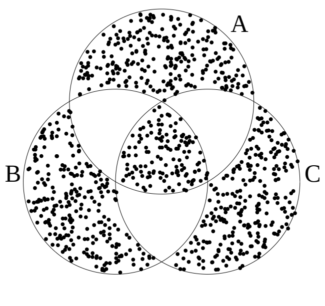
Finite Mathematics Answers

1. (i) $A \cup B = \{1, 2, 3, 4, 5, 7, 10\}$
(ii) $A \cap C = \{4\}$
(iii) $A \setminus B = \{7, 10\}$
(iv) $\emptyset$
(v) $\{1, 2, 3, 4, 5, 7, 10\} \setminus \{6, 8\} = \{1, 2, 3, 4, 5, 7, 10\}$.

2. (i) $\Longleftrightarrow$ (b), (ii) $\Longleftrightarrow$ (a), (iii) $\Longleftrightarrow$ (b), (iv) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (a), (v) $\Longleftrightarrow$ (c).

3. (i) T (any set is a subset of itself)
(ii) T ($\{x, \{x\}\}$ has exactly two members, one of which is $\{x\}$)
(iii) F (the set on the right, $\{x\}$, has exactly one member, which is x, not $\{x\}$)
(iv) T (the set on the left, $\{x\}$, has just one member x, which belongs to the set on the right)
(v) F (if, for instance, $A = \{1\}$ and $B = \{2\}$, then $A \times B = \{(1, 2)\}$, while $B \times A = \{(2, 1)\}$)
(vi) F (take the same sets A, B as in (v), and $\mathcal{U} = \{1, 2\}$; then $\mathcal{U} \setminus (A \setminus B) = \{2\}, \mathcal{U} \setminus (B \setminus A) = \{1\}$)
(vii) T (draw a Venn diagram)
(viii) T ($\emptyset$ has exactly one subset, which is $\emptyset$)
(ix) F (try $A = \{1\}$, $B = \{2\}$, $C = \{3\}$; then $A \setminus (B \cup C) = \{1\}$, $(A \setminus B) \cup C = \{1, 3\}$.

4.



Yes, they are the same.

5.

$$
\begin{aligned}
(\mathcal{U} \cap A) \cup (B \cap A) &= (A \cap \mathcal{U}) \cup (A \cap B) && \text{(3b)}\\
&= A \cap (\mathcal{U} \cup B) && \text{(4b)}\\
&= A \cap (B \cup \mathcal{U}) && \text{(3a)}\\
&= A \cap \mathcal{U} && \text{(6a)}\\
&= A. && \text{(5b)}
\end{aligned}
$$

Dual identity: $(\emptyset \cup A) \cap (B \cup A) = A$.

6.

$$
\begin{aligned}
(A^c \cap B)^c \cap (A \cup B) &= ((A^c)^c \cup B^c) \cap (A \cup B) && \text{(10b)}\\
&= (A \cup B^c) \cap (A \cup B) && \text{(7)}\\
&= A \cup (B^c \cap B) && \text{(4a)}\\
&= A \cup \emptyset && \text{(8b)}\\
&= A. && \text{(5a)}
\end{aligned}
$$

7. (i) $A \times B = \{(1,1),(1,b),(a,1),(a,b),(x,1),(x,b)\}$
(ii) $B \times A = \{(1,1),(b,1),(1,a),(b,a),(1,x),(b,x)\}$
(iii) $(A \times B) \setminus (B \times A) = \{(1,b),(a,1),(a,b),(x,1),(x,b)\}$
(at least, provided none of a, b, x is equal to 1).

8. All are true except the last (try $A = \{1\}$, $B = \{2\}$; then $(A \triangle B) \times (B \triangle A) = \{1,2\} \times \{1,2\} = \{(1,1),(1,2),(2,1),(2,2)\}$, while $(A \times B) \triangle (B \times A) = \{(1,2)\} \triangle \{(2,1)\} = \{(1,2),(2,1)\}$.)

9. (a) $12 + 11 + 7 - 5 - 3 - 5 + 2 = 19$.

(b) $\#(A \times B) = mn$ so $\#(\mathcal{P}(A \times B)) = 2^{mn}$.

10. The total number of subsets of the group of 12 people is $2^{12} = 4096$. The number of subsets with *less* than two members is $1 + 12 = 13$ (one set with no members, 12 sets with just one member). So the number of sets with 2 or more members is $4096 - 13 = 4083$.

11. (i) Yes, because, for any X,

$$\begin{aligned} X \in \mathcal{P}(A \cap B) &\iff X \subseteq A \cap B \\ &\iff X \subseteq A \text{ and } X \subseteq B \\ &\iff X \in \mathcal{P}(A) \text{ and } X \in \mathcal{P}(B) \\ &\iff X \in \mathcal{P}(A) \cap \mathcal{P}(B). \end{aligned}$$

(ii) No; if, for instance, $A = \{1,2\}$ and $B = \{2,3\}$, then $A \cup B = \{1,2,3\}$,

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}, \quad \mathcal{P}(B) = \{\emptyset, \{2\}, \{3\}, \{2,3\}\},$$

$$\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}\},$$

$$\mathcal{P}(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

contains $\{1,3\}$ and $\{1,2,3\}$, which don't belong to either $\mathcal{P}(A)$ or $\mathcal{P}(B)$.

(iii) The Principle of Duality doesn't apply to formulae including the power set operation $\mathcal{P}$.

12. (a)(i) R is reflexive iff every vertex of the digraph is linked to itself by a loop.
(ii) R is symmetric iff every arc in the digraph is two-way.
(iii) R is transitive iff whenever there is a connected path from a to b in the digraph, there is an arc directly from a to b.

(b)(i) R is reflexive iff every coefficient on the principal diagonal of $\mathbf{M}$ is 1.
(ii) R is symmetric iff $\mathbf{M}$ is symmetric, that is, is equal to its transpose $\mathbf{M}^T$, that is, is equal to its reflection in its principal diagonal.

(iii) R is transitive iff $m_{ik} \geq m_{ij} m_{jk}$ for all i, j, $k \leq n$, where we think of $\mathbf{M}$ as $\begin{pmatrix} m_{11} & \dots & m_{1n} \\ & \dots & \\ m_{n1} & \dots & m_{nn} \end{pmatrix}$.

13. (i) (ii)

(iii) $\mathbf{M}_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$.

(i) $R^{-1} = \{(3,1),(4,1),(2,3),(3,3),(4,3)\}$.

