Version of 31.10.17

Chapter 65

Applications

At long last I turn again to some of the results for which the theory outlined in this volume was developed. I start with some relatively elementary ideas using nothing more advanced than §624, showing that locally jump-free virtual local martingales are associated with 'exponential' processes of the same kind (651C). These in turn are associated with identities for integral equations (651G, 651K) and change-of-law results (651I). Ideas at the same level take us to Lévy's characterization of Brownian motion (653F); going deeper, and using the time-changes of §635, we can represent many locally jump-free local martingales in terms of Brownian motion (653G).

The exponential processes of §651 can be thought of as solutions of a particularly simple kind of stochastic differential equation. Working very much harder, we find that we have versions of Picard's theorem, for integral equations with a Lipschitz condition on the integrand, for both the Riemann-sum integral (654G) and the S-integral (654L). A twist in the theory of exponential processes, with a refinement inspired by the theory of financial markets, leads us to the famous Black-Scholes equation (655D).

Version of 11.9.23

651 Exponential processes

Associated with any jump-free integrator is an 'exponential process' (651B); if the integrator is a martingale, the exponential process may be a uniformly integrable martingale (651D-651E). This gives us an important class of non-negative martingales which we can use in change-of-law results (651J).

651A Notation

651B Theorem Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$. Suppose that $\boldsymbol{v}$ is a locally jump-free local integrator, and $\boldsymbol{u}$ a locally moderately oscillatory process, both with domain $\mathcal{S}$. Set $\boldsymbol{z}=\overline{\exp}(\boldsymbol{v}-v_{\downarrow}\mathbf{1}-\frac{1}{2}\boldsymbol{v}^*)$. Then $\boldsymbol{z}$ is a locally jump-free local integrator, $\boldsymbol{z}=\mathbf{1}+ii_{\boldsymbol{v}}(\boldsymbol{z})$ and $ii_{\boldsymbol{z}}(\boldsymbol{u})=ii_{\boldsymbol{v}}(\boldsymbol{u}\times\boldsymbol{z})$. If $\boldsymbol{v}$ is in fact a jump-free integrator and $\boldsymbol{u}$ a moderately oscillatory process, then $\boldsymbol{z}$ is a jump-free integrator

651C Corollary Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, and $\boldsymbol{v}$ a locally jump-free virtually local martingale with domain $\mathcal{S}$. Then $\boldsymbol{z}=\overline{\exp}(\boldsymbol{v}-v_{\downarrow}\mathbf{1}-\frac{1}{2}\boldsymbol{v}^*)$ is a locally jump-free virtually local martingale.

651D Theorem Let $\mathcal{S}$ be a non-empty finitely full sublattice of $\mathcal{T}$ and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally jump-free virtually local martingale. Let $\boldsymbol{z}=\overline{\exp}(\boldsymbol{v}-v_{\downarrow}\mathbf{1}-\frac{1}{2}\boldsymbol{v}^*)$ be the associated exponential process. If $\sup_{\sigma\in\mathcal{S}}\mathbb{E}(\overline{\exp}(\frac{1}{2}(v_\sigma-v_{\downarrow})))$ is finite, then $\boldsymbol{z}$ is a uniformly integrable martingale.

651E Corollary Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a locally jump-free virtually local martingale with domain $\mathcal{S}$. Let $\boldsymbol{z}=\overline{\exp}(\boldsymbol{v}-v_{\downarrow}\mathbf{1}-\frac{1}{2}\boldsymbol{v}^*)$ be the associated exponential process.

(a) If $\overline{\exp}(\frac{1}{2}\boldsymbol{v}^*)$ is $\| \|_1$-bounded, then $\boldsymbol{z}$ is a uniformly integrable martingale.

(b) If $\boldsymbol{v}^*$ is an L^∞-process, then $\boldsymbol{z}$ is a martingale.

651F Corollary If $\boldsymbol{w}$ is Brownian motion, then $\overline{\exp}(\boldsymbol{w}-\frac{1}{2}\boldsymbol{\iota})\restriction\mathcal{T}_b$ is a martingale.

651G Theorem Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, and $\boldsymbol{v}$, $\boldsymbol{w}$ locally jump-free local integrators with domain $\mathcal{S}$. Set $\boldsymbol{z}=\overline{\exp}(\boldsymbol{v}-v_{\downarrow}\mathbf{1}-\frac{1}{2}\boldsymbol{v}^*)$ and $\boldsymbol{y}=\boldsymbol{w}-[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]$. Then

$$ii_{\boldsymbol{y}\times\boldsymbol{z}}(\boldsymbol{u})=ii_{\boldsymbol{w}}(\boldsymbol{u}\times\boldsymbol{z})+ii_{\boldsymbol{v}}(\boldsymbol{u}\times\boldsymbol{y}\times\boldsymbol{z})$$

for any locally moderately oscillatory process $\boldsymbol{u}$ with domain $\mathcal{S}$.

651H Corollary Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, and $\boldsymbol{v}$, $\boldsymbol{w}$ locally jump-free virtually local martingales. Set $\boldsymbol{z} = \overline{\exp}(\boldsymbol{v} - v_{\downarrow}\mathbf{1} - \frac{1}{2}\boldsymbol{v}^*)$ and $\boldsymbol{y} = \boldsymbol{w} - [\boldsymbol{w} \overset{*}{|} \boldsymbol{v}]$. Then $\boldsymbol{y} \times \boldsymbol{z}$ is a virtually local martingale.

651I Proposition Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$, $\boldsymbol{w}$ locally jump-free virtually local martingales such that $\boldsymbol{z} = \overline{\exp}(\boldsymbol{v} - v_{\downarrow}\mathbf{1} - \frac{1}{2}\boldsymbol{v}^*)$ is a uniformly integrable martingale and $\lim_{\sigma \uparrow \mathcal{S}}(v_\sigma - \frac{1}{2}v_\sigma^*)$ is defined in L^0, where $\boldsymbol{v}^* = \langle v_\sigma^* \rangle_{\sigma \in \mathcal{S}}$. Set $\boldsymbol{y} = \boldsymbol{w} - [\boldsymbol{w} \overset{*}{|} \boldsymbol{v}]$. Then there is a change of law on $\mathfrak{A}$ rendering $\boldsymbol{y}$ a virtually local martingale.

651J Corollary Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, $\boldsymbol{u}$ a locally moderately oscillatory process such that $\gamma = \|\boldsymbol{u}\|_\infty$ is finite, and $\boldsymbol{w}$ a locally jump-free virtually local martingale with quadratic variation $\boldsymbol{w}^*$ such that $\overline{\exp}(\frac{1}{2}\gamma^2\boldsymbol{w}^*)$ is a $\| \, \|_1$-bounded process. Set $\boldsymbol{y} = \boldsymbol{w} + ii_{\boldsymbol{w}^*}(\boldsymbol{u})$. Then there is a change of law on $\mathfrak{A}$ rendering $\boldsymbol{y}$ a virtually local martingale.

651K S-integrals: Theorem Suppose that $\langle \mathfrak{A}_t \rangle_{t \in T}$ is right-continuous, and that $\mathcal{S}$ is a non-empty order-convex sublattice of $\mathcal{T}$. Let $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ be a jump-free integrator, and $\boldsymbol{x}$ an S-integrable process with domain $\mathcal{S}$. Set $\boldsymbol{z} = \overline{\exp}(\boldsymbol{v} - v_{\downarrow}\mathbf{1} - \frac{1}{2}\boldsymbol{v}^*)$.

(a)

$$\oint_{\mathcal{S}} \boldsymbol{x} \, d\boldsymbol{z} = \oint_{\mathcal{S}} \boldsymbol{x} \times \boldsymbol{z} \, d\boldsymbol{v}.$$

(b) Suppose that $\boldsymbol{w}$ is another jump-free integrator with domain $\mathcal{S}$. Set $\boldsymbol{y} = \boldsymbol{w} - [\boldsymbol{v} \overset{*}{|} \boldsymbol{w}]$. Then

$$\oint_{\mathcal{S}} \boldsymbol{x} \, d(\boldsymbol{y} \times \boldsymbol{z}) = \oint_{\mathcal{S}} \boldsymbol{x} \times \boldsymbol{z} \, d\boldsymbol{w} + \oint_{\mathcal{S}} \boldsymbol{x} \times \boldsymbol{y} \times \boldsymbol{z} \, d\boldsymbol{v}.$$

Version of 7.1.23/18.2.23

652 Lévy processes

When defining the Poisson process in 612U, I referred to 455P in Volume 4. §455 is hard work; it is the longest section in the whole treatise and bristles with technical difficulties. Some of these are exacerbated by the generality which seemed natural at that point – for instance, it is meant to support the treatment of multidimensional Brownian motion in Chapter 47. The processes described in the last quarter of §455 take values in Polish groups which need not even be abelian. But these processes have always been the standard-bearers for the theory of stochastic processes I have set out to describe in the present volume, and the time has come to link the approaches.

