many different total orderings are there on A?
(c) How many different total orderings are there on A which have $q < p$? [FG]

39. Set $A = \{1, 2, 3\}$.
(a) How many injective functions are there from A to itself?
(b) How many surjective functions are there from A to itself?
(c) How many reflexive relations are there on A?
(d) How many symmetric relations are there on A?
(e) How many equivalence relations are there on A?
(f) How many partial orders are there on A? [FH]

40. (a) If we know that A is an infinite countable set and B is an uncountable set, what can we say about the size of $A \cap B$?

(b) If we know that A and B are uncountable sets, what can we say about the size of $A \cap B$?

(c) If we know that A and B are uncountable sets, what can we say about the size of $A \cup B$?

(d) If we know that A and B are countable sets, what can we say about the size of $A \triangle B$?

(e) If we know that A and B are uncountable sets and $A \cap B \subseteq \mathbb{Q}$ (the set of rational numbers), what can we say about the size of $A \setminus B$?

(f) If we know that A and C are infinite countable sets, while B is uncountable, what can we say about the size of $A \cup (B \cap C)$?

[FZ]

41. For each of the following sets, say whether it is finite, countably infinite, or uncountable.

(a) The set of all words of length 17 in the alphabet $\mathbb{Z}$.

(b) The set of all words of any length in the alphabet $\mathbb{Z}$.

(c) The set of all polynomial equations $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$, where $n \in \mathbb{P}$ and all the coefficients a_i are integers and $a_n \neq 0$.

(d) The set of all algebraic numbers, where a real number x is **algebraic** if there are integers $a_0, \ldots, a_n$ such that $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0$ and $a_n \neq 0$. (*Hint*: you need to know that for any particular coefficients $a_0, \ldots, a_n$ where $a_n \neq 0$, there can be at most n different numbers x such that $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0$.)

(e) The set of transcendental numbers, where a real number x is **transcendental** if it is not algebraic. [GAB]

42. For each of the following sets, say whether it is finite, countably infinite, or uncountable.

(a) The set of all quadratic equations $ax^2 + bx + c = 0$, where a, b and c are integers.

(b) The set of all real numbers with decimal expansions beginning 0·12....

(c) The set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers.

(d) The set of all words in the alphabet $\{a, b, c, \ldots, z\}$ which have 17 letters.

(e) The set of all words (of any length) in the alphabet $\{a, b, c, \ldots, z\}$.

[GA]

43. Let A and B be two sets. Then each of them can be either (i) empty (ii) finite but not empty (iii) countably infinite or (iv) uncountable, so we have 16 possibilities in all.

(a) If we know that there is a function $f : A \to B$, are any of these possibilities excluded, and if so which?

(b) If we know that there is an injective function $f : A \to B$, are any possibilities excluded, and if so which?

(c) If we know that there is a surjective function $f : A \to B$, are any possibilities excluded, and if so which?

[GB]

44. Find an integer $n \geq 0$ such that $m = n^2 + n + 41$ is not a prime number; for your solution, express m as a product of primes. [HA]

45. If a and b are real numbers, their *arithmetic mean* is $\frac{a+b}{2}$; if a and b are positive real numbers, their *geometric mean* is $\sqrt{ab}$. (i) Find the arithmetic and geometric means of the pair $1, 2$. (ii) Which is greater? (iii) Show that, for any real numbers a, $b \geq 0$, $\frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}(\sqrt{a} - \sqrt{b})^2$. (iv) Show that if a, $b \geq 0$ then their arithmetic mean is greater than or equal to their geometric mean. (v) Under what circumstances are the two means equal? [HB]

46. Use induction to prove that $1 + 3 + 5 + \ldots + (2n-1) = n^2$ for every integer $n \geq 1$. [HC]

47. Use induction to prove that the following statements are true for all integers $n \geq 1$.

(i) $2 + 4 + 6 + \ldots + 2n = n(n+1)$;

(ii) $1 \times 2 + 2 \times 3 + \ldots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$; [HD]

48. Use induction to prove that the following statements are true for all integers $n \geq 1$.
(i) $1 + 4 + 7 + \ldots + (3n - 2) = \frac{1}{2}n(3n - 1)$;
(ii) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{1}{4}n^2(n + 1)^2$. [HE]

49. Use induction to prove that

$$1 + a + \ldots + a^n = \frac{1 - a^{n+1}}{1 - a}$$

whenever $a \neq 1$ and $n \in \mathbb{N}$. [HF]

50. A colony of bacteria starts with 6 individuals and doubles in size every hour. Find a_n, the number of bacteria after n hours have passed. About how long will you have to wait before the population reaches a million? [IA]

51. Prove by induction on n that if $A_1, \ldots, A_n$ are countable sets then the set $A_1 \times \ldots \times A_n = \{(x_1, \ldots, x_n) : x_1 \in A_1, \ldots, x_n \in A_n\}$ is countable. (*Hint*: remember that $A \times B$ is countable if A and B are.) [IB]