(ii) (iii)

(iv) $\mathbf{M}_{R^{-1}} = \begin{array}{c c} & \begin{array}{cccc} {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 4} \end{array} \\ \begin{array}{c} {\scriptstyle 1} \\ {\scriptstyle 2} \\ {\scriptstyle 3} \\ {\scriptstyle 4} \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}.$

14. $R = \{(2,3),(2,5),(2,7),(2,9),(2,11),(3,2),(3,5),(3,8),(3,11),(4,3),(4,7),(4,11),(5,4),(5,9),(6,5),(6,11),(7,6),(8,7),(9,8),(10,9),(11,10),(12,11)\}$ has 22 members.

15. (i) $R \circ S = \{(a,c),(a,d),(c,a),(d,a)\}$.

(ii) $S \circ R = \{(b,a),(b,c),(c,b),(c,d),(d,a),(d,c)\}$.

(iii) $R \circ R = \{(a,a),(a,b),(a,c),(a,d),(c,b)\}$.

(iv) $S \circ S = \{(c,a),(c,c),(c,d)\}$.

16. (i) $R = \{(3,3),(6,2),(9,1)\}$; its domain is $\{3,6,9\}$ and its range is $\{1,2,3\}$.

(ii) $R \circ R = \{(3,3)\}$, $R^{-1} = \{(3,3),(2,6),(1,9)\}$, $R \circ R^{-1} = \{(3,3),(6,6),(9,9)\}$, $R^{-1} \circ R = \{(1,1),(2,2),(3,3)\}$.

17. (a) Not reflexive (for instance, $0 \not R\, 0$)
Symmetric (if $ab = 1$ then $ba = 1$)
Transitive ($R = \{(1,1),(-1,-1)\}$, so if $a\,R\,b$ and $b\,R\,c$ then $a = b \in \{-1,1\}$ and $b = c$ so $a = c$ and $a\,R\,c$).

(b) Not reflexive (for instance, $2 \not R\, 2$)
Symmetric (the 'highest common factor of a and b' must also be 'the highest common factor of b and a')
Not transitive (for instance, $2\,R\,3$ and $3\,R\,4$ but $2 \not R\, 4$).

18. (a) $\mathbf{A}\,R\,\mathbf{A}$ for every $\mathbf{A}$, because we can get from $\mathbf{A}$ to itself by swapping rows 0 times and swapping columns 0 times. So R is reflexive.

(b) If $\mathbf{A}\,R\,\mathbf{B}$ we can get from $\mathbf{A}$ to $\mathbf{B}$ by legal operations; now using the same operations in the reverse order we can get from $\mathbf{B}$ to $\mathbf{A}$, so $\mathbf{B}\,R\,\mathbf{A}$. Thus R is symmetric.

(c) If $\mathbf{A}\,R\,\mathbf{B}$ and $\mathbf{B}\,R\,\mathbf{C}$ then we can get from $\mathbf{A}$ to $\mathbf{B}$ by some sequence of legal operations, and then from $\mathbf{B}$ to $\mathbf{C}$ by some other sequence; so putting the two lists together, we have a path from $\mathbf{A}$ to $\mathbf{C}$, and $\mathbf{A}\,R\,\mathbf{C}$. So R is transitive.

(d) Putting these three together, R is an equivalence relation.

19.

$$\left\{\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix}\right\},\quad \left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\},\right.$$

$$\left\{\begin{pmatrix}1&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&1\end{pmatrix}\right\},\quad \left\{\begin{pmatrix}1&0\\1&0\end{pmatrix},\begin{pmatrix}0&1\\0&1\end{pmatrix}\right\},\quad \left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix}\right\},$$

$$\left.\left\{\begin{pmatrix}0&1\\1&1\end{pmatrix},\begin{pmatrix}1&0\\1&1\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}1&1\\1&0\end{pmatrix}\right\},\quad \left\{\begin{pmatrix}1&1\\1&1\end{pmatrix}\right\}\right\}.$$

(As long as all sixteen matrices are listed and grouped correctly, give full marks, even if the brackets $\{$ and $\}$ aren't in their proper places.)

Note however that the 'partition' is the set of equivalence classes, and each equivalence class is a set; so each equivalence class is listed in a pair $\{\dots\}$ of brackets, and the set of equivalence classes is listed in a pair $\Big\{\dots\Big\}$ of brackets.

20. (a) If $(a,b)\in A\times A$ then $(a,b)\,R\,(a,b)$ because $a+b=b+a$, so R is reflexive.

(b) If $(a,b)\,R\,(c,d)$ then $a+d=b+c$ so $c+b=d+a$ and $(c,d)\,R\,(a,b)$. Thus R is symmetric.

(c) If $(a,b)\,R\,(c,d)$ and $(c,d)\,R\,(e,f)$ then $a+d=b+c$ and $c+f=d+e$; so that $a+d+c+f=b+c+d+e$ and $a+f=b+e$, that is, $(a,b)\,R\,(e,f)$. So R is transitive.

(d) Putting these together, R is an equivalence relation.

(Note that if we express R in the form

$$(a,b)\,R\,(c,d) \text{ iff } a-b=c-d,$$

then we have a sentence in the form

$$\text{`}x\,R\,y \text{ iff } f(x)=f(y)\text{'},$$

and any such relation is necessarily an equivalence relation.)

21. The equivalence class of $(2,5)$ is

$$\{(1,4),(2,5),(3,6),(4,7),(5,8),(6,9)\}.$$

22. (i) Reflexive symmetric not transitive

(ii) Not reflexive symmetric not transitive

(iii) Not reflexive symmetric transitive

(iv) Reflexive symmetric transitive

23. (i) R is reflexive, transitive, a partial order, not a total order.
(To see that R is transitive, observe that if $(a,b)\,R\,(c,d)\,R\,(e,f)$ then $a \le c \le e$ and $f \le d \le b$, so $a \le e$ and $f \le b$, that is, $(a,b)\,R\,(e,f)$.)

(ii) R is reflexive, transitive, not a partial order, not a total order.
(R is not a partial order because $-1\,R\,1$ and $1\,R\,-1$ but $1 \neq -1$.)

(iii) R is reflexive, transitive, a partial order, a total order.
(To see that R is a partial order, take X, Y, $Z \subseteq \mathbb{N}$ such that $X\,R\,Y$ and $Y\,R\,Z$. If either $X = Y$ or $Y = Z$ or $X = Z$ then of course $X\,R\,Z$. Otherwise, write m for the smallest element of $X \triangle Y$. and n for the smallest element of $Y \triangle Z$. Note that $m \in X$, so $m \notin Y$, and $n \in Y$; it follows that m and n are different. Let $p = \min(m, n)$ be the smaller of the two.
If $i < p$, then $i \notin X \triangle Y$ (because $i < m$ and m is the smallest element of $X \triangle Y$), and similarly $i \notin Y \triangle Z$ (because $i < n$). But this means that $i \in X \iff i \in Y \iff i \in Z$, and $i \notin X \triangle Z$.
On the other hand, p is equal to one of m, n and strictly less than the other, so it belongs to exactly one of the sets $X \triangle Y$ and $Y \triangle Z$. It therefore must belong to $(X \triangle Y) \triangle (Y \triangle Z) = X \triangle Z$, and is the least element of $X \triangle Z$.
If $m < n$, then $p = m \in X$, so $X\,R\,Z$. If $n < m$, then $p = n \in Y$; but also $p \notin X \triangle Y$, so again $p \in X$ and $X\,R\,Z$.
As X, Y and Z are arbitrary, R is transitive.)