652B Independence (a) If $(\mathfrak{A}, \bar{\mu})$ is a probability algebra, two subalgebras $\mathfrak{B}$, $\mathfrak{C}$ of $\mathfrak{A}$ are (stochastically) independent if $\bar{\mu}(b \cap c) = \bar{\mu}b \cdot \bar{\mu}c$ whenever $b \in \mathfrak{B}$ and $c \in \mathfrak{C}$; more generally, a family $\langle \mathfrak{B}_i \rangle_{i \in I}$ of subalgebras of $\mathfrak{A}$ is independent if $\bar{\mu}(\inf_{i \in J} b_i) = \prod_{i \in J} \bar{\mu}b_i$ whenever $J \subseteq I$ is finite and $b_i \in \mathfrak{B}_i$ for every $i \in J$. Turning to families in $L^0 = L^0(\mathfrak{A})$, $\langle u_i \rangle_{i \in I}$ is independent if $\langle \mathfrak{B}_i \rangle_{i \in I}$ is independent where $\mathfrak{B}_i = \{[\![u_i \in E]\!] : E \subseteq \mathbb{R}$ is Borel$\}$ is the closed subalgebra generated by u_i for each $i \in I$. $u \in L^0$ is independent of an algebra $\mathfrak{C} \subseteq \mathfrak{A}$ if $\mathfrak{B}$ and $\mathfrak{C}$ are independent where $\mathfrak{B}$ is the closed subalgebra generated by u. More elaborately, a family $\langle u_i \rangle_{i \in I}$ in L^0 is independent of $\mathfrak{C}$, where $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$, if $\mathfrak{B}$ and $\mathfrak{C}$ are independent, where $\mathfrak{B}$ is the smallest closed subalgebra of $\mathfrak{A}$ including all the subalgebras generated by the u_i.

(b) Using the Monotone Class Theorem, we see that if B, $C \subseteq \mathfrak{A}$ are such that both B and C are closed under $\cap$ and $\bar{\mu}(b \cap c) = \bar{\mu}b \cdot \bar{\mu}c$ for all $b \in B$ and $c \in C$, then the closed subalgebras $\mathfrak{B}$, $\mathfrak{C}$ generated by B, C respectively are independent.

(c) If $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, the set $C = \{u : u \in L^0, u$ is independent of $\mathfrak{B}\}$ is closed for the topology of convergence in measure.

652C Definition Let $(\mathfrak{A}, \bar{\mu}, [0, \infty[, \langle \mathfrak{A}_t \rangle_{t \geq 0}, \mathcal{T}, \langle \mathfrak{A}_\tau \rangle_{\tau \in \mathcal{T}})$ be a right-continuous real-time stochastic integration structure. I will say that a fully adapted process $\langle v_\sigma \rangle_{\sigma \in \mathcal{T}_f}$ is a **Lévy process** if it is locally near-simple and whenever $s,\,t \ge 0$, $v_{(s+t)\check{}} - v_{\check{s}}$ is independent of $\mathfrak{A}_s$ and has the same distribution as $v_{\check{t}}$.

Examples (i) The identity process is a Lévy process.

(ii) Brownian motion, is a Lévy process.

(iii) The standard Poisson process is a Lévy process.

652D Lemma If $(\mathfrak{A},\bar{\mu},[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ is a right-continuous real-time stochastic integration structure and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ a Lévy process, then $v_{\check{0}} = \lim_{t\downarrow0} v_{\check{t}} = 0$.

652E Proposition Let $(\mathfrak{A},\bar{\mu},[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a right-continuous real-time stochastic integration structure, and $\boldsymbol{v} = \langle v_\tau\rangle_{\tau\in\mathcal{T}_f}$, $\boldsymbol{w} = \langle w_\tau\rangle_{\tau\in\mathcal{T}_f}$ two Lévy processes. If $v_{\check{t}} = w_{\check{t}}$ for every $t \ge 0$, then $\boldsymbol{v} = \boldsymbol{w}$.

652F Classical Lévy processes: Proposition Let C_{dlg} be the space of càdlàg real-valued functions on $[0,\infty[$, endowed with its topology of pointwise convergence. Let $\langle\lambda_t\rangle_{t>0}$ be a family of distributions such that the convolution $\lambda_s * \lambda_t$ is equal to λ_{s+t} for all $s,\,t>0$, and $\lim_{t\downarrow0}\lambda_t G = 1$ for every open subset G of $\mathbb{R}$ containing 0. Let $\ddot{\mu}$ be the completion regular quasi-Radon probability measure on C_{dlg} defined by saying that

$$\begin{aligned}\ddot{\mu}\{\omega : \omega\in C_{\text{dlg}},\ \omega(s_0)\in E_0,\ \omega(s_i)-\omega(s_{i-1})\in E_i \text{ for } 1\le i\le n\}\\ = \delta_0E_0\cdot\prod_{i=1}^n \lambda_{s_i-s_{i-1}}E_i \qquad (*)\end{aligned}$$

whenever $0 = s_0 < \ldots < s_n$ in $[0,\infty[$ and $E_0,\ldots,E_n \subseteq \mathbb{R}$ are Borel sets, and $\ddot{\Sigma}$ its domain. For $t\ge0$, set

$$\ddot{\Sigma}_t = \{F : F\in\ddot{\Sigma},\ \omega'\in F \text{ whenever } \omega\in F,\ \omega'\in C_{\text{dlg}} \text{ and } \omega'{\restriction}[0,t] = \omega{\restriction}[0,t]\},$$

$$\hat{\ddot{\Sigma}}_t = \{F\triangle A : F\in\ddot{\Sigma}_t,\ A\in\mathcal{N}(\ddot{\mu})\}$$

so that $\langle\hat{\ddot{\Sigma}}_t\rangle_{t\ge0}$ is a right-continuous filtration of σ-subalgebras of $\ddot{\Sigma}$. Let $(\mathfrak{C},\bar{\ddot{\mu}})$ be the measure algebra of $(C_{\text{dlg}},\ddot{\Sigma},\ddot{\mu})$ and set $\mathfrak{C}_t = \{E^\bullet : E\in\hat{\ddot{\Sigma}}_t\}$ for $t\ge0$, so that $(\mathfrak{C},\bar{\ddot{\mu}},[0,\infty[\,,\langle\mathfrak{C}_t\rangle_{t\ge0})$ is a right-continuous stochastic integration structure with associated set $\mathcal{T}$ of stopping times and family $\langle\mathfrak{C}_\tau\rangle_{\tau\in\mathcal{T}}$ of closed subalgebras. Setting $X_t(\omega) = \omega(t)$ for $\omega\in\Omega$ and $t\ge0$, $\langle X_t\rangle_{t\ge0}$ is progressively measurable and gives rise to a locally near-simple process $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ such that if $\sigma\in\mathcal{T}_f$ is represented by a stopping time $h : C_{\text{dlg}}\to[0,\infty[$ adapted to $\langle\hat{\ddot{\Sigma}}_t\rangle_{t\ge0}$, and $X_h(\omega) = X_{h(\omega)}(\omega)$ for $\omega\in C_{\text{dlg}}$, then $u_\sigma = X_h^\bullet$ in $L^0(\mu)$. Now $\boldsymbol{u}$ is a Lévy process, and the distribution of $u_{\check{t}}$ is λ_t for every $t\ge0$.

652G Sums of stopping times Let $(\mathfrak{A},\bar{\mu},[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a real-time stochastic integration structure.