52. The Fibonacci sequence $a_0, a_1, \ldots$ is defined by the rule

$$a_0 = 0,\ a_1 = 1,\ a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 2.$$

(i) Find $a_0, \ldots, a_{10}$.
(ii) Give a formula for a_n in terms of a_{n-2} and a_{n-3}. For which values of n is your formula valid?
(iii) Prove by induction on n that a_n is even if n is divisible by 3. [IC]

53. (a) Simplify the expression $x(x^k - y^k) + (x - y)y^k$.
(b) Show by induction on n that $5^n - 2^n$ is divisible by 3 for every $n \geq 1$. [ID]

54. (a) Show that $\sqrt{k+1} - \sqrt{k} \leq \frac{1}{\sqrt{k+1}}$ for every $k \geq 1$.
(b) Show that $\sqrt{n} \leq 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}}$ for every integer $n \geq 1$. [IE]

55. Show that, for any integers $i, j \geq 2$, if you have 2^{i+j-3} or more people gathered together then *either* there is a group of i of them all of whom know each other, *or* there is a group of j of them none of whom know each other. (*Hint*: induce on $n = i + j$.) [JA]

56. Suppose that $a_0 = a_1 = -1$ and $a_n = 5a_{n-1} - 6a_{n-2}$ for every $n \geq 2$. Show that $a_n = 3^n - 2^{n+1}$ for every $n \in \mathbb{N}$. [JB]

57. Suppose that $a_0 = 3$ and that $a_n \leq a_0 + a_1 + \ldots + a_{n-1}$ for every $n \geq 1$. Show that $a_n \leq 3 \times 2^n$ for every n. [JC]

58. Suppose that $x_1, x_2, \ldots$ are real numbers between 0 and 1. For each $n \geq 1$, write s_n for $x_1+x_2+\ldots+x_n$ and p_n for $(1 - x_1) \times (1 - x_2) \times \ldots \times (1 - x_n)$. Show by induction that
(a) $p_n \geq 0$;
(b) $s_n \geq 0$;
(c) $p_n \geq 1 - s_n$;
(d) if $s_n \leq 1$ then $p_n \leq 1 - \frac{1}{2}s_n$. [JD]

59. You have n Christmas cards to put into n envelopes. There having been a power cut, you are doing it in the dark.

(a) How many ways are there of doing it?

(b) Let q_n be the number of ways of doing it such that every card is in the *wrong* envelope. Find q_1, q_2. Show that

$$q_n = (n-1)(q_{n-1} + q_{n-2})$$

for $n \geq 3$.

(c) Setting $a_n = \frac{q_n}{n!}$, show that $a_n - a_{n-1} = \frac{(-1)^n}{n!}$ for every $n \geq 2$.

(d) (for mathematicians only!)(i) Find $\lim_{n\to\infty} a_n$. (ii) If you have a large pile of Christmas cards, what is the approximate probability that at least one will find its way into the right envelope? [JE]

60. (This question concerns the problem of sorting a list of different numbers into order with the smallest possible number of comparisons.)

(a) Show that, given two sorted lists of lengths i and j, you can arrange them into one sorted list of length $i+j$ by making at most $i+j-1$ comparisons.

(b) Show that if $a_1 = 1$ and $a_{n+1} \leq 2a_n + 2^{n+1}$ for every n, then $a_n \leq 2^n n$ for every $n \geq 1$.

(c) Show that, for any $n \geq 1$, a list of length 2^n can be sorted into order with at most $2^n n$ comparisons. [JF]

61. Let $a_0, a_1, \ldots$ be the Fibonacci sequence.

(a) Show that $1 + a_0 + a_2 + \ldots + a_{2n} = a_{2n+1}$ for every $n \in \mathbb{N}$.

(b) Define $b_0, b_1, \ldots$ by the rules

$$b_0 = 1,$$

$$b_n = nb_0 + (n-1)b_1 + (n-2)b_2 + \ldots + 2b_{n-2} + b_{n-1}$$

for every $n \geq 1$. Show that $b_n = a_{2n}$ for every $n \geq 1$. [JG]

62. What is wrong with the following proof that '$2^n = 1$ for every $n \in \mathbb{N}$'?

Base step P(0) says that '$2^0 = 1$', which is true, so the induction starts.

Inductive step Assuming that P(i) is true whenever $0 \leq i < k$, we have to prove P(k), that is, that $2^k = 1$. Now

$$2^k = \frac{2^{k-1} \times 2^{k-1}}{2^{k-2}},$$

and (by the inductive hypothesis) $2^{k-2} = 2^{k-1} = 1$, so $2^k = \frac{1\times 1}{1} = 1$, and the induction continues.

[JH]

63. Show by induction on m that if $0 < m \leq n$ then $\dfrac{m}{n}$ can be expressed as a sum of distinct reciprocals. (*Hint*: start with $\frac{1}{r}$ where r is the smallest number such that $\frac{1}{r} \leq \frac{m}{n}$.) [JI]