24. For any $(a,b) \in \mathbb{R}^2$, $a^2 + b^2 = a^2 + b^2$, so $(a,b)\,R\,(a,b)$, and R is reflexive.

If $(a,b)\,R\,(c,d)$ then $a^2 + b^2 = c^2 + d^2$ so $c^2 + d^2 = a^2 + b^2$ and $(c,d)\,R\,(a,b)$; thus R is symmetric.

If $(a,b)\,R\,(c,d)$ and $(c,d)\,R\,(e,f)$, then $a^2 + b^2 = c^2 + d^2$ and $c^2 + d^2 = e^2 + f^2$, so $a^2 + b^2 = e^2 + f^2$ and $(a,b)\,R\,(e,f)$; thus R is transitive.

Being reflexive, symmetric and transitive, R is an equivalence relation.

The equivalence classes of R are circles centre $(0,0)$ together with the singleton set $\{(0,0)\}$ (This is because two points of the plane are related iff they are the same distance from $(0,0)$.)

25. (Note that you are being asked to prove an ‘if and only if’ sentence. So break this into its two halves: first prove that ‘if R is both symmetric and anti-symmetric, then it is included in the equality relation’, and then ‘if R is included in the equality relation, it is both symmetric and anti-symmetric’.)

(a) Suppose that R is both symmetric and anti-symmetric. If $(a,b) \in R$, then (because R is symmetric) $(b,a) \in R$, and now (because R is anti-symmetric) $a = b$. So $R \subseteq \{(a,b) : a = b\}$.

(b) Suppose that $R \subseteq \{(a,b) : a = b\}$. If $(a,b) \in R$ then $a = b$ so $(b,a) = (a,b)$ belongs to R; thus R is symmetric. If (a,b) and (b,a) both belong to R then of course $a = b$; thus R is anti-symmetric.

26. $! < @ < + < L$.
($! < @$ because $+L+!! < +L+!@$; $@ < +$ because $@@+L < +L+!!$; $+ < L$ because $@@++ < @@+L$.)

27. (i) The relation R is a function on A iff for every $a \in A$ there is exactly one b such that $a\,R\,b$, that is, iff for every vertex a of the digraph there is exactly one arc leaving a, that is, iff every vertex of the digraph has outdegree exactly 1.

(ii) Given that R is a function, it is injective iff every vertex has indegree at most 1, and surjective iff every vertex has indegree at least 1.

28. (i) This is a functionbut is not injectivenor surjective its set of values is $\{1\}$ (Here $f(a) = 1 = f(b) = f(c) = f(d) = 1$.)

(ii) This is not a function defined on the given set A(because $f(d)$ is undefined).

(iii) This is not a function defined on $\mathbb{R}$(because (α) $f(4)$ is not defined (β) 2 and -2 are both possible values for $f(0)$).

(iv) This is a functionwhich is injectivebut not surjective its set of values is the set of odd integers (If $2m + 1 = 2m' + 1$ then $2m = 2m'$ so $m = m'$; thus the function is injective. On the other hand, there is no $m \in \mathbb{Z}$ such that $0 = 2m + 1$, so the function is not surjective.)

29. $(f \circ g)(n) = f(g(n)) = 5(7n + k) + 2 = 35n + 5k + 2$,
$(g \circ f)(n) = g(f(n)) = 7(5n + 2) + k = 35n + 14 + k$,
$f \circ g = g \circ f$ iff $k = 3$
(because $5k + 2 = k + 14$ iff $4k = 12$ iff $k = 3$).

30. (a) f is injective; its set of values is the set E of even integers; $f^{-1}(m) = \frac{1}{2}m$ for every $m \in E$.

(b) g is not injective (because $g(1) = g(-1)$).

(c) h is injective; its set of values is $\mathcal{P}\mathbb{Z}$; $h^{-1}(A) = A \triangle \mathbb{N}$ for every $A \subseteq \mathbb{Z}$.

31. (a) Set $f(n) = n + 1$ for even n, $n - 1$ for odd n.

(b) Define a relation R on $\{0, 1, 2, 3, 4\}$ by saying that $m\,R\,n$ if either $n = m$ or $n = f(m)$. Check that R is an equivalence relation. Now $\{0, 1, 2, 3, 4\}$ has five members, so at least one equivalence class has an odd number of members. But the equivalence class of n is $\{n, f(n)\}$ so it can have an odd number of members only when it has just one member, that is, $f(n) = n$.

32.

$$M_{R^*} = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

.

33. R is transitive, so $M_{R^*} = M_R$.

$$M_{R^\sim} = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}$$; the corresponding partition of A is $\{\{a, c\}, \{b\}\}$.

34. $$M_{R^*} = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}.$$

$R^\sim = A \times A$; the corresponding 'partition' of A is just the single set $\{A\}$.

35. $R^* = \{(a, b), (a, c), (a, d), (b, b), (b, c), (b, d), (d, c), (e, c)\}$.

36. (a) Infinite.
(b) Finite.
(c) Finite. (Your calculator has only finitely many transistors in it.)
(d) Infinite.
(e) Infinite. (All numbers of the form $-1-\frac{1}{n}$, where $n \in \mathbb{P}$, are in the set.)
(f) Finite. (The set is $\{0\}$.)
(g) Infinite. (You can go on as long as you like.)
(h) Finite. (The computer has to store the entire program before it can confidently compile it.)
(i) Finite. (It has $(10^{10}+1)^{10^{10}+1}$ members.)
(j) Infinite. (If we write $f_n(0) = f_n(1) = n$, then f_n belongs to the set for every n.)

37. (a) Infinite. (For any strictly positive integers $a < b$, $(b^2 - a^2, 2ab, b^2 + a^2)$ belongs to the set.)
(b) Finite. (By Fermat's Last Theorem, proved by Wiles, the set is empty.)
(c) Finite. (We must have $f(0) = 0$, $f(1) = 1$, $\dots$, so there is just one function in the set.)
(d) Infinite. (If $f_k(n) = n - k$, f_k belongs to the set for every $k \in \mathbb{N}$.)