(a) If $\tau\in\mathcal{T}$ and $s\ge0$ we have an element $\tau+\check{s}$ of $\mathcal{T}$ defined by saying that

$$\begin{aligned}[\![\tau+\check{s}>t]\!] &= [\![\tau>t-s]\!] \text{ if } s\le t,\\ &= 1 \text{ if } s>t.\end{aligned}$$

(b)(i) $\tau\vee\check{s}\le\tau+\check{s}$ whenever $\tau\in\mathcal{T}$ and $s\ge0$.

(ii) If $\tau\in\mathcal{T}$ and $s,\,s'\ge0$ then $\tau+(s+s')\check{} = (\tau+\check{s})+\check{s}'$.

(iii) $\tau+\check{0} = \tau$ for every $\tau\in\mathcal{T}$.

(iv) If $s,\,s'\ge0$ then $\check{s}+\check{s}' = (s+s')\check{}$.

(v) If $\tau,\,\tau'\in\mathcal{T}$ and $s\ge0$ then $[\![\tau\le\tau']\!]\subseteq[\![\tau+\check{s}\le\tau'+\check{s}]\!]$.

(vi) For any $\tau\in\mathcal{T}$, $\{\tau+\check{s}: s\ge0\}$ separates $\mathcal{T}\vee\tau$.

(vii) Suppose that $\langle\mathfrak{A}_t\rangle_{t\ge0}$ is right-continuous. If $A\subseteq\mathcal{T}$, $s\ge0$ and we write $A+\check{s}$ for $\{\sigma+\check{s}:\sigma\in A\}$, then $\inf(A+\check{s}) = (\inf A)+\check{s}$.

652H Proposition Let $(\mathfrak{A}, \bar{\mu}, [0,\infty[\,, \langle\mathfrak{A}_t\rangle_{t\ge 0}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a right-continuous real-time stochastic integration structure, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ a Lévy process.

(a) For any $s \ge 0$ and $\tau \in \mathcal{T}_f$, $v_{\tau+\check{s}} - v_\tau$ is independent of $\mathfrak{A}_\tau$ and has the same distribution as $v_{\check{s}}$.

(b) For any $\tau \in \mathcal{T}_f$, $\langle v_{\tau+\check{s}} - v_\tau\rangle_{s\ge 0}$ is independent of $\mathfrak{A}_\tau$ and has the same distribution as $\langle v_{\check{s}}\rangle_{s\ge 0}$.

652I Theorem Let $(\mathfrak{A}, \bar{\mu}, [0,\infty[\,, \langle\mathfrak{A}_t\rangle_{t\ge 0}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a right-continuous real-time stochastic integration structure, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ a Lévy process such that $\mathrm{Osclln}(\boldsymbol{v}\restriction[\check{0},\tau]) \in L^\infty(\mathfrak{A})$ for every $\tau \in \mathcal{T}_b$. Then $\boldsymbol{v}\restriction\mathcal{T}_b$ is an L^1-process.

652J Proposition Let $(\mathfrak{A}, \bar{\mu}, [0,\infty[\,, \langle\mathfrak{A}_t\rangle_{t\ge 0}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a right-continuous real-time stochastic integration structure, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ a Lévy process such that $\boldsymbol{v}\restriction\mathcal{T}_b$ is an L^1-process. Then there is an $\alpha \in \mathbb{R}$ such that $\boldsymbol{v} - \alpha\boldsymbol{\iota}$ is a local martingale.

652K Theorem Let $(\mathfrak{A}, \bar{\mu}, [0,\infty[\,, \langle\mathfrak{A}_t\rangle_{t\ge 0}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a right-continuous real-time stochastic integration structure, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ a Lévy process. Then $\boldsymbol{v}$ is a semi-martingale, therefore a local integrator.

652L Proposition Let $(\mathfrak{A}, \bar{\mu}, [0,\infty[\,, \langle\mathfrak{A}_t\rangle_{t\ge 0}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a right-continuous real-time stochastic integration structure, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ a Lévy process. Then its quadratic variation $\boldsymbol{v}^*$ is a Lévy process.

652M The Cauchy process (a) If $t > 0$, then we have a distribution λ_t on $\mathbb{R}$ with density function $\xi \mapsto \frac{t}{\pi(t^2+\xi^2)}$, the **Cauchy distribution** with centre 0 and scale parameter t.

(b)(i) For $t > 0$ the characteristic function of λ_t is $\xi \mapsto e^{-t|\xi|}$.

(ii) $\int_0^\infty \frac{1-\cos\eta}{\eta^2}d\eta = \frac{\pi}{2}$.

(c) If s, $t > 0$ $\lambda_s * \lambda_t = \lambda_{s+t}$. We can apply the construction of 652F with the family $\langle\lambda_t\rangle_{t>0}$ to obtain a probability space $(C_{\mathrm{dlg}}, \ddot{\Sigma}, \ddot{\mu})$, a stochastic integration structure $(\mathfrak{C}, \bar{\ddot{\mu}}, [0,\infty[\,, \langle\mathfrak{C}_t\rangle_{t\ge 0})$ and a classical Lévy process $\boldsymbol{u}$, the **Cauchy process**.

(d) Writing $C([0,\infty[) \subseteq C_{\mathrm{dlg}}$ for the set of continuous functions from $[0,\infty[$ to $\mathbb{R}$, $\ddot{\mu}C([0,\infty[) = 0$.

652N Alternative description of the Cauchy process Let $\ddot{\mu}$ be the probability measure on C_{dlg} defined in 652M. Let μ_0 be the indefinite-integral measure over μ_L corresponding to the function $\xi \mapsto \dfrac{1}{\pi\xi^2}$: $\mathbb{R}\setminus\{0\} \to [0,\infty[$, μ_1 the subspace measure on $[0,\infty[$ induced by μ_L, and $\mu = \mu_0 \times \mu_1$ the c.l.d. product measure on $S = \mathbb{R}\times[0,\infty[$. Let ν be the Poisson point process on (S,μ) with intensity 1, and $\ddot{\mu}$ the measure on C_{dlg} described in 652Mc. For $\omega \in C_{\mathrm{dlg}}$ write $\mathrm{Jump}(\omega)$ for

$$\{(\xi,t) : \xi \in \mathbb{R}\setminus\{0\},\, t > 0,\, \omega(t) = \lim_{s\uparrow t}\omega(s) + \xi\}.$$

Then $\mathrm{Jump} : C_{\mathrm{dlg}} \to \mathcal{P}S$ is inverse-measure-preserving for $\ddot{\mu}$ and ν. If $E \subseteq S$ is such that μE is defined and finite then $\ddot{\mu}\{\omega : E \cap \mathrm{Jump}(\omega) = \emptyset\} = e^{-\mu E}$; in particular, if $t \ge 0$ and $\alpha > 0$, then

$$\ddot{\mu}\{\omega : \mathrm{Jump}(\omega) \cap ([\alpha,\infty[\times[0,t]) \ne \emptyset\} = 1 - e^{-t/\alpha}.$$

Suppose that μE is defined and finite, and $(-\xi, s) \in E$ whenever $(\xi, s) \in E$. Setting $X_E(\omega) = \sum_{(\xi,s)\in E\cap\mathrm{Jump}(\omega)}\xi$, $\mathbb{E}(X_E) = 0$ and $\mathbb{E}(X_E^2) = \int_E \xi^2\mu(d\xi)$ is finite.

For $\ddot{\mu}$-almost every $\omega \in C_{\mathrm{dlg}}$, $\mathrm{Jump}(\omega) \cap (\mathbb{R}\times[0,t])$ is countably infinite and can be enumerated as $\langle(\xi_{nt}(\omega), s_{nt}(\omega))\rangle_{n\in\mathbb{N}}$ where $|\xi_{n+1,t}(\omega)| < |\xi_{nt}(\omega)|$ for every n, for every $t > 0$. For any particular $t > 0$, $\omega(t) = \sum_{n=0}^\infty \xi_{nt}(\omega)$ for $\ddot{\mu}$-almost every ω.

652O Third construction for the Cauchy process (a) Write $\mu_W = \mu_{W2}$ for two-dimensional Wiener measure on the space $\Omega = C([0,\infty[\,;\mathbb{R}^2)_0$ of continuous functions from $[0,\infty[$ to $\mathbb{R}^2$ starting at zero. For $\omega \in \Omega$, write ω_0, ω_1 for its coordinates in $C([0,\infty[)_0$.