38. (a) 2^{16} or 65536. ($A \times A$ has $4 \times 4 = 16$ members and 2^{16} subsets.)

(b) 4! or 24. You have 4 choices for the biggest element, then 3 choices for the next biggest, then 2 for the next biggest.)

(c) 12. Exactly half of the total orders will have $q < p$ and half will have $p < q$.)

39. (a) 6. (You have 3 choices for $f(1)$, then 2 choices for $f(2)$.

(b) 6. (Because A is finite, a function from A to itself is surjective iff it is injective.)

(c) 2^6. (You are told that $1R1$, $2R2$, $3R3$ so you have to decide for each of the 6 pairs $(1,2)$, $(1,3)$, $(2,1)$, $(2,3)$, $(3,1)$, $(3,2)$ whether to put it into R.)

(d) 2^6. (You have to decide for each of the 6 pairs $(1,1)$, $(1,2)$, $(1,3)$, $(2,2)$, $(2,3)$, $(3,3)$ whether to put it into R.)

(e) 5. (Count the partitions. You have one partition $\{A\}$ into a single set, 3 partitions $\{\{1\},\{2,3\}\}$, $\{\{2\},\{1,3\}\}$, $\{\{3\},\{1,2\}\}$ into 2 sets, one partition $\{\{1\},\{2\},\{3\}\}$ into three sets.)

(f) 16. (We have the trivial partial order in which no element is less than any other; then 3 partial orders in which we have a pair $a < b$ but no other inequality; 3 more in which $a < c$ and $b < c$ but $a \not< b$ and $b \not< a$; 3 in which $a < b$ and $a < c$ but $b \not< c$ and $c \not< b$; and 6 total orders.)

40. (a) $A \cap B$ is countable.

(b) Nothing.

(c) $A \cup B$ must be uncountable.

(d) $A \triangle B$ must be countable (because it is included in $A \cup B$).

(e) $A \setminus B$ must be uncountable (because $(A \setminus B) \cup (A \cap B) = A$ is uncountable and $A \cap B \subseteq \mathbb{Q}$ is countable).

(f) $A \cup (B \cap C)$ must be countably infinite (because A is infinite and both A and $B \cap C$ are countable). (for either 'countable' or 'infinite'.)

41. (a) Countably infinite.

(b) Uncountable. (This set is either the closed interval $[0{\cdot}12, 0{\cdot}13]$ or the half-open interval $[0{\cdot}12, 0{\cdot}13[$, depending on whether you allow '$0{\cdot}12999999\ldots$' as a decimal expansion. But in either case it is uncountable.)

(c) Uncountable. (Since $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ is uncountable, at least one of $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$ is uncountable; but $\mathbb{Q}$ is countable, so $\mathbb{R} \setminus \mathbb{Q}$ is uncountable.)

(d) Finite. (It has 26^{17} members.)

(e) Countably infinite. (It is the union of a sequence of finite sets.)

42. (a) Countably infinite.

(b) Countably infinite.

(c) Countably infinite. (It can be regarded as a subset of the set in (b).)

(d) Countably infinite. (There are only countably many equations, and each has only finitely many solutions, so we have the union of a sequence of finite sets, which must be countable.)

(e) Uncountable. (Like $\mathbb{R} \setminus \mathbb{Q}$.)

43. (a) The three cases in which $B = \emptyset$, $A \neq \emptyset$ are impossible.

(b) The three possibilities in which $B = \emptyset$, $A \neq \emptyset$ are still impossible; in addition, we can no longer have the two cases in which A is infinite and B is finite and not empty, nor the case in which A is uncountable and B is countably infinite.

(c) The three possibilities in which $B = \emptyset$, $A \neq \emptyset$ are still impossible; in addition, we can no longer have the three cases in which A is empty and B is not empty, nor the two cases in which A is finite but not empty and B is infinite, nor the case in which A is countably infinite and B is uncountable.

44. $n = 40$, $m = 1681 = 41 \times 41$.
(Alternatively $n = 41$, $m = 41 \times 43$. There are infinitely many other solutions.)

45. (i) Arithmetic mean is $1{\cdot}5$, geometric mean is $\sqrt{2} \simeq 1{\cdot}414$.

(ii) The arithmetic mean is greater.

(iii) $\frac{1}{2}(\sqrt{a} - \sqrt{b})^2 = \frac{1}{2}((\sqrt{a})^2 - 2\sqrt{a}\sqrt{b} + (\sqrt{b})^2) = \frac{1}{2}(a - 2\sqrt{ab} + b) = \frac{1}{2}(a + b) - \sqrt{ab}$.

(iv) Writing μ_{A} for the arithmetic mean, μ_{G} for the geometric mean, $\mu_{\mathrm{A}} - \mu_{\mathrm{G}} = \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 \geq 0$ because squares are always positive, so $\mu_{\mathrm{A}} \geq \mu_{\mathrm{G}}$.

(v) The two means are equal exactly when $\frac{1}{2}(\sqrt{a} - \sqrt{b})^2 = 0$, that is, when $\sqrt{a} - \sqrt{b} = 0$, that is, when $\sqrt{a} = \sqrt{b}$, that is, when $a = b$.

46. **Base step** If $n = 1$, then the LHS is 1 and the RHS is $1^2 = 1$; these are equal, so the induction starts.

Inductive step Assuming that the formula is true for $n = k$, that is, that $1+3+5+\ldots+(2k-1) = k^2$, then we look at the formula for $n = k+1$. The two sides of the equation are

$$1+3+5+\ldots+(2(k+1)-1), \qquad (k+1)^2.$$

Expanding the LHS, we have

$$\begin{aligned} 1+3+5+&\ldots+(2(k+1)-1) \\ &= 1+3+5+\ldots+(2k-1)+(2k+2-1) \\ &= k^2+(2k+1) \end{aligned}$$

(using the inductive hypothesis)

$$= (k+1)^2,$$

as required. Thus the induction proceeds.

47. (i) **Base step** If $n = 1$, then the LHS is 2 and the RHS is $1 \times 2 = 2$; these are equal, so the induction starts.

Inductive step Assuming that the formula is true for $n = k$, that is, that $2+4+6+\ldots+2k = k(k+1)$, then

$$\begin{aligned} 2+4+6+\ldots+2(k+1) &= 2+4+6+\ldots+2k+2(k+1) \\ &= k(k+1)+2(k+1) \end{aligned}$$

(by the inductive hypothesis)

$$= (k+2)(k+1) = (k+1)((k+1)+1),$$

so the induction proceeds.

(ii) **Base step** If $n = 1$, then the LHS is $1 \times 2 = 2$ and the RHS is $\frac{1}{3} \times 1 \times 2 \times 3 = 2$; these are equal, so the induction starts.

Inductive step Assuming that the formula is true for $n = k$, that is, that $1 \times 2 + \ldots + k(k+1) = \frac{1}{3}k(k+1)(k+2)$, then

$$\begin{aligned} 1 \times 2 + 2 \times 3 + \ldots + (k+1)(k+2) &= 1 \times 2 + \ldots + k(k+1) + (k+1)(k+2) \\ &= \tfrac{1}{3}k(k+1)(k+2) + (k+1)(k+2) \end{aligned}$$

(by the inductive hypothesis)

$$\begin{aligned} &= (k+1)(k+2)(\tfrac{1}{3}k+1) \\ &= \tfrac{1}{3}(k+1)(k+2)(k+3) \\ &= \tfrac{1}{3}(k+1)((k+1)+1)((k+1)+2), \end{aligned}$$

so the induction proceeds.

(i) **Base step** If $n = 1$, then the LHS is 1 and the RHS is $\frac{1}{2} \times 1 \times (3-1) = 1$; these are equal, so the induction starts.

Inductive step Assuming that the formula is true for $n = k$, that is, that $1+4+\ldots+(3k-2) = \frac{1}{2}k(3k-1)$, then

$$1 + 4 + \ldots + (3(k+1)-2) = 1 + \ldots + (3k-2) + (3(k+1)-2)$$
$$= \frac{1}{2}k(3k-1) + (3(k+1)-2)$$

(by the inductive hypothesis)

$$= \frac{1}{2}(3k^2 - k) + (3k + 3 - 2)$$
$$= \frac{1}{2}(3k^2 - k) + (3k+1)$$
$$= \frac{1}{2}(3k^2 - k + 6k + 2) = \frac{1}{2}(3k^2 + 5k + 2).$$

On the other hand,

$$\frac{1}{2}(k+1)(3(k+1)-1) = \frac{1}{2}(k+1)(3k+3-1)$$
$$= \frac{1}{2}(k+1)(3k+2) = \frac{1}{2}(3k^2 + 3k + 2k + 2)$$
$$= \frac{1}{2}(3k^2 + 5k + 2)$$
$$= 1 + 4 + \ldots + (3(k+1)-2),$$

48. so the induction proceeds.

(ii) **Base step** If $n = 1$, then the LHS is 1 and the RHS is $\frac{1}{4} \times 1^2 \times 2^2 = 1$; these are equal, so the induction starts.

Inductive step Assuming that the result is true for $n = k$, that is, that $1^3 + 2^3 + \ldots + k^3 = \frac{1}{4}k^2(k+1)^2$, then

$$\frac{1}{4}(k+1)^2((k+1)+1)^2 = \frac{1}{4}(k+1)^2(k+2)^2$$
$$= \frac{1}{4}(k^2 + 4k + 4)(k+1)^2$$
$$= \frac{1}{4}k^2(k+1)^2 + \frac{1}{4}(4k+4)(k+1)^2$$
$$= \frac{1}{4}k^2(k+1)^2 + (k+1)(k+1)^2$$
$$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$
$$= 1^3 + 2^3 + \ldots + k^3 + (k+1)^3$$

(by the inductive hypothesis)

$$= 1^3 + 2^3 + \ldots + (k+1)^3,$$

so the induction proceeds.

49. **Base step** If $n = 0$ the LHS is 1 and the RHS is $\frac{1-a}{1-a} = 1$; these are equal so the induction starts.

(for 'if $n = 1$ the LHS is $1 + a$ and the RHS is $\frac{1-a^2}{1-a} = 1 + a$'.)

Inductive step Assuming that the result is true when $n = k$, that is, that $1 + a + a^2 + \ldots + a^k = \frac{1-a^{k+1}}{1-a}$, then

$$1 + a + \ldots + a^{k+1} = 1 + a + \ldots + a^k + a^{k+1}$$
$$= \frac{1-a^{k+1}}{1-a} + a^{k+1}$$

(by the inductive hypothesis)

$$= \frac{1}{1-a}(1 - a^{k+1} + a^{k+1} - a^{k+2})$$
$$= \frac{1-a^{k+2}}{1-a}$$

so the induction continues

50. $a_n = 6 \times 2^n$.
You will have to wait between 17 and 18 hours. (Allow 17·3466 or other approximations between 17 and 18.)

51. **Base step** If $n = 2$, then we just need to check that $A_1 \times A_2$ is countable, which is true because A_1 and A_2 are countable.
Inductive step Assuming the result for $n = k$, where $k \geq 2$, examine the proposition when $n = k + 1$. We are given countable sets $A_1, \ldots, A_k, A_{k+1}$ and have to show that $A_1 \times \ldots \times A_k \times A_{k+1}$ is countable. Now we know that $A_1 \times \ldots \times A_k$ is countable, by the inductive hypothesis, so $(A_1 \times \ldots \times A_k) \times A_{k+1}$ is countable. But we have a bijection $f : (A_1 \times \ldots \times A_k) \times A_{k+1} \to A_1 \times \ldots \times A_{k+1}$ defined by writing

$$f((x_1, \ldots, x_k), x_{k+1}) = (x_1, \ldots, x_k, x_{k+1})$$

whenever $x_1 \in A_1, \ldots, x_{k+1} \in A_{k+1}$. So $\#(A_1 \times \ldots \times A_{k+1}) = \#((A_1 \times \ldots \times A_k) \times A_{k+1})$, and $A_1 \times A_{k+1}$ must be countable because $(A_1 \times \ldots \times A_k) \times A_{k+1}$ is. Thus the induction proceeds.

52. (i) $a_0 = 0$, $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, $a_4 = 3$, $a_5 = 5$, $a_6 = 8$, $a_7 = 13$, $a_8 = 21$, $a_9 = 34$, $a_{10} = 55$.
or each of $a_6, \ldots, a_{10}$ inclusive.

(ii) $a_n = 2a_{n-2} + a_{n-3}$ for any $n \geq 3$.
($a_{n-1} + a_{n-2} = a_{n-2} + a_{n-3} + a_{n-2} = 2a_{n-2} + a_{n-3}$.)

(iii) **Base step** If $n = 0$ then $a_{3n} = a_0 = 0$ is even.
Inductive step Suppose the result is true for $n = k$, where $k \geq 0$. Then

$$a_{3(k+1)} = a_{3k+3} = 2a_{(3k+3)-2} + a_{(3k+3)-3}$$
$$= 2a_{3k+1} + a_{3k}$$

is the sum of two even numbers, by the inductive hypothesis, so is even. Thus the induction continues.