(b) For $t \ge 0$, let $\tilde{h}_t$ be the Brownian hitting time to $]t, \infty[\times \mathbb{R}$, and set

$$\begin{aligned}\tilde{Z}_t(\omega) &= \omega_1(\tilde{h}_t(\omega)) \text{ if } \omega_0[[0,\infty[\,] = \mathbb{R},\\ &= 0 \text{ otherwise.}\end{aligned}$$

Then $\langle \tilde{Z}_t \rangle_{t \ge 0}$ has the same distribution as the process $\langle Z_t \rangle_{t \ge 0}$ of 652N, and $t \mapsto \tilde{Z}_t(\omega)$ is càdlàg for every $\omega \in \Omega$.

(c) Writing Σ for the domain of μ_W,

$$\Sigma_t = \{E : E \in \Sigma,\ \omega' \in E \text{ whenever } \omega \in E,\ \omega' \in \Omega \text{ and } \omega' \restriction [0,t] = \omega \restriction [0,t]\},$$

$$\Sigma_{\tilde{h}_t} = \{E : E \in \Omega,\ E \cap \{\omega : \tilde{h}_t(\omega) \le s\} \in \Sigma_s \text{ for every } s \ge 0\},$$

$$\mathrm{T}_t = \{E \triangle A : E \in \textstyle\bigcap_{s > t} \Sigma_{\tilde{h}_s},\ A \in \mathcal{N}(\mu_W)\}$$

for $t \ge 0$, $\langle \mathrm{T}_t \rangle_{t \ge 0}$ is a right-continuous filtration of σ-algebras. $\tilde{Z}_t$ is T_t-measurable and $\tilde{Z}_s - \tilde{Z}_t$ is independent of T_t whenever $0 \le t \le s$.

(d) Applying the construction of 631D to (Ω, Σ, μ_W), $\langle \mathrm{T}_t \rangle_{t \ge 0}$ and $\langle \tilde{Z}_t \rangle_{t \ge 0}$ we get a Lévy process $\langle \tilde{z}_\sigma \rangle_{\sigma \in \mathcal{T}_f}$ in which the distribution of $\tilde{z}_{\check{t}}$ is the Cauchy distribution with scale parameter t for every $t \ge 0$.

652Z Problem Is the Cauchy process, as described in 652Mc, a local martingale?

Version of 22.9.20/30.9.23

653 Brownian processes

In 624F, we saw that the quadratic variation of Brownian motion is the identity process. In fact, under suitable conditions, this characterizes Brownian motion among local martingales (653F). Elaborating on the argument, we can show that (again under suitable conditions) general locally jump-free local martingales can be described in terms of Brownian motion and a time-change of the type considered in §635.

653B Distributions (a) If $k \ge 1$ and $u_1, \dots, u_k \in L^0$, we have a sequentially order-continuous function $E \mapsto [\![(u_1, \dots, u_k) \in E]\!]$ from the Borel σ-algebra $\mathcal{B}_k$ of $\mathbb{R}^k$ to $\mathfrak{A}$. This leads us to a Borel probability measure $E \mapsto \bar{\mu}[\![u \in E]\!] : \mathcal{B}_k \to [0,1]$; the completion of this measure is a Radon probability measure ν_U on $\mathbb{R}^k$, the **distribution** of $U = (u_1, \dots, u_k)$.

If $h : \mathbb{R}^k \to \mathbb{R}$ is a bounded Borel measurable function, then $\mathbb{E}(\bar{h}(U)) = \int h\, d\nu_U$.

(b) We can now speak of the corresponding characteristic function φ_{ν_U} where

$$\begin{aligned}\varphi_{\nu_U}(y) = \int e^{iy\,\boldsymbol{.}\,x} \nu_U(dx) &= \int \cos(y\,\boldsymbol{.}\,x)\nu_U(dx) + i \int \sin(y\,\boldsymbol{.}\,x)\nu_U(dx)\\ &= \mathbb{E}(\overline{\cos}(\eta_1 u_1 + \ldots + \eta_k u_k)) + i\mathbb{E}(\overline{\sin}(\eta_1 u_1 + \ldots + \eta_k u_k))\end{aligned}$$

for $y = (\eta_1, \dots, \eta_k) \in \mathbb{R}^k$, the characteristic function of $U = (u_1, \dots, u_k)$. If now $V \in (L^0)^k$ has the same characteristic function as U, it must have the same distribution as U.

(c) If $u_1, \dots, u_k$ are stochastically independent, then the distribution ν_U of $U = (u_1, \dots, u_k)$ is the product of the distributions ν_{u_i} of u_i.

653C Lemma Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a locally jump-free virtually local martingale such that its quadratic variation $\boldsymbol{v}^*$ is an L^∞-process. Writing $v_\downarrow$ for the starting value of $\boldsymbol{v}$,

$$\boldsymbol{z}_1 = \overline{\sin}(\boldsymbol{v} - v_\downarrow \mathbf{1}^{(\mathcal{S})}) \times \overline{\exp}(\tfrac{1}{2}\boldsymbol{v}^*), \quad \boldsymbol{z}_2 = \overline{\cos}(\boldsymbol{v} - v_\downarrow \mathbf{1}^{(\mathcal{S})}) \times \overline{\exp}(\tfrac{1}{2}\boldsymbol{v}^*)$$

are martingales.

653D Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ with least element τ and greatest element τ', and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a locally jump-free virtually local martingale with quadratic variation $\boldsymbol{v}^* = \langle v^*_\sigma \rangle_{\sigma \in \mathcal{S}}$. If $v_\tau = 0$ and $v^*_{\tau'} = \gamma\chi 1$ for some $\gamma > 0$, then $v_{\tau'}$ has a normal distribution with mean 0 and variance γ and is independent of $\mathfrak{A}_\tau$.

653E Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ with least element τ and greatest element τ', and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a locally jump-free virtually local martingale with quadratic variation $\boldsymbol{v}^* = \langle v^*_\sigma \rangle_{\sigma \in \mathcal{S}}$, starting from $v_\tau = 0$. If $\tau = \tau_0 \le \ldots \le \tau_k$ in $\mathcal{S}$ and $v^*_{\tau_j} = \gamma_j \chi 1$ for $j \le k$, where $0 = \gamma_0 \le \gamma_1 \le \ldots \le \gamma_k$, then $(v_{\tau_0}, \ldots , v_{\tau_k})$ has a centered Gaussian distribution with covariance matrix $\langle \gamma_{\min(j,l)} \rangle_{j,l \le k}$.

653F Theorem Let $(\mathfrak{A}, \bar\mu, [0, \infty[\,, \langle \mathfrak{A}_t \rangle_{t \ge 0}, \mathcal{T}, \langle \mathfrak{A}_\tau \rangle_{\tau \in \mathcal{T}})$ be a right-continuous real-time stochastic integration structure and $\boldsymbol{v} = \langle v_\tau \rangle_{\tau \in \mathcal{T}_f}$ a locally jump-free local martingale such that

(α) the quadratic variation of $\boldsymbol{v}$ is the identity process,

(β) $\mathfrak{A}$ is the closed subalgebra of itself defined by $\{v_{\check t} : t \ge 0\}$,

(γ) for each $t \ge 0$, $\mathfrak{A}_t$ is the closed subalgebra of $\mathfrak{A}$ defined by $\{v_{\check s} : s \le t\}$.

Then $(\mathfrak{A}, \bar\mu, [0, \infty[\,, \langle \mathfrak{A}_t \rangle_{t \ge 0}, \mathcal{T}, \langle \mathfrak{A}_\tau \rangle_{\tau \in \mathcal{T}}, \boldsymbol{v})$ is isomorphic to Brownian motion.

653G Theorem Suppose that $\langle \mathfrak{A}_t \rangle_{t \in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ with a least member, $\boldsymbol{v} = \langle v_\tau \rangle_{\tau \in \mathcal{S}}$ a locally jump-free local martingale such that $v_{\min \mathcal{S}} = 0$, and $\boldsymbol{v}^* = \langle v^*_\tau \rangle_{\tau \in \mathcal{S}}$ the quadratic variation of $\boldsymbol{v}$. Suppose that for every $n \in \mathbb{N}$ there is a $\tau \in \mathcal{S}$ such that $v^*_\tau \ge n\chi 1_{\mathfrak{A}}$. Let $(\mathfrak{C}, \bar\nu, \langle \mathfrak{C}_r \rangle_{r \ge 0}, \mathcal{Q}, \boldsymbol{w})$ be Brownian motion as described in 612T, again writing $\mathcal{Q}$ for the set of stopping times associated with $\langle \mathfrak{C}_r \rangle_{r \ge 0}$. Express $\boldsymbol{w}$ as $\langle w_\sigma \rangle_{\sigma \in \mathcal{Q}_f}$. Then there are ϕ, $\hat\pi$ and $\mathcal{Q}'$ such that

$\phi : \mathfrak{C} \to \mathfrak{A}$ is a measure-preserving Boolean homomorphism,

$\hat\pi : \mathcal{Q} \to \mathcal{T}$ is a right-continuous lattice homomorphism,

$\mathcal{Q}' = \{\rho : \rho \in \mathcal{Q}_f,\ \hat\pi(\rho) \in \mathcal{S}\}$ is an ideal in $\mathcal{Q}$ including the ideal $\mathcal{Q}_b$ of bounded stopping times,

taking $T_\phi : L^0(\mathfrak{C}) \to L^0(\mathfrak{A})$ to be the f-algebra homomorphism associated with ϕ and $\langle \iota_\rho \rangle_{\rho \in \mathcal{Q}_f}$ to be the identity process on $\mathcal{Q}_f$, $v_{\hat\pi(\rho)} = T_\phi(w_\rho)$ and $v^*_{\hat\pi(\rho)} = T_\phi(\iota_\rho)$ for every $\rho \in \mathcal{Q}'$,

if $\boldsymbol{u} = \langle u_\tau \rangle_{\tau \in \mathcal{S}}$ and $\boldsymbol{z} = \langle z_\rho \rangle_{\rho \in \mathcal{Q}'}$ are locally moderately oscillatory processes such that $T_\phi(z_\rho) = u_{\hat\pi(\rho)}$ for every $\rho \in \mathcal{Q}'$, then $\int_{\mathcal{S} \wedge \tau} \boldsymbol{u}\, d\boldsymbol{v} = T_\phi(\int_{\mathcal{Q} \wedge \rho} \boldsymbol{z}\, d\boldsymbol{w})$ whenever $\tau \in \mathcal{S}$ and $\rho \in \mathcal{Q}'$ are such that $v^*_\tau = T_\phi(\iota_\rho)$.

653I Corollary Suppose that $\langle \mathfrak{A}_t \rangle_{t \in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ with a least member, $\boldsymbol{v} = \langle v_\tau \rangle_{\tau \in \mathcal{S}}$ a locally jump-free local martingale such that $v_{\min \mathcal{S}} = 0$, and $\boldsymbol{v}^* = \langle v^*_\tau \rangle_{\tau \in \mathcal{S}}$ the quadratic variation of $\boldsymbol{v}$. Suppose that for every $n \in \mathbb{N}$ there is a $\tau \in \mathcal{S}$ such that $v^*_\tau \ge n\chi 1_{\mathfrak{A}}$. Let $(\mathfrak{C}, \bar\nu, \langle \mathfrak{C}_r \rangle_{r \in [0,\infty[}, \mathcal{Q}, \boldsymbol{w})$ be Brownian motion. Then there are ϕ and $\hat\pi$ such that

$\phi : \mathfrak{C} \to \mathfrak{A}$ is a measure-preserving Boolean homomorphism,

$\hat\pi : \mathcal{Q} \to \mathcal{T}$ is a lattice homomorphism,

if $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, then, taking $T_\phi : L^0(\mathfrak{C}) \to L^0(\mathfrak{A})$ to be the f-algebra homomorphism associated with ϕ, $\int_{\mathcal{S} \wedge \hat\pi(\rho)} \bar f(\boldsymbol{v}, \boldsymbol{v}^*)\, d\boldsymbol{v} = T_\phi(\int_{\mathcal{Q} \wedge \rho} \bar f(\boldsymbol{w}, \boldsymbol{\iota})\, d\boldsymbol{w})$ whenever $\rho \in \mathcal{Q}_f \cap \hat\pi^{-1}[\mathcal{S}]$.

653J Corollary Suppose that $\langle \mathfrak{A}_t \rangle_{t \in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ with a least member, $\boldsymbol{v} = \langle v_\tau \rangle_{\tau \in \mathcal{S}}$ a locally jump-free local martingale such that $v_{\min \mathcal{S}} = 0$, and $\boldsymbol{v}^* = \langle v^*_\tau \rangle_{\tau \in \mathcal{S}}$ the quadratic variation of $\boldsymbol{v}$. Suppose that for every $n \in \mathbb{N}$ there is a $\tau \in \mathcal{S}$ such that $v^*_\tau \ge n\chi 1_{\mathfrak{A}}$. Let $(\mathfrak{C}, \bar\nu, \langle \mathfrak{C}_r \rangle_{r \in [0,\infty[}, \mathcal{Q}, \boldsymbol{w})$ be Brownian motion. Then there are ϕ and $\hat\pi$ such that

$\phi : \mathfrak{C} \to \mathfrak{A}$ is a measure-preserving Boolean homomorphism,

$\hat\pi : \mathcal{Q} \to \mathcal{T}$ is a lattice homomorphism,

if $h : \mathbb{R}^2 \to \mathbb{R}$ is locally bounded and Borel measurable then, taking $T_\phi : L^0(\mathfrak{C}) \to L^0(\mathfrak{A})$ to be the f-algebra homomorphism associated with ϕ, $\oint_{\mathcal{S} \wedge \hat\pi(\rho)} \bar h(\boldsymbol{v}, \boldsymbol{v}^*)\, d\boldsymbol{v} = T_\phi(\oint_{\mathcal{Q} \wedge \rho} \bar h(\boldsymbol{w}, \boldsymbol{\iota})\, d\boldsymbol{w})$ whenever $\rho \in \mathcal{Q}_f \cap \hat\pi^{-1}[\mathcal{S}]$.

653K Brownian processes: Definition A **Brownian-type process** is a locally jump-free virtually local martingale $\boldsymbol{v}$, defined on a lattice $\mathcal{S}$ of stopping times based on a real-time stochastic integration structure, such that the quadratic variation of $\boldsymbol{v}$ is $\boldsymbol{\iota}\restriction\mathcal{S}$.

Version of 26.9.24

654 Picard's theorem

The general theory of solutions of ordinary differential equations begins with a classical existence and uniqueness theorem: if h is a continuous function of two variables which is Lipschitz in the first variable, then the differential equation

$$x'(t) = h(x(t), t), \quad x(0) = x_\star$$

or, equivalently, the integral equation

$$x(t) = x_\star + \int_0^t h(x(s), s)ds$$

has a unique solution. In this section I present corresponding results for stochastic integral equations of this type, first for the Riemann-sum integral (654G) and then for the S-integral (654L).

654B Lemma (a) Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ such that $\inf_{\tau\in\mathcal{S}}\sup_{\sigma\in\mathcal{S}}[\![\tau < \sigma]\!] = 0$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process.

(a) If $\boldsymbol{u}$ is locally order-bounded, it is order-bounded.

(b) If $\boldsymbol{u}$ is locally moderately oscillatory, it is moderately oscillatory.

(c) If $\boldsymbol{u}$ is locally near-simple, it is near-simple.

654C Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $h : \mathbb{R}^k \to \mathbb{R}$ a continuous function. Then $\bar{h}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k) \in M_{\text{o-b}} = M_{\text{o-b}}(\mathcal{S})$ whenever $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k \in M_{\text{o-b}}$, and $\bar{h} : M_{\text{o-b}}^k \to M_{\text{o-b}}$ is continuous for the ucp topology on $M_{\text{o-b}}$.

654D Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ with a greatest element, and define $\boldsymbol{z} = \langle z_\sigma\rangle_{\sigma\in\mathcal{S}}$ by setting $z_\sigma = \chi[\![\sigma < \max\mathcal{S}]\!]$ for $\sigma \in \mathcal{S}$.