53. (a) $x(x^k - y^k) + (x - y)y^k = x \times x^k - x \times y^k + x \times y^k - y \times y^k = x^{k+1} - y^{k+1}$.

(b)**Base step** If $n = 1$ then $5^n - 2^n = 5 - 2 = 3$ is divisible by 3.
alternative If $n - 0$ then $5^n - 2^n = 1 - 1 = 0$ is divisible by 3.
Inductive step Suppose that the result is true for $n = k$. Then

$$\begin{aligned} 5^{k+1} - 2^{k+1} &= 5(5^k - 2^k) + (5 - 2)2^k \\ &= 5(5^k - 2^k) - 3 \times 2^k \end{aligned}$$

is the difference of two multiples of 3, by the inductive hypothesis, so is a multiple of 3. Thus the induction continues.

54. (a)

$$\begin{aligned} \frac{1}{\sqrt{k+1}} - \sqrt{k+1} + \sqrt{k} &= \frac{1}{\sqrt{k+1}}(1 - (k+1) + \sqrt{k}\sqrt{k+1}) \\ &= \frac{1}{\sqrt{k+1}}(\sqrt{k}\sqrt{k+1} - k) \\ &\geq \frac{1}{\sqrt{k+1}}(\sqrt{k}\sqrt{k} - k) = 0, \end{aligned}$$

so $\frac{1}{\sqrt{k+1}} \geq \sqrt{k+1} - \sqrt{k}$.

(b) Use induction.
Base step If $n = 1$ the proposition is

$$\sqrt{1} \leq 1,$$

which is true.
Inductive step Assume that the result is true for $n = k$. Then

$$\sqrt{k+1} \leq \sqrt{k} + \frac{1}{\sqrt{k+1}}$$

(by part (a))

$$\leq 1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$$

by the inductive hypothesis. Thus the induction proceeds.

55. **Base step** If $n = 4$ then $i = j = 2$ and $2^{i+j-3} = 2$, so we have at least two people. Take any pair of them; either they know each other or they don't, so the induction starts.

Inductive step Assume that the result is true for $n = k \geq 4$, and that we have a set A of people with $\#(A) = 2^{i+j-3}$ people, where $i + j = k + 1$.

The case $i = 2$ is easy. In this case, if there is any pair who know each other, we can stop; but otherwise the whole group A don't know each other, and since $2^{i+j-3} = 2^{j-1} \geq j$ we have a group of j of them who don't know each other. Similarly, if $j = 2$, either there is some pair who don't know each other, or the whole group know each other and there are at least i of them.

So suppose, for the rest of the argument, that $i \geq 3$ and $j \geq 3$.

Take any member of A, call him/her z, and look at the sets

$$B = \{x : x \in A \setminus \{z\}, x \text{ knows } z\},$$

$$C = \{x : x \in A \setminus \{z\}, x \text{ doesn't know } z\}.$$

case 1 If $\#(B) \geq 2^{i+j-4} = 2^{(i-1)+j-3}$ and $i \geq 3$, then, applying the inductive hypothesis, we either have a group of $i - 1$ members of B who know each other, or a group of j members of B who don't know each other. In the former case, adding z to the group, we have i members of A who know each other; in the latter case we already have j members of A who don't know each other.

case 2 If $\#(B) \leq 2^{i+j-4} - 1$ then

$$\#(C) = \#(A) - \#(B) - 1 \geq 2^{i+j-3} - 2^{i+j-4} + 1 - 1 = 2^{i+j-4} = 2^{i+(j-1)-3}.$$

This time apply the inductive hypothesis to see that either there is a group of i members of C who know each other, or a group of $j - 1$ members of C who don't know each other. In the former case, we have a group of i members of A who know each other; in the latter case we add z to get a group of j members of A who don't know each other.

Thus the induction continues.

56. **Base step** $3^0 - 2^{0+1} = 1 - 2 = -1 = a_0$, $3^1 - 2^{1+1} = 3 - 4 = -1 = a_1$, so the result is true if $n = 0$ or $n = 1$.

Inductive step Assuming that $a_i = 3^i - 2^{i+1}$ for $0 \leq i < k$, we have to prove that $a_k = 3^k - 2^{k+1}$.

Now

$$a_k = 5a_{k-1} - 6a_{k-2}$$

(by the recurrence relation)

$$= 5 \times (3^{k-1} - 2^{(k-1)+1}) - 6 \times (3^{k-2} - 2^{(k-2)+1})$$

(by the inductive hypothesis)

$$\begin{aligned}
&= 5 \times 3^{k-1} - 5 \times 2^k - 6 \times 3^{k-2} + 6 \times 2^{k-1} \\
&= 5 \times 3^{k-1} - 5 \times 2^k - 2 \times 3 \times 3^{k-2} + 3 \times 2 \times 2^{k-1} \\
&= 5 \times 3^{k-1} - 5 \times 2^k - 2 \times 3^{k-1} + 3 \times 2^k \\
&= 5 \times 3^{k-1} - 2 \times 3^{k-1} + 3 \times 2^k - 5 \times 2^k \\
&= 3 \times 3^{k-1} - 2 \times 2^k = 3^k - 2^{k+1},
\end{aligned}$$

as required. So the induction continues.

57. **Base step** If $n = 0$, then $a_0 = 3 = 3 \times 2^0$, so the induction starts.
Inductive step Assuming that $a_i \leq 3 \times 2^i$ for every $i < k$, where $k \geq 1$, we have to show that $a_k \leq 3 \times 2^k$.
Now

$$a_k \leq a_0 + a_1 + \ldots + a_{k-1}$$

(by the recurrence relation)

$$\leq 3 \times 2^0 + 3 \times 2^1 + \ldots + 3 \times 2^{k-1}$$

(by the inductive hypothesis)

$$= 3 \times (2^0 + 2^1 + \ldots + 2^{k-1})$$
$$= 3 \times \tfrac{1-2^k}{1-2} = 3 \times \tfrac{2^k-1}{2-1}$$
$$\leq 3 \times 2^k,$$

as required. Thus the induction continues.

58. (We can use the First Principle in every case.) Note that $s_{n+1} = s_n + x_{n+1}$, $p_{n+1} = p_n(1 - x_{n+1})$ for each n.

(a) **Base step** If $n = 1$ then $p_1 = 1 - x_1 \geq 0$ because $x_1 \leq 1$.
Inductive step Assuming that $p_k \geq 0$, we have to show that $p_{k+1} \geq 0$. But $p_{k+1} = p_k(1 - x_{k+1})$ is the product of two non-negative numbers, because $p_k \geq 0$ by the inductive hypothesis, and $1 - x_{k+1} \geq 0$ because $x_{k+1} \leq 1$. Since the product of two non-negative numbers is non-negative, the induction continues.