(a) Suppose that $\boldsymbol{u}$ is a moderately oscillatory process and $\boldsymbol{v}$ an integrator. Then

$$\boldsymbol{z} \times ii_{\boldsymbol{v}}(\boldsymbol{u}) = \boldsymbol{z} \times ii_{\boldsymbol{v}}(\boldsymbol{z} \times \boldsymbol{u}) = \boldsymbol{z} \times ii_{\boldsymbol{z}\times\boldsymbol{v}}(\boldsymbol{u})$$

and $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{z} \times \boldsymbol{u}\, d\boldsymbol{v}$.

(b) Let $\boldsymbol{w}$ be a process of bounded variation, and $\boldsymbol{w}^\uparrow$ its cumulative variation. Then $\int_{\mathcal{S}} |d(\boldsymbol{z} \times \boldsymbol{w})| \le \sup|\boldsymbol{z} \times (\boldsymbol{w}^\uparrow + |\boldsymbol{w}|)|$.

(c) Suppose that $\mathcal{S}$ has a least member and that $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ is an order-bounded fully adapted process starting from $w_{\min\mathcal{S}} = 0$. Then $\sup|\boldsymbol{w}| \le \sup|\boldsymbol{z} \times \boldsymbol{w}| + \operatorname{Osclln}(\boldsymbol{w})$.

(d) Suppose that $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a fully adapted process and that $\alpha \ge 0$ is such that $[\![\sigma < \max\mathcal{S}]\!] \subseteq [\![|w_\sigma| \le \alpha]\!]$ for $\sigma \in \mathcal{S}$. Then $\boldsymbol{w}$ is order-bounded, $\|\sup|\boldsymbol{z} \times \boldsymbol{w}|\|_\infty \le \alpha$ and $\|\boldsymbol{w}\|_\infty \le \alpha + \|\operatorname{Osclln}(\boldsymbol{w})\|_\infty$.

(e) If $\boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})$ then $\boldsymbol{u}_< = (\boldsymbol{z} \times \boldsymbol{u})_<$.

654E Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. Write M^0_{mo} for the space of moderately oscillatory processes $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ with starting value 0. For an integrator $\boldsymbol{v}$, write $\boldsymbol{v}^*$ for its quadratic variation. Suppose that $\boldsymbol{u} \in M_{\text{mo}} = M_{\text{mo}}(\mathcal{S})$ and that $\boldsymbol{w}$, $\boldsymbol{w}' \in M^0_{\text{mo}}$ are such that $\boldsymbol{w}$ is a virtually local martingale and $\boldsymbol{w}'$ is of bounded variation; set $\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{w}'$. Then

$$\|\sup|ii_{\boldsymbol{v}}(\boldsymbol{u})|\|_2 \le 2(\sqrt{\|\boldsymbol{w}^*\|_\infty} + \|\int_{\mathcal{S}}|d\boldsymbol{w}'|\|_\infty)\|\sup|\boldsymbol{u}|\|_2.$$

654F Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ with a greatest element. Suppose that $h : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and that $K \ge 0$ is such that $|h(\alpha, \beta) - h(\alpha', \beta)| \le K|\alpha - \alpha'|$ for all α, α', $\beta \in \mathbb{R}$; let $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{w}' = \langle w'_\sigma\rangle_{\sigma\in\mathcal{S}}$ be processes with domain $\mathcal{S}$ such that $\boldsymbol{w}$ is a virtually local martingale, $\boldsymbol{w}'$ is of bounded variation and both start fom 0. Write $\boldsymbol{w}^*$ for the quadratic variation of $\boldsymbol{w}$, $\boldsymbol{w}'^\uparrow$ for the cumulative variation of $\boldsymbol{w}'$, and $\boldsymbol{z}$ for $\langle\chi[\![\sigma < \max\mathcal{S}]\!]\rangle_{\sigma\in\mathcal{S}}$. Suppose that

$$2K(\sqrt{\|\boldsymbol{w}^*\|_\infty} + 2\|\boldsymbol{z} \times \boldsymbol{w}'^\uparrow\|_\infty) < 1.$$

Set $\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{w}'$. Then for any $\boldsymbol{u}_\star$, $\boldsymbol{y} \in M_{\text{mo}} = M_{\text{mo}}(\mathcal{S})$ there is a unique $\boldsymbol{u} \in M_{\text{mo}}$ such that

$$\boldsymbol{u} = \boldsymbol{u}_\star + ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{u}, \boldsymbol{y})).$$

654G Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ with a least member. Suppose that $h:\mathbb{R}^2\to\mathbb{R}$ is a continuous function and that $K\ge 0$ is such that $|h(\alpha,\beta)-h(\alpha',\beta)|\le K|\alpha-\alpha'|$ for all α, α', $\beta\in\mathbb{R}$. Let $\boldsymbol{v}$ be a locally near-simple local integrator with domain $\mathcal{S}$. Then for any locally moderately oscillatory processes $\boldsymbol{u}_\star$, $\boldsymbol{y}$ with domain $\mathcal{S}$ there is a unique locally moderately oscillatory process $\boldsymbol{u}$ with domain $\mathcal{S}$ such that

$$\boldsymbol{u}=\boldsymbol{u}_\star+ii_{\boldsymbol{v}}(\bar h(\boldsymbol{u},\boldsymbol{y})).$$

654H Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous, and that $\mathcal{S}=[\min\mathcal{S},\max\mathcal{S}]$ is an interval in $\mathcal{T}$. Define $\boldsymbol{z}=\langle z_\sigma\rangle_{\sigma\in\mathcal{S}}$ by setting $z_\sigma=\chi[\![\sigma<\max\mathcal{S}]\!]$ for $\sigma\in\mathcal{S}$. Suppose that $\boldsymbol{x}\in M_{\text{S-i}}(\mathcal{S})$, and that $\boldsymbol{v}\in M_{\text{n-s}}=M_{\text{n-s}}(\mathcal{S})$ is an integrator.

(a) $\boldsymbol{z}\times\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})=\boldsymbol{z}\times\mathrm{S}ii_{\boldsymbol{z}\times\boldsymbol{v}}(\boldsymbol{x})$.

(b) $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})_<=\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{z}\times\boldsymbol{x})_<$.

654I Lemma Suppose $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and that $\mathcal{S}=[\min\mathcal{S},\max\mathcal{S}]$ is an interval in $\mathcal{T}$. Let $M^0_{\text{n-s}}$ be the space of near-simple processes $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ with domain $\mathcal{S}$ such that $u_{\min\mathcal{S}}=0$. For a near-simple integrator $\boldsymbol{v}$, write $\boldsymbol{v}^*$ for its quadratic variation. Suppose that $\boldsymbol{w}$, $\boldsymbol{w}'\in M^0_{\text{n-s}}$ are such that $\boldsymbol{w}$ is a martingale and $\boldsymbol{w}'$ is of bounded variation; set $\boldsymbol{v}=\boldsymbol{w}+\boldsymbol{w}'$. If $\boldsymbol{x}\in M^0_{\text{S-i}}(\mathcal{S})$, $\boldsymbol{u}\in M_{\text{mo}}(\mathcal{S})$ and $|\boldsymbol{x}|\le\boldsymbol{u}_<$, then

$$\|\sup|\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})|\|_2\le 2(\sqrt{\|\boldsymbol{w}^*\|_\infty}+\|\textstyle\int_{\mathcal{S}}|d\boldsymbol{w}'|\|_\infty)\|\sup|\boldsymbol{u}|\|_2.$$