(b) **Base step** If $n = 1$ then $s_1 = x_1 \geq 0$.
Inductive step Assuming that $s_k \geq 0$, we have to show that $s_{k+1} \geq 0$. But $s_{k+1} = s_k + x_{k+1}$ is the sum of two non-negative numbers, because $s_k \geq 0$ by the inductive hypothesis, and $x_{k+1} \geq 0$ by the original conditions. Since the sum of two non-negative numbers is non-negative, the induction continues.

(c) **Base step** $p_1 = 1 - x_1 = 1 - s_1$.
Inductive step Assuming that $p_k \geq 1 - s_k$, we have to show that $p_{k+1} \geq 1 - s_{k+1}$. But

$$\begin{aligned} p_{k+1} &= p_k(1 - x_{k+1}) \\ &\geq (1 - s_k)(1 - x_{k+1}) \end{aligned}$$

(because $p_k \geq 1 - s_k$ by the inductive hypothesis, and $1 - x_{k+1} \geq 0$ because $x_{k+1} \leq 1$)

$$\begin{aligned} &= 1 - s_k - x_{k+1} + s_k x_{k+1} \\ &= 1 - s_{k+1} + s_k x_{k+1} \geq 1 - s_{k+1} \end{aligned}$$

because (by (b)) $s_k \geq 0$, so $s_k x_{k+1} \geq 0$. Thus the induction continues.

(d) **Base step** $p_1 = 1 - x_1 \leq 1 - \frac{1}{2}x_1 = 1 - \frac{1}{2}s_1$ because $x_1 \geq 0$.
Inductive step Assuming that 'if $s_k \leq 1$, then $p_k \leq 1 - \frac{1}{2}s_k$', we have to show that 'if $s_{k+1} \leq 1$ then $p_{k+1} \leq 1 - \frac{1}{2}s_{k+1}$'.
Now since $s_{k+1} \leq 1$, $s_k = s_{k+1} - x_{k+1} \leq s_{k+1} \leq 1$. So the inductive hypothesis really does tell us that $p_k \leq 1 - \frac{1}{2}s_k$. Accordingly

$$\begin{aligned} p_{k+1} &= p_k(1 - x_{k+1}) \\ &\leq (1 - \frac{1}{2}s_k)(1 - x_{k+1}) \end{aligned}$$

(because $p_k \leq 1 - \frac{1}{2}s_k$ and $1 - x_{k+1} \geq 0$)

$$\begin{aligned} &= 1 - \frac{1}{2}s_k - x_{k+1} + \frac{1}{2}s_k x_{k+1} \\ &= 1 - \frac{1}{2}s_k - \frac{1}{2}x_{k+1} - \frac{1}{2}x_{k+1} + \frac{1}{2}s_k x_{k+1} \\ &= 1 - \frac{1}{2}s_{k+1} - \frac{1}{2}x_{k+1}(1 - s_k) \leq 1 - \frac{1}{2}s_{k+1} \end{aligned}$$

because $x_{k+1} \geq 0$ and $s_k \leq 1$, so $1 - s_k \geq 0$. Thus the induction continues.

(a) $n!$.

(b) $q_1 = 0$ (you can't miss) and $q_2 = 1$ (they must be switched over).

For the general formula, look at the first card. There are $n-1$ envelopes it could go into, since it's not allowed to go into its own.

Now look at the card belonging to the envelope the first card went into. There are two possibilities. Either it goes into the envelope belonging to the first card, or it doesn't. (i) If the second card goes into the first card's envelope, we now have $n-2$ cards to be put into $n-2$ envelopes; since they must all be wrong, there are q_{n-2} ways of doing it. (ii) Otherwise, we have $n-1$ cards to put into $n-1$ envelopes, and each card is barred from just one of them; its 'own' envelope in most cases, but in the case of the second card it's the first card's envelope which is forbidden. But this means that there are q_{n-1} possibilities.

So for each assignment of the first card to an envelope, there are $q_{n-1} + q_{n-2}$ ways of assigning the rest. Accordingly there are $(n-1)(q_{n-1} + q_{n-2})$ ways of arranging all the cards such that none is in the right
59. envelope.

(c) Induce on n (using the First Principle).

Base step $a_2 - a_1 = \frac{q_2}{2!} - \frac{q_1}{1!} = \frac{1}{2} - \frac{0}{1} = \frac{1}{2} = \frac{(-1)^2}{2!}$.

Inductive step Assuming that $a_k - a_{k-1} = \frac{(-1)^k}{k!}$, we have to show that $a_{k+1} - a_k = \frac{(-1)^{k+1}}{(k+1)!}$. Now

$$a_{k+1} - a_k = \frac{q_{k+1}}{(k+1)!} - \frac{q_k}{k!} = \frac{k(q_k+q_{k-1})}{(k+1)!} - \frac{q_k}{k!}$$

(by part (b))

$$= \frac{1}{(k+1)!}(k(q_k + q_{k-1}) - (k+1)q_k)$$
$$= \frac{1}{(k+1)!}(kq_k + kq_{k-1} - kq_k - q_k)$$
$$= \frac{1}{(k+1)!}(kq_{k-1} - q_k)$$
$$= \frac{1}{(k+1)!}(k(k-1)!a_{k-1} - k!a_k)$$
$$= \frac{1}{(k+1)!}k!(a_{k-1} - a_k) = \frac{1}{(k+1)!}(-(-1)^k)$$

(by the inductive hypothesis)

$$= \frac{(-1)^{k+1}}{(k+1)!}.$$

So the induction proceeds.

(d)(i)

$$a_n = (a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \ldots + (a_2 - a_1) + a_1$$
$$= \sum_{r=2}^{n} a_r - a_{r-1}$$

(because $a_1 = 0$)

$$= \sum_{r=2}^{n} \frac{(-1)^r}{r!}$$

(by part (c))

$$= \sum_{r=0}^{n} \frac{(-1)^r}{r!}$$

because $\frac{(-1)^0}{0!} + \frac{(-1)^1}{1!} = \frac{1}{1} - \frac{1}{1} = 0$. So

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sum_{r=0}^{n} \frac{(-1)^r}{r!} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} = \exp(-1) = \frac{1}{e} \simeq 0{\cdot}37.$$

(ii) For large values of n, the probability that at least one will find its way into the right envelope is $\frac{n!-q_n}{n!} = 1 - a_n \simeq 1 - \frac{1}{e} \simeq 0{\cdot}63$.

60. (a) Induce on $i + j$ (using the First Principle).

Base step If $i+j = 1$ then one of the lists is empty and we need do nothing, that is, we use $i+j-1 = 0$ comparisons.