654J Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}=[\min\mathcal{S},\max\mathcal{S}]$ be an interval in $\mathcal{T}$. Suppose that $h:\mathbb{R}^2\to\mathbb{R}$ is a locally bounded Borel measurable function and that $K\ge 0$ is such that $|h(\alpha,\beta)-h(\alpha',\beta)|\le K|\alpha-\alpha'|$ for all α, α', $\beta\in\mathbb{R}$; let $\boldsymbol{w}=\langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{w}'=\langle w'_\sigma\rangle_{\sigma\in\mathcal{S}}$ be near-simple processes with domain $\mathcal{S}$ such that $\boldsymbol{w}$ is a martingale, $\boldsymbol{w}'$ is of bounded variation and $w_{\min\mathcal{S}}=w'_{\min\mathcal{S}}=0$. Write $\boldsymbol{w}^*$ for the quadratic variation of $\boldsymbol{w}$, $\boldsymbol{w}'^{\uparrow}$ for the cumulative variation of $\boldsymbol{w}'$, and $\boldsymbol{z}$ for $\langle\chi[\![\sigma<\max\mathcal{S}]\!]\rangle_{\sigma\in\mathcal{S}}$. Suppose that $2K(\sqrt{\|\boldsymbol{w}^*\|_\infty}+2\|\boldsymbol{z}\times\boldsymbol{w}'^{\uparrow}\|_\infty)<1$. Set $\boldsymbol{v}=\boldsymbol{w}+\boldsymbol{w}'$. Then for any $\boldsymbol{x}_\star$, $\boldsymbol{y}\in M_{\text{S-i}}=M_{\text{S-i}}(\mathcal{S})$ there is a unique process $\boldsymbol{x}\in M_{\text{S-i}}$ such that

$$\boldsymbol{x}=\boldsymbol{x}_\star+\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))_<.$$

654K Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous, and that $\mathcal{S}\subseteq\mathcal{T}$ is an order-convex sublattice with a least member. Let $h:\mathbb{R}^2\to\mathbb{R}$ be a locally bounded Borel measurable function. Take processes $\boldsymbol{x}$, $\boldsymbol{x}_\star$, $\boldsymbol{y}\in M_{\text{S-i}}(\mathcal{S})$ and an integrator $\boldsymbol{v}\in M_{\text{n-s}}(\mathcal{S})$. Set $\boldsymbol{u}=\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))$ and express $\boldsymbol{u}$ as $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{u}_<$ as $\langle u_{<\sigma}\rangle_{\sigma\in\mathcal{S}}$.

Fix $\tau\in\mathcal{S}$. Set

$$\mathcal{S}'=\mathcal{S}\wedge\tau,\quad \boldsymbol{x}'=\boldsymbol{x}\restriction\mathcal{S}',\quad \boldsymbol{v}'=\boldsymbol{v}\restriction\mathcal{S}',\quad \boldsymbol{y}'=\boldsymbol{y}\restriction\mathcal{S}',\quad \boldsymbol{x}'_\star=\boldsymbol{x}_\star\restriction\mathcal{S}',$$

$$\mathcal{S}''=\mathcal{S}\vee\tau,\quad \boldsymbol{x}''=\boldsymbol{x}\restriction\mathcal{S}'',\quad \boldsymbol{v}''=\boldsymbol{v}\restriction\mathcal{S}'',\quad \boldsymbol{y}''=\boldsymbol{y}\restriction\mathcal{S}''\quad \boldsymbol{x}''_\star=\boldsymbol{x}_\star\restriction\mathcal{S}''+\tilde{\boldsymbol{x}}_*$$

where $\tilde{\boldsymbol{x}}_*=u_{<\tau}\mathbf{1}^{(\mathcal{S}'')}+(u_\tau-u_{<\tau})\mathbf{1}^{(\mathcal{S}'')}_<$.

(a) $\mathrm{S}ii_{\boldsymbol{v}'}(\bar h(\boldsymbol{x}',\boldsymbol{y}'))=\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))\restriction\mathcal{S}'$, so $\mathrm{S}ii_{\boldsymbol{v}'}(\bar h(\boldsymbol{x}',\boldsymbol{y}'))_<=\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))_<\restriction\mathcal{S}'$.

(b) $\boldsymbol{x}''_\star\in M_{\text{S-i}}(\mathcal{S}'')$.

(c) $\boldsymbol{x}=\boldsymbol{x}_\star+\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))_<$ if and only if

$$\boldsymbol{x}'=\boldsymbol{x}'_\star+\mathrm{S}ii_{\boldsymbol{v}'}(\bar h(\boldsymbol{x}',\boldsymbol{y}'))_<\quad\text{and}\quad\boldsymbol{x}''=\boldsymbol{x}''_\star+\mathrm{S}ii_{\boldsymbol{v}''}(\bar h(\boldsymbol{x}'',\boldsymbol{y}''))_<.$$

654L Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ with a least member. Suppose that $h:\mathbb{R}^2\to\mathbb{R}$ is a locally bounded Borel measurable function and that $K\ge 0$ is such that $|h(\alpha,\beta)-h(\alpha',\beta)|\le K|\alpha-\alpha'|$ for all α, α', $\beta\in\mathbb{R}$. Let $\boldsymbol{v}$ be a locally near-simple local integrator with domain $\mathcal{S}$. Then for any locally S-integrable processes $\boldsymbol{x}_\star$, $\boldsymbol{y}$ with domain $\mathcal{S}$ there is a unique locally S-integrable process $\boldsymbol{x}$ with domain $\mathcal{S}$ such that

$$\boldsymbol{x}=\boldsymbol{x}_\star+\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))_<.$$

Version of 13.2.21

655 The Black-Scholes model

This volume is supposed to be an introduction to stochastic integration for a mathematically sophisticated but otherwise untutored readership. You will find it difficult to persuade anyone else to take your efforts seriously if you do not have something to say about its most famous applications, starting with Black & Scholes 73. I will therefore take the space to give a very short account of the simplest model derived by their method, even though the mathematical content is no more than direct quotes from the work so far, and all the interesting ideas relate to the theory of financial markets.

655A Stochastic differential equations In §§651 and 653, I expressed every result in terms of integral equations; so that Theorem 651B, for instance, reads

$$\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{z} = \int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{z}\, d\boldsymbol{v}.$$

But the mnemonic for it, in the style of §617, would be

$$d\boldsymbol{z} \sim \boldsymbol{z}\, d\boldsymbol{v},$$

and some authors are happy to express this in the form $\frac{d\boldsymbol{z}}{d\boldsymbol{v}} = \boldsymbol{z}$. Similarly, where in Theorem 654G I write

$$\boldsymbol{u} = \boldsymbol{u}_\star + ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{u}, \boldsymbol{y})),$$

others might write

$$u_{\min \mathcal{S}} = u_{\star,\min \mathcal{S}}, \quad d\boldsymbol{u} \sim d\boldsymbol{u}_\star + \bar{h}(\boldsymbol{u}, \boldsymbol{y})d\boldsymbol{v}$$

or perhaps

$$u_{\min \mathcal{S}} = u_{\star,\min \mathcal{S}}, \quad \frac{d\boldsymbol{u}}{d\boldsymbol{v}} = \frac{d\boldsymbol{u}_\star}{d\boldsymbol{v}} + \bar{h}(\boldsymbol{u}, \boldsymbol{y}).$$

655B A model of stock prices For the rest of this section, I will suppose that $(\mathfrak{A}, \bar{\mu}, [0, \infty[\,, \langle \mathfrak{A}_t \rangle_{t \ge 0}, \mathcal{T}, \langle \mathfrak{A}_\tau \rangle_{\tau \in \mathcal{T}})$ is a right-continuous real-time stochastic integration structure and $\mathcal{S}$ is a non-empty ideal of $\mathcal{T}$. Let $\boldsymbol{\iota}$ be the identity process and $\boldsymbol{w}$ a Brownian-type process on $\mathcal{S}$. Consider the differential equation

$$d\boldsymbol{u} \sim \alpha \boldsymbol{u} d\boldsymbol{\iota} + \beta \boldsymbol{u} d\boldsymbol{w}, \quad u_{\check{0}} = u_\star$$

or

$$\boldsymbol{u} = u_\star \mathbf{1} + \alpha ii_{\boldsymbol{\iota}}(\boldsymbol{u}) + \beta ii_{\boldsymbol{w}}(\boldsymbol{u}) = u_\star \mathbf{1} + ii_{\tilde{\boldsymbol{w}}}(\boldsymbol{u})$$

where $\tilde{\boldsymbol{w}} = \alpha \boldsymbol{\iota} + \beta \boldsymbol{w}$ and $u_\star \in L^0(\mathfrak{A}_0)$. Then $\tilde{\boldsymbol{w}}$ is a locally jump-free local integrator and its quadratic variation $\tilde{\boldsymbol{w}}^*$ is $\beta^2 \boldsymbol{\iota}$. So the equation has solution

$$\boldsymbol{u} = u_\star \, \overline{\exp}(\tilde{\boldsymbol{w}} - \tfrac{1}{2}\beta^2 \boldsymbol{\iota}),$$

which is a locally jump-free local integrator and is unique.