Inductive step Assuming that whenever $i+j = k$ we can sort the lists with at most $k-1$ comparisons, we need to show that whenever $i + j = k + 1$ we can sort the lists with at most k comparisons.

If one of the lists is empty, we do not need any comparisons; we just take the other list as it stands. Otherwise, compare the first elements of the lists. Take the smaller, and cut it off that list. Now our lists are either of lengths $i - 1$ and j or of lengths i and $j - 1$. In either case the sum of their lengths is $(i - 1) + j = i + (j - 1) = k$, so we can sort the rest with at most $k - 1$ comparisons (by the inductive hypothesis). So the whole lot has been sorted with at most $1 + (k - 1) = k$ comparisons. Thus the induction continues.

(b) (Use the First Principle.)

Base step If $n = 1$ then $a_1 = 1 \leq 2 = 2^1 \times 1$.

Inductive step Assuming that $a_k \leq 2^k k$ we have to show that $a_{k+1} \leq 2^{k+1}(k + 1)$. But

$$a_{k+1} \leq 2a_k + 2^{k+1} \leq 2 \times 2^k k + 2^{k+1}$$

(by the inductive hypothesis)

$$= 2^{k+1}k + 2^{k+1} = 2^{k+1}(k + 1).$$

Thus the induction continues.

(c) Let a_n be the number of comparisons needed to sort a list with 2^n members. If $n = 1$ we just have 2 numbers to sort, so we can do it with one comparison, and $a_1 = 1$. If we have to sort 2^{n+1} numbers, we can break them into two sets of 2^n numbers each, sort each of them, and then amalgamate the lists as in (a). This will take at most a_n comparisons (for the first half-list), plus a_n comparisons (for the second half-list), plus $2^{n+1} - 1$ comparisons (for the amalgamation, using part (a)). So

$$a_{n+1} \leq a + n + a_n + 2^{n+1} - 1 \leq 2a_n + 2^{n+1}.$$

And this works for every $n \geq 1$. So (b) tells us that $a_n \leq 2^n n$ for every n.

(In fact a_n is rather less than $2^n n$. For instance, $a_2 = 5$ rather than $2^2 \times 2 = 8$.)

61. (a) Induce on n (using the First Principle).
Base step If $n = 0$ then $a_{2n+1} = a_1 = 1 = 1 + 0 = 1 + a_0$.
Inductive step Assuming that $1 + a_0 + \ldots + a_{2k} = a_{2k+1}$ we need to prove that $1 + a_0 + \ldots + a_{2(k+1)} = a_{2(k+1)+1}$.
Now

$$a_{2(k+1)+1} = a_{2k+3} = a_{2k+1} + a_{2k+2}$$

(by the definition of the Fibonacci numbers)

$$= 1 + a_0 + \ldots + a_{2k} + a_{2k+2}$$

(by the inductive hypothesis)

$$= 1 + a_0 + \ldots + a_{2(k+1)},$$

as required. So the induction continues.

(b) Induce on n (using the Second Principle).
Base step If $n = 1$ then $b_1 = 1 \times b_0 = 1 = a_2 = a_{2\times 1}$.
Inductive step Assuming that $b_i = a_{2i}$ whenever $1 \leq i < k$, where $k > 1$, we have to show that $b_k = a_{2k}$.
Now

$$b_k = kb_0 + (k-1)b_1 + \ldots + 2b_{k-2} + b_{k-1}$$

(by the definition of the b_n)

$$= ((k-1)b_0 + b_0) + ((k-2)b_1 + b_1) + \ldots + (b_{k-2} + b_{k-2}) + b_{k-1}$$
$$= ((k-1)b_0 + ((k-2)b_1 + \ldots + b_{k-2}) + (b_0 + b_1 + \ldots + b_{k-2} + b_{k-1})$$
$$= b_{k-1} + (b_0 + b_1 + \ldots + b_{k-2} + b_{k-1})$$

(by the definition of b_{k-1})

$$= a_{2(k-1)} + 1 + a_2 + \ldots + a_{2(k-1)}$$

(by the inductive hypothesis)

$$= a_{2(k-1)} + 1 + a_0 + a_2 + \ldots + a_{2(k-1)}$$

(because $a_0 = 0$)

$$= a_{2(k-1)} + a_{2(k-1)+1}$$

(by part (a) above)

$$= a_{2k-2} + a_{2k-1} = a_{2k}$$

(by the definition of a_{2k}). So the induction continues.

62. In the line
(by the inductive hypothesis) $2^{k-2} = 2^{k-1} = 1$
we didn't check that $0 \leq k-2 < k$, $0 \leq k-1 < k$. This is nearly true! but if $k = 1$ then $k - 2 < 0$, and we can't safely use the inductive hypothesis. In fact, of course, $2^{-1} \neq 1$, so

$$2^1 = \frac{2^0 \times 2^0}{2^{-1}}$$

isn't equal to 1 either.

63. **Base step** If $m = 1$ then $\frac{m}{n} = \frac{1}{n}$ is already a reciprocal and we can stop. So the induction starts.

Inductive step Assuming that the result is true whenever $1 \le m < k$, where $k > 1$, we have to show that it is true when $m = k$.

So take any $n \ge k$ and look at $\frac{k}{n}$. This is greater than 0 and at most 1, so there is a least r such that $\frac{1}{r} \le \frac{k}{n}$, that is, $n \le rk$. If $\frac{1}{r} = \frac{k}{n}$, we can stop. Otherwise, $n < rk$, and $\frac{k}{n} - \frac{1}{r} = \frac{rk-n}{nr}$. But note that r is the least number such that $\frac{1}{r} \le \frac{k}{n}$, so $\frac{k}{n} < \frac{1}{r-1}$, that is, $(r-k)k < n$, that is, $rk - n < k$. So $0 < rk - n < k$, and by the inductive hypothesis we can express $\frac{rk-n}{nr}$ as a sum of distinct reciprocals; say $\frac{rk-n}{nr} = \frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_j}$, where $r_1, \ldots, r_j$ are all diifferent. Now our original fraction $\frac{k}{n} = \frac{1}{r} + \frac{1}{r_1} + \ldots + \frac{1}{r_j}$ is expressed as a sum of reciprocals.

We still have to check that r, $r_1, \ldots, r_j$ are all distinct. But we know that for $1 \le i \le j$, $\frac{1}{r_i} \le \frac{rk-n}{nr} < \frac{n}{nr} = \frac{1}{r}$, because $rk - n < k \le n$; so we must have $r_i > r$. And all the $r_1, \ldots, r_j$ are different because that's how we picked them when we applied the inductive hypothesis.

Thus the induction continues.