655C A model for options Now suppose that we have an 'option' in a 'stock' whose value is accurately modelled by the process $\boldsymbol{u}$. Our objective is to find a rational approach leading to a way of determining the value $\boldsymbol{v}$ of this option. We assume that there is a function h such that $\boldsymbol{v} = \bar{h}(\boldsymbol{u}, \boldsymbol{\iota})$. (If $\boldsymbol{u}$ and $\boldsymbol{v}$ are represented by real-valued processes $\langle U_t \rangle_{t \ge 0}$ and $\langle V_t \rangle_{t \ge 0}$ with càdlàg sample paths, we are supposing that $V_t(\omega) = h(U_t(\omega), t)$ for most pairs (ω, t).) The terms of the option will give us some information about the function h; for instance, a call option, allowing us to buy a quantity c of the stock for a strike price x_1 at expiry time t_1, will then have value $h(x, t_1) = c \max(x - x_1, 0)$, because we shall be able to buy the stock at price x_1 and sell it at price x; if $x \le x_1$, we just do nothing.

We assume that h is twice continuously differentiable, with partial derivatives h_1, h_2 and second partial derivatives $h_{11}, \dots, h_{22}$. Then

$$\boldsymbol{v} = \bar{h}(u_\star, 0) + ii_{\boldsymbol{u}}(\bar{h}_1(\boldsymbol{u}, \boldsymbol{\iota})) + ii_{\boldsymbol{\iota}}(\bar{h}_2(\boldsymbol{u}, \boldsymbol{\iota}) + \frac{1}{2}\beta^2 \boldsymbol{u}^2 \times \bar{h}_{11}(\boldsymbol{u}, \boldsymbol{\iota})).$$

655D Hedging and a risk-free portfolio Still supposing that there is such a function h, consider a hedged version of the option. In addition to the option, we 'hedge' by a suitably varying quantity $\bar{g}(\boldsymbol{u},\boldsymbol{\iota})$ in the stock $\boldsymbol{u}$ to give us a portfolio $\tilde{\boldsymbol{v}} = \boldsymbol{v} - ii_{\boldsymbol{u}}(\bar{g}(\boldsymbol{u},\boldsymbol{\iota}))$ in which the value of $\boldsymbol{v}$ is modified by the accumulated losses and gains of our hedging strategy. The idea of a 'hedge' is that we can 'sell the stock short', that is, sell stock we don't necessarily possess; we take cash now, and promise to buy the stock back soon *at the price then ruling*. This is not an option, it is a contract. From the point of view of our counterparty, it is just like buying real stock. The idea behind this is that if we possess a call option, we stand to make money if the stock rises in value, and not if it falls; by going short, we hedge our bet to make our prospects more even.

We allow g to take negative values; this is because we can 'go long', that is, buy some stock with the intention of selling it again if our strategy calls on us to do so. Note that we believe that $\boldsymbol{u}$ is jump-free, so can imagine adjusting the hedge rapidly compared with changes in $\boldsymbol{u}$.

We shall have

$$\tilde{\boldsymbol{v}} = \bar{h}(u_\star,0) + ii_{\boldsymbol{u}}(\bar{h}_1(\boldsymbol{u},\boldsymbol{\iota}) - \bar{g}(\boldsymbol{u},\boldsymbol{\iota})) + ii_{\boldsymbol{\iota}}(\bar{h}_2(\boldsymbol{u},\boldsymbol{\iota}) + \frac{1}{2}\beta^2\boldsymbol{u}^2 \times \bar{h}_{11}(\boldsymbol{u},\boldsymbol{\iota})).$$

So if we set $g = h_1$, we get

$$\tilde{\boldsymbol{v}} = \bar{h}(u_\star,0) + ii_{\boldsymbol{\iota}}(\bar{h}_2(\boldsymbol{u},\boldsymbol{\iota}) + \frac{1}{2}\beta^2\boldsymbol{u}^2 \times \bar{h}_{11}(\boldsymbol{u},\boldsymbol{\iota})),$$

$$d\tilde{\boldsymbol{v}} \sim (\bar{h}_2(\boldsymbol{u},\boldsymbol{\iota}) + \frac{1}{2}\beta^2\boldsymbol{u}^2 \times \bar{h}_{11}(\boldsymbol{u},\boldsymbol{\iota}))d\boldsymbol{\iota}.$$

Now this is 'risk-free'; for a short time interval $[\sigma,\sigma']$,

$$\tilde{v}_{\sigma'} - \tilde{v}_\sigma \doteqdot (\bar{h}_2(u_\sigma,\sigma) + \frac{1}{2}\beta^2\bar{h}_{11}(u_\sigma,\sigma)) \times (\sigma'-\sigma)$$

is well approximated by something calculable from knowledge of the stopping times σ, σ' and the situation at the starting time σ, but not requiring any foreknowledge of the evolution of $\boldsymbol{u}$ or $\boldsymbol{w}$. Suppose that we can be sure of being able to borrow, or lend, cash, at an interest rate ρ, with complete safety. We are supposed to be operating in a perfect market, in which every agent knows just what we know about $\boldsymbol{w}$ and $\boldsymbol{u}$, and can do the same calculations, so that the process $\bar{h}(\boldsymbol{u},\boldsymbol{\iota})$ describes the evolution of the market price, either buying or selling, of the option. We therefore expect $\tilde{v}_{\sigma'} - \tilde{v}_\sigma$, the agreed expected profit from holding the portfolio $\tilde{v}$ from time σ to time σ', to be very close to the expected income over that time period if we sell our option and our holding in the stock $\boldsymbol{u}$, and invest the net proceeds in a bond at interest rate ρ.

At this point I need to remark that these net proceeds will not be the current value $\tilde{v}_\sigma$. The process $\tilde{\boldsymbol{v}}$ takes past gains and losses into account in the term $ii_{\boldsymbol{u}}(\bar{g}(\boldsymbol{u},\boldsymbol{\iota}))$. Our holding at, and immediately after, the time σ is $v_\sigma - \bar{g}(u_\sigma,\sigma) \times u_\sigma$, because if g is positive, that is, we are shorting the stock, we shall have to buy it back at once to liquidate our position, while if g is negative, that is, we are holding some stock, we will sell it. So we expect that

$$\tilde{v}_{\sigma'} - \tilde{v}_\sigma \doteqdot \rho(v_\sigma - \bar{h}_1(u_\sigma,\sigma) \times u_\sigma) \times (\sigma'-\sigma),$$

that is,

$$d\tilde{\boldsymbol{v}} \sim \rho(\boldsymbol{v} - \bar{h}_1(\boldsymbol{u},\boldsymbol{\iota}) \times \boldsymbol{u})d\boldsymbol{\iota},$$

and matching the two formulae for $d\tilde{\boldsymbol{v}}$ we get

$$\bar{h}_2(\boldsymbol{u},\boldsymbol{\iota}) + \frac{1}{2}\beta^2\boldsymbol{u}^2 \times \bar{h}_{11}(\boldsymbol{u},\boldsymbol{\iota}) = \rho(\boldsymbol{v} - \boldsymbol{u} \times \bar{h}_1(\boldsymbol{u},\boldsymbol{\iota})) = \rho(\bar{h}(\boldsymbol{u},\boldsymbol{\iota}) - \boldsymbol{u} \times \bar{h}_1(\boldsymbol{u},\boldsymbol{\iota})).$$

To ensure this, we shall have to have

$$h_2(x,t) + \frac{1}{2}\beta^2x^2h_{11}(x,t) = \rho(h(x,t) - xh_1(x,t))$$

for all relevant x and t, that is,

$$\frac{\partial h}{\partial t} + \frac{1}{2}\beta^2x^2\frac{\partial^2 h}{\partial x^2} + \rho x\frac{\partial h}{\partial x} - \rho h = 0$$

which is the **Black-Scholes equation** for the evolution of the value $h(x, t)$ of an option in a stock with price x at time